

# Optimal Two-stage Auctions with Costly Information Acquisition\*

Jingfeng Lu<sup>†</sup> Lixin Ye<sup>‡</sup>

December 2014

## Abstract

We consider an auction environment with costly entry wherein the cost mainly stems from information acquisition. Bidders are endowed with original estimates (“types”) about their private values and can further learn their true values of the object for sale by incurring an entry cost. We first derive an integral form of the envelope formula as required by incentive compatible two-stage mechanisms, based on which we demonstrate that optimality of the generalized Myerson allocation rule is robust to our setting with costly information acquisition. Optimal entry is thus to admit the set of bidders that maximizes expected *virtual surplus* adjusted by both the second-stage signal and entry cost. We show that our optimal entry and allocation rules are both IR and IC implementable, and furthermore, in an important class of environments, they can be implemented via a two-stage auction that is essentially a handicap auction augmented with an entry mechanism.

Keywords: Two-stage auctions, entry, information acquisition, sequential screening, handicap auctions, optimal mechanisms.

JEL Classification: D44, D80, D82.

## 1 INTRODUCTION

In high-valued asset sales, buyers often need to go through a due diligence process before developing final bids. Due diligence is usually a process to update or acquire information about the value of the asset for sale or to prepare for the bidding process (e.g., to establish qualifications to bid). This process is costly and is usually modeled as entry as it is closely monitored by the auctioneer. For a sale of an asset worth billions of dollars, the entry cost can run from tens of thousands to millions of dollars.<sup>1</sup>

---

\*We thank seminar participants at University of Michigan, the Conference in Honor of Paul Milgrom’s 65th Birthday, the Midwest Economic Theory Conference, NSF/CEME Decentralization Conference, and in particular, Dirk Bergemann, Tilman Börgers, Yeon-Koo Che, Jeff Ely, Li Hao, Preston McAfee, David Miller, Ilya Segal, Xianwen Shi and Juuso Toikka for very helpful comments and suggestions. All remaining errors are our own.

<sup>†</sup>Department of Economics, National University of Singapore, 10 Kent Ridge Crescent, Singapore 119260. Tel: (65)65166026, Email: ecljlf@nus.edu.sg.

<sup>‡</sup>Department of Economics, The Ohio State University, 449A Arps Hall, 1945 North High Street, Columbus, OH 43210. Tel.: (614)292-6883. Email: ye.45@osu.edu.

<sup>1</sup>A more detailed description of a typical due diligence process is provided in Section 4.

Given the substantial entry cost, it is unrealistic to assume that whoever is interested would necessarily go through the costly entry process. The success of a sale thus very much relies on whether the most qualified bidders would commit to the due diligence process and participate in the final sale. Mainly motivated by the need for entry screening, variants of two-stage selling mechanisms have emerged in the real world. A leading example of the two-stage auction procedure is known as indicative bidding, which is commonly used in sales of complicated business assets with very high values. It works as follows: the auctioneer actively markets the assets to a large group of potentially interested buyers. The potential buyers are then asked to submit non-binding bids, based on which a final set of bidders is shortlisted to advance to the second stage. The auctioneer then communicates only with these final bidders, providing them with extensive access to information about the assets,<sup>2</sup> and finally runs the auction (typically using binding sealed bids). The use of this two-stage auction procedure is quite widespread. For example, in response to the restructuring of the electric power industry in the U.S. – which was designed to separate power generation from transmission and distribution – billions of dollars of electrical generating assets were divested through this two-stage auction procedure over the last two decades.<sup>3</sup> This two-stage auction procedure is also commonly used in privatization, takeover, and merger and acquisition contests.<sup>4</sup> Finally, it is commonly used in the institutional real estate market, which has an annual sales volume in the order of \$60 to \$100 billion.<sup>5</sup>

Ye (2007) was the first study of indicative bidding based on the assumption of costly information acquisition.<sup>6</sup> Ye’s analysis suggests that the current design of indicative bidding cannot reliably select the most qualified bidders for the final sale, as there does not exist a symmetric, strictly increasing equilibrium bid function in the indicative bidding stage. In a more recent paper, by restricting indicative bids to a finite discrete domain, Quint and Hendricks (2013) show that a symmetric equilibrium exists in weakly-monotone strategies. But again, the highest-value bidders are not always selected, as bidder types “pool” over a finite number of bids. Without safely selecting the most qualified bidders for the final sale, the mechanism is hardly optimal in maximizing expected revenue. What the optimal mechanism is in this two-stage auction environment remains an open question in the literature, and this paper seeks to provide an answer.

We model the two-stage auction environment as follows. Before entry, each potential bidder is endowed with a private signal,  $\alpha_i$ , which can be regarded as her pre-entry “type.” After entry (by incurring a common entry cost,  $c$ ), each bidder  $i$  fully observes her (private) value  $v_i$ , which is positively correlated to her pre-entry type. Given costly entry, it is not feasible for all potential bidders to be included in the final sale. As such, we consider the class of two-stage mechanisms with the first stage allocating entry

---

<sup>2</sup>Data rooms, which are described in Section 4, are typically set up to facilitate bidders’ due diligence process.

<sup>3</sup>A list of industry examples using this two-stage auction design can be found in Ye (2007).

<sup>4</sup>Leading examples include the privatization of the Italian Oil and Energy Corporation (ENI), the acquisition of Ireland’s largest cable television provider Cablelink Limited, and the takeover contest for South Korea’s second largest conglomerate Daewoo Motors.

<sup>5</sup>See Foley (2003) for a detailed account.

<sup>6</sup>Boone and Goeree (2009) provide an analysis of pre-qualifying auctions, which are similar to indicative bidding.

rights and the second stage allocating the asset.<sup>7</sup>

Despite the potential complication due to both sequential screening and costly information acquisition, we are able to completely characterize the optimal revenue-maximizing two-stage mechanisms. Our analysis benefits greatly from recent developments in the literature of sequential screening (e.g., Courty and Li, 2000; Esö and Szentes, 2007; Bergemann and Wambach, 2014; and Pavan, Segal, and Toikka, 2014).<sup>8</sup> In particular, our analysis follows Esö and Szentes closely, and our technical contribution is to extend their analysis to dynamic auctions with costly information acquisition. We first derive an integral form of the envelope formula as a necessary condition for incentive compatibility for our two-stage mechanisms, which extends the validity of the envelope theorem to dynamic auctions with costly information acquisition. Based on this derived envelope formula, we are able to show that the optimal allocation rule of the asset in our second stage is the same as that identified by Esö and Szentes, which requires that, among the shortlisted bidders, the asset be allocated to the bidder with the highest *virtual value* adjusted by the second-stage signal. Our analysis thus suggests that the optimality of the generalized Myerson optimal allocation rule (adjusted by second-round signals) is robust to the dynamic auction setting with costly entry. The first-stage entry right allocation mechanism is new to the original Esö-Szentes framework, and we show that the optimal entry rule is to admit the set of bidders that gives rise to the maximum expected *virtual surplus* (adjusted by both the second-stage signal and entry cost). Alternatively, given the regularity assumption and that buyers are *ex ante* symmetric in our model, the optimal entry rule is to admit the bidders in descending order of their pre-entry “types,” the highest type first, the second highest type second, etc., provided that their marginal contribution to the expected virtual surplus is positive. Therefore, the optimal number of shortlisted bidders typically depends on the reported type profile from the potential bidders, which is endogenously determined. We then show that specific payment rules can be constructed in each stage to implement both optimal entry and allocation rules truthfully.

For an important setting where one’s value is linear in her first signal, Esö and Szentes show that their optimal mechanism can be implemented over two rounds via a so-called handicap auction: in the first round (before observing the second-stage signal), each buyer selects a “price premium” by paying a fee according to a pre-announced schedule. In the second round (after observing the second-stage signals), buyers compete in a second-price or English auction, where the winner obtains the object at a price equal to the second-highest bid plus the price premium selected from the first round. In our setting with entry, the implementation is presumably more complicated, as optimal entry needs to be implemented prior to the final auction. Indeed, now the implementation requires that an (optimal) entry rule be augmented to the handicap auction. So in our case the optimal mechanism is implemented via a two-stage auction, with the first stage being an auction for entry rights (as well as the price premia) and the second stage

---

<sup>7</sup>The focus on two-stage mechanisms should be regarded as a constraint, which is discussed in Section 4.

<sup>8</sup>Early work on dynamic contracting with a single agent are due to Baron and Besanko (1984) and Riordan and Sappington (1987).

being a second-price or English auction for the asset.

Other than the connection with sequential screening and dynamic auctions mentioned above, our paper is related to the literature on information acquisition in auctions (see, for example, Persico, 2000; Compte and Jehiel, 2001; and Rezende, 2013). These papers focus on bidders' incentives to acquire information in different auction formats. Our paper differs from theirs in that we follow the normative approach to identify optimal mechanisms with information acquisition.

To the extent that information acquisition is modeled as entry, our paper is closely related to the growing literature on auctions with costly entry.<sup>9</sup> This literature can be summarized into three branches. In the first branch, bidders are assumed to possess no private information before entry and they learn their private values or signals only after entry (see, for example, McAfee and McMillan, 1987; Engelbrecht-Wiggans, 1993; Tan, 1992; Levin and Smith, 1994; and Ye, 2004). In the second branch, it is assumed that bidders are endowed with private information about their values but have to incur entry costs to participate in an auction (see, for example, Samuelson, 1985; Stegeman, 1996; Campbell, 1998; Menezes and Monteiro, 2000; Tan and Yilankaya, 2006; Cao and Tian, 2009; and Lu, 2009). Finally, in the third branch, bidders are endowed with some private information before entry, and are able to acquire additional private information after entry (Ye, 2007; Quint and Hendricks, 2013). The framework in this current paper nests all the models mentioned above as special cases. Our paper thus characterizes optimal mechanisms for a very general framework in the literature on auctions with costly entry.

Our research is also related to a small literature on auctions of entry rights. In a pioneering work, Fullerton and McAfee (1999) introduce auctions for entry rights to shortlist contestants for a final tournament. Ye (2007) extends their approach to the setting of two-stage auctions described above. Our current approach differs from theirs in the way the set of finalists is determined: while in their approach the number of finalists to be selected is fixed and pre-announced, in our entry right allocation mechanism the selection of shortlisted bidders is contingent on the reported bid profile, making the number of finalists endogenously determined. For this reason the entry right allocation mechanism examined in this research is more general.<sup>10</sup>

In another relevant paper, Lu and Ye (2013) explore optimal two-stage mechanisms in an environment where bidders are characterized by heterogenous and private information acquisition costs before entry. In that setting the pre-entry "type" is the entry cost, which is neither correlated to nor part of the value of the asset for sale. As such, there is no benefit to make the second-stage mechanism contingent on the reports of the pre-entry types, resulting in a much simpler characterization of optimal mechanisms. The setting in this current paper is different, as the pre-entry "type" is correlated to the value of the asset, hence there are potential gains to make the second-stage mechanism contingent on first-stage reports. Indeed, in our current setting, the optimal allocation and payment rules in the second stage do depend on the first-stage reports. Therefore the characterization of optimal mechanisms is more demanding in

---

<sup>9</sup>See Bergemann and Välimäki (2006) for a thoughtful survey of this literature.

<sup>10</sup>In fact, it resemble multi-unit auctions with endogenously determined supply (see, e.g., McAdams, 2007).

this work, and the implementation of the optimal mechanism is also more sophisticated.

The rest of the paper is organized as follows. Section 2 presents the model. Section 3 characterizes the optimal mechanism and its auction implementation. Section 4 discusses some restrictions in our analysis. Section 5 concludes.

## 2 THE MODEL

The information structure in our model is closest to that in Esö and Szentes (2007). The main differences are that in Esö and Szentes, the additional information is controlled by the seller, and they focus on the seller’s incentive to *disclose* (without observing) additional signals to the buyers. In our setting, however, it is costly for the bidders to *acquire* additional information, and we focus on the bidders’ incentive for information acquisition (entry). In addition, all buyers are included in the final sale in Esö and Szentes, but due to costly entry in our setting, not all buyers will be willing to participate in the final auction. As such, we will additionally consider entry mechanisms – which is the major difference from the analysis in Esö and Szentes.

Formally, a single indivisible asset is offered for sale to  $N$  potentially interested buyers. The seller and bidders are assumed to be risk neutral. The seller’s own valuation for the asset is normalized to 0. Buyer  $i$ ’s true valuation for the asset is  $v_i$ . However, initially she only observes a noisy signal of it,  $\alpha_i$ , which is her private information and can be interpreted as her original “type.” After incurring a common information acquisition cost (or entry cost) of  $c$ , bidder  $i$  fully observes her ex post value,  $v_i$ . The pairs  $(\alpha_i, v_i)$  are assumed to be independent across  $i$ .<sup>11</sup>

Ex ante,  $\alpha_i$  follows distribution  $F(\cdot)$  with its associated density  $f(\cdot)$  on support  $[\underline{\alpha}, \bar{\alpha}]$ .<sup>12</sup> We assume that  $f$  is positive on the interval  $[\underline{\alpha}, \bar{\alpha}]$  and satisfies the monotone hazard rate condition; that is,  $f/(1 - F)$  is weakly increasing. Given  $\alpha_i$ , the ex post value  $v_i$  follows distribution  $H_{\alpha_i} \equiv H(\cdot|\alpha_i)$  with its density  $h_{\alpha_i} \equiv h(\cdot|\alpha_i)$  over support  $[\underline{v}, \bar{v}] \subset \mathbb{R}$ .<sup>13</sup> The values  $N$  and  $c$  and distributions  $F$  and  $H_{\alpha_i}$  are all common knowledge.

Following the signal orthogonalization technique introduced by Esö and Szentes (2007),<sup>14</sup> there exist functions  $u$  and  $s_i$ , such that  $u(\alpha_i, s_i) \equiv v_i$ , where  $u$  is strictly increasing in both arguments, and  $s_i$  is independent of  $\alpha_i$ . In particular,  $s_i$  can be constructed as follows:

$$s_i = H(v_i|\alpha_i),$$

---

<sup>11</sup>As in Esö and Szentes (2007) and Pavan, Segal, and Toikka (2014), this assumption rules out the possibility of full rent extraction (Crémer and McLean, 1988).

<sup>12</sup>Esö and Szentes allow  $\alpha_i$ ’s to be drawn from different distributions. Our procedure can be extended to accommodate asymmetric distributions for  $\alpha_i$ ’s. For ease of characterizing our optimal entry right allocation rule, we assume that  $\alpha_i$ ’s are drawn from a common distribution, so that bidders are *ex ante* symmetric. Note that with different realizations of  $\alpha_i$ ’s, bidder heterogeneity before entry is still captured in our model.

<sup>13</sup>Following the dynamic mechanism design literature, we assume that the support of  $v_i$  is independent of the first-stage signal  $\alpha_i$ .

<sup>14</sup>The use of this technique becomes standard in the literature (see, e.g., Pavan, Segal, and Toikka, 2014, and Bergemann and Wambach, 2014).

which is the percentile of the value realization to bidder  $i$ .<sup>15</sup> Thus given type  $\alpha_i$  and signal  $s_i$ , the valuation can be computed as

$$v_i = H_{\alpha_i}^{-1}(s_i) \equiv u(\alpha_i, s_i).$$

We will denote the c.d.f. of  $s_i$  by  $G_i$ .<sup>16</sup>

We maintain the following assumptions that are adopted in Esö and Szentes (2007):

**Assumption 1.**  $(\partial H_\alpha(v)/\partial \alpha)/h_\alpha(v)$  is increasing in  $v$ .

**Assumption 2.**  $(\partial H_\alpha(v)/\partial \alpha)/h_\alpha(v)$  is increasing in  $\alpha$ .

Esö and Szentes show that Assumption 1 is equivalent to  $u_{12} \leq 0$  and Assumption 2 is equivalent to  $u_{11}/u_1 \leq u_{12}/u_2$ . Assumption 1 thus states that the marginal impact of the new information on buyer  $i$ 's value is decreasing in her type  $\alpha_i$ . Assumption 2 implies that an increase in  $\alpha_i$ , holding  $u(\alpha_i, s_i)$  constant, weakly decreases the marginal value of  $\alpha_i$ . Assumptions 1 and 2 can thus be interpreted as a kind of substitutability in buyer  $i$ 's posterior valuation between  $\alpha_i$  and  $s_i$ .

**Example 1.** (Ye, 2007): Each potential bidder is endowed with a private value component  $\alpha_i$  before entry; after entry, each buyer learns another private value component  $s_i$ , where  $s_i$  is independent of  $\alpha_i$ . The ex post value  $u(\alpha_i, s_i) = \alpha_i + s_i$ . By the linearity of  $u(\alpha_i, s_i)$ , Assumptions 1 and 2 hold.

**Example 2.** (Adapted from Esö and Szentes, 2007):  $v_i$  is drawn from a normal distribution with mean  $\mu$  and precision (inverse variance)  $\tau_0$ . The pre-entry type,  $\alpha_i$ , is normally distributed with mean  $v_i$  and precision  $\tau_v$ . After entry, the buyer can observe her true value,  $v_i$ . It is clear that  $v_i$  and  $\alpha_i$  are strictly affiliated. The distribution of  $\alpha_i$ , which is normal, satisfies the hazard rate condition. The cdf of  $v_i$  conditional on  $\alpha_i$ ,  $H_{\alpha_i}$ , is normal with mean  $(\tau_0\mu + \tau_v\alpha_i)/(\tau_0 + \tau_v)$  and precision  $\tau_0 + \tau_v$ . Define  $s_i = H_{\alpha_i}(v_i)$  and let  $u(\alpha_i, s_i) = H_{\alpha_i}^{-1}(s_i) \equiv v_i$ . Obviously  $u$  is strictly increasing in  $s_i$ . It can be verified that  $u_1(\alpha_i, s_i) = \tau_v/(\tau_0 + \tau_v)$ , which is a constant. Therefore,  $u$  is linear and strictly increasing in  $\alpha_i$ . Hence Assumptions 1 and 2 hold.

Since information acquisition is modeled as entry in our setting, we consider a mechanism design framework in which the seller exercises entry control. In addition, we restrict our analysis to two-stage mechanisms: the first stage is the entry right allocation mechanism, and the second stage is the private good provision mechanism. Note that in this mechanism design framework, the second-stage mechanism can be made contingent on the first-stage reports.

Following Myerson (1986) and Pavan, Segal, and Toikka (2014), we restrict to direct mechanisms where agents report their types truthfully at each stage on the equilibrium path. We assume that the first-stage reported profile  $\alpha$  is revealed to all admitted bidders so that the first-stage entry allocation

<sup>15</sup>It is easily seen that  $s_i$  is uniformly distributed over  $[0, 1]$ , and is hence statistically independent of the initial information  $\alpha_i$ .

<sup>16</sup> $G_i$  could be assumed to be uniform on  $[0, 1]$ . More generally, all  $s_i$ 's satisfying  $u(\alpha_i, s_i) \equiv v_i$  are positive monotonic transformation of each other (Lemma 1 in Esö and Szentes).

and payments are immediately verifiable.<sup>17</sup> This revelation policy turns out to be “optimal,” in the sense that no other revelation policy (e.g., not revealing or partially revealing  $\alpha$ ) can generate higher expected revenue to the seller. For this reason, our restriction to fully reveal  $\alpha$  is without loss of generality in our search for optimal mechanisms. A detailed discussion is relegated to Section 4. As in Pavan, Segal, and Toikka (2014), the revelation policy concerned in this paper is about the first-stage information and outcome, including the agents’ first-stage reports, their payments, and the agents being shortlisted. In our paper, the principal has no control over the ways in which the second-stage new information is revealed to bidders. A shortlisted bidder will be fully informed about her true value  $v_i$  after incurring the entry cost. As such, we are not concerned about the discriminatory information disclosure issue studied in Li and Shi (2013).

As in Esö and Szentes, we can focus attention on equivalent direct mechanisms that require bidders to report  $s_i$ ’s, rather than  $v_i$ ’s. Note that reporting  $(\alpha'_i, v'_i)$  is equivalent to reporting  $(\alpha'_i, s'_i = H_{\alpha'_i}(v'_i))$ .

Let  $\mathbf{N} = \{1, 2, \dots, N\}$  denote the set of all the potential buyers and  $2^{\mathbf{N}}$  denote the collection of all the subsets (subgroups) of  $\mathbf{N}$ , including the empty set,  $\phi$ . The first-stage mechanism is characterized by the shortlisting rule  $A^g(\alpha)$  and payment rule  $x_i(\alpha), i = 1, 2, \dots, N$ . Given the reported profile  $\alpha$ , the shortlisting rule,  $A^g : [\underline{\alpha}, \bar{\alpha}]^N \rightarrow [0, 1]$ , assigns a probability to each subgroup  $g \in 2^{\mathbf{N}}$ , where  $\sum_{g \in 2^{\mathbf{N}}} A^g(\alpha) = 1$ . The payment rule  $x_i : [\underline{\alpha}, \bar{\alpha}]^N \rightarrow \mathbb{R}$ , specifies bidder  $i$ ’s first-stage payment given the reported profile  $\alpha$ .<sup>18</sup>

Given the first-stage reported profile  $\alpha$ , the second-stage mechanism is characterized by  $p_i^g(\alpha, \mathbf{s}^g)$ , the probability that the asset is allocated to buyer  $i \in g$ , and  $t_i^g(\alpha, \mathbf{s}^g)$ , the payment to the seller made by buyer  $i \in g, \forall g \in 2^{\mathbf{N}}$ .

### 3 THE ANALYSIS

We start with the second stage. Suppose group  $g$  is shortlisted, and the profile  $\tilde{\alpha}$  reported in the first stage is revealed as public information to the shortlisted bidders.

First, suppose  $\alpha$  is truthfully reported at the first stage and group  $g$  is shortlisted. Assume that they follow the recommendation and incur the information acquisition cost  $c$  to discover  $\mathbf{s}^g$ .<sup>19</sup>

Given the announced  $\alpha$  and  $s_i$ , define the interim winning probability and expected payment rule as  $P_i^g(\alpha, s_i) = E_{\mathbf{s}_{-i}^g} p_i^g(\alpha, \mathbf{s}^g)$  and  $T_i^g(\alpha, s_i) = E_{\mathbf{s}_{-i}^g} t_i^g(\alpha, \mathbf{s}^g)$ , where  $\mathbf{s}_{-i}^g = \mathbf{s}^g \setminus \{s_i\}, \forall i \in g$  and  $\forall g \in 2^{\mathbf{N}}$ . Then bidder  $i$ ’s second-stage interim expected payoff when she observes  $s_i$  but reports  $\hat{s}_i$  is as follows:

$$\tilde{\pi}_i^g(\alpha; s_i, \hat{s}_i) = E_{\mathbf{s}_{-i}^g} [u(\alpha_i, s_i) p_i^g(\alpha, \hat{s}_i, \mathbf{s}_{-i}^g) - t_i^g(\alpha, \hat{s}_i, \mathbf{s}_{-i}^g)] = u(\alpha_i, s_i) P_i^g(\alpha, \hat{s}_i) - T_i^g(\alpha, \hat{s}_i).$$

<sup>17</sup>In Esö and Szentes, there is no such need for interim verification, as their allocation and payment rules are executed at the end of the mechanism.

<sup>18</sup>Note that our shortlisting rule  $A^g(\alpha)$  is more general than specifying each bidder’s probability of being shortlisted given their reported type profile  $\alpha$ .

<sup>19</sup>As will be shown, the equilibrium expected profit from going forward is positive for a buyer upon entry, so in equilibrium, a bidder does have an incentive to follow the recommendation to acquire (costly) information and participate in the final auction once admitted (as dropping out only results in zero profit).

The second-stage incentive compatibility (IC) conditions require that

$$\tilde{\pi}_i^g(\alpha; s_i, \hat{s}_i) \leq \tilde{\pi}_i^g(\alpha; s_i, s_i), \forall g, \alpha, s_i, \hat{s}_i. \quad (1)$$

First, the following lemma is standard in the traditional screening literature:

**Lemma 1.** Suppose  $\alpha$  is truthfully revealed from the first stage and  $P_i^g(\alpha, s_i), \forall i \in g$ , is continuous and weakly increasing in  $s_i$  where  $g$  denotes the group being shortlisted, then the second-stage incentive compatibility condition (1) holds if and only if

$$\tilde{\pi}_i^g(\alpha; s_i, s_i) = \tilde{\pi}_i^g(\alpha; \hat{s}_i, \hat{s}_i) + \int_{\hat{s}_i}^{s_i} u_2(\alpha_i, \tau) P_i^g(\alpha, \tau) d\tau, \forall s_i > \hat{s}_i, \forall i \in g. \quad (2)$$

(2) is an integral form of the envelope formula. Next, we consider the case when  $\hat{\alpha}_i$  instead of  $\alpha_i$  is reported by bidder  $i$  while others report their types truthfully. As demonstrated in Esö and Szentes (2007), whenever a bidder had misrepresented her type in the first stage, she would “correct” her lie in the second stage. Formally in our setting, suppose  $\alpha_{-i}$  is truthfully revealed from the first stage and the second-stage mechanism is incentive-compatible given a truthfully revealed  $\alpha$ . Then buyer  $i$  of type  $\alpha_i$  who reported  $\hat{\alpha}_i$  in the first round will report  $\hat{s}_i = \sigma_i(\alpha_i, \hat{\alpha}_i, s_i)$  if she observes  $s_i$  in the second stage such that<sup>20</sup>

$$u(\alpha_i, s_i) = u(\hat{\alpha}_i, \sigma_i(\alpha_i, \hat{\alpha}_i, s_i)). \quad (3)$$

Reporting  $\hat{s}_i$  after a lie  $\hat{\alpha}_i$  is equivalent to revealing  $v_i$  truthfully regardless of the first-stage report. The optimality of this strategy has been established in general for the Markov environments by Pavan, Segal and Toikka (2014). Our two-stage setting resembles the Markov environment defined in Pavan, Segal, and Toikka since the agents’ payoffs only depend on their second-stage true types ( $v_i$ ’s) and the allocation outcome, but not on their first-stage true types. For this reason, an agent’s reporting incentive in the second stage depends only on her current type and her first-stage report, but not on her first-stage true type. Since it is optimal for the agent to report her value truthfully when the past report has been truthful, it is also optimal for her to report her value truthfully even if she has lied in the first stage.

Note that  $\hat{s}_i$  does not depend on  $\alpha_{-i}$ ,  $g$ , or  $\mathbf{s}_{-i}^g$ . Define

$$\begin{aligned} \tilde{\pi}_i^g(\alpha, \hat{\alpha}_i; s_i, \hat{s}_i) &= E_{\mathbf{s}_{-i}^g} [u(\alpha_i, s_i) p_i^g(\alpha_{-i}, \hat{\alpha}_i, \hat{s}_i, \mathbf{s}_{-i}^g) - t_i^g(\alpha_{-i}, \hat{\alpha}_i, \hat{s}_i, \mathbf{s}_{-i}^g)] \\ &= u(\alpha_i, s_i) P_i^g(\alpha_{-i}, \hat{\alpha}_i, \hat{s}_i) - T_i^g(\alpha_{-i}, \hat{\alpha}_i, \hat{s}_i); \\ \tilde{\pi}_i^g(\alpha_i, \hat{\alpha}_i; \alpha_{-i}) &= E_{s_i} \tilde{\pi}_i^g(\alpha, \hat{\alpha}_i; s_i, \hat{s}_i = \sigma_i(\alpha_i, \hat{\alpha}_i, s_i)). \end{aligned}$$

$\tilde{\pi}_i^g(\alpha_i, \hat{\alpha}_i; \alpha_{-i})$  is the expected second-stage payoff for the type- $\alpha_i$  bidder if she reported  $\hat{\alpha}_i$  in the first

<sup>20</sup>The existence of  $\sigma_i(\cdot, \cdot, \cdot)$  relies on the assumption that the support of  $v_i$  does not depend on the first-stage signal  $\alpha_i$ .



stage (and everyone else reported truthfully). Parallel to Lemma 5 in Esö and Szentes, we can show the following lemma:

**Lemma 2.** Suppose  $\alpha_{-i}$  is truthfully revealed from the first stage and the second-stage mechanism is incentive-compatible given a truthfully revealed  $\alpha$ . If buyer  $i$  of type  $\alpha_i$  who reported  $\hat{\alpha}_i$  in the first stage is shortlisted in group  $g_i$ , her expected payoff from the second stage is given by

$$\tilde{\pi}_i^{g_i}(\alpha_i, \hat{\alpha}_i; \alpha_{-i}) = \tilde{\pi}_i^{g_i}(\hat{\alpha}_i, \hat{\alpha}_i; \alpha_{-i}) + \int \int_{\hat{\alpha}_i}^{\alpha_i} u_1(y, s_i) P_i^{g_i}(\hat{\alpha}_i, \alpha_{-i}, \sigma_i(y, \hat{\alpha}_i, s_i)) dy dG_i(s_i). \quad (4)$$

Throughout,  $g_i$  will be used to denote the group including bidder  $i$ . (4) should again be regarded as an integral form of the envelope formula: the winning probability ( $P_i^{g_i}$ ) is now obtained when evaluating at  $\hat{s}_i = \sigma_i(y, \hat{\alpha}_i, s_i)$  (which is optimal given the first-round “lie”  $\hat{\alpha}_i$ ). We are now ready to consider the first-stage IC mechanism.

Let  $\pi_i(\alpha_i, \hat{\alpha}_i)$  be the expected payoff (net of the entry cost) for a type- $\alpha_i$  bidder who reports  $\hat{\alpha}_i$  in the first stage. By (3), we have

$$\begin{aligned} \pi_i(\alpha_i, \hat{\alpha}_i) &= E_{\alpha_{-i}} \left\{ \sum_{g_i} A^{g_i}(\hat{\alpha}_i, \alpha_{-i}) [\tilde{\pi}_i^{g_i}(\alpha_i, \hat{\alpha}_i; \alpha_{-i}) - c] - x_i(\hat{\alpha}_i, \alpha_{-i}) \right\} \\ &= E_{\alpha_{-i}} \left\{ \sum_{g_i} A^{g_i}(\hat{\alpha}_i, \alpha_{-i}) [E_{s_i} (u(\alpha_i, s_i) P_i^{g_i}(\alpha_{-i}, \hat{\alpha}_i, \hat{s}_i) - T_i^{g_i}(\alpha_{-i}, \hat{\alpha}_i, \hat{s}_i)) - c] \right\} - x_i(\hat{\alpha}_i), \end{aligned} \quad (5)$$

where  $\hat{s}_i = \sigma_i(\alpha_i, \hat{\alpha}_i, s_i)$  and  $x_i(\hat{\alpha}_i) = E_{\alpha_{-i}} x_i(\hat{\alpha}_i, \alpha_{-i})$ .

The following lemma characterizes the bidder’s expected payoff in an IC two-stage mechanism with costly entry.

**Lemma 3.** If the two-stage mechanism is incentive compatible and  $E_{\alpha_{-i}} A^{g_i}(\alpha_i, \alpha_{-i}) P_i^{g_i}(\alpha_i, \alpha_{-i}, s_i)$  is continuous in  $\alpha_i$  then buyer  $i$ ’s expected payoff (as a function of her pre-entry type) can be expressed as

$$\pi_i(\alpha_i, \alpha_i) = \pi_i(\underline{\alpha}, \underline{\alpha}) + \int_{\underline{\alpha}}^{\alpha_i} \int u_1(y, s_i) \cdot \sum_{g_i} [E_{\alpha_{-i}} A^{g_i}(y, \alpha_{-i}) P_i^{g_i}(y, \alpha_{-i}, s_i)] dG_i(s_i) dy. \quad (6)$$

*Proof.* See Appendix. □

Note that  $\sum_{g_i} [E_{\alpha_{-i}} A^{g_i}(y, \alpha_{-i}) P_i^{g_i}(y, \alpha_{-i}, s_i)]$  is buyer  $i$ ’s equilibrium probability of eventually winning the asset with signals  $(y, s_i)$  in our setting. Thus (6) is also an integral form of the envelope formula. Under a set of *regular* conditions, Pavan, Segal, and Toikka (2014) show that the envelope formula continues to hold in the dynamic mechanism design setting. Lemma 3 can be regarded as an extension of their result to a dynamic mechanism design setting with costly information acquisition.

### 3.1 The Optimal Two-stage Mechanisms

We are now ready to derive the seller's expected payoff from an IC two-stage mechanism. By Lemma 3, we have

$$\begin{aligned}
& E\pi_i(\alpha_i, \alpha_i) \\
&= \pi_i(\underline{\alpha}, \underline{\alpha}) + \int_{\underline{\alpha}}^{\bar{\alpha}} \int_{\underline{\alpha}}^{\alpha_i} \int u_1(y, s_i) \cdot \sum_{g_i} [E_{\alpha_{-i}} A^{g_i}(y, \alpha_{-i}) P_i^{g_i}(y, \alpha_{-i}, s_i)] dG_i(s_i) dy dF(\alpha_i) \\
&= \pi_i(\underline{\alpha}, \underline{\alpha}) + \int_{\underline{\alpha}}^{\bar{\alpha}} \frac{1-F(\alpha_i)}{f(\alpha_i)} \int u_1(\alpha_i, s_i) \cdot \sum_{g_i} [E_{\alpha_{-i}} A^{g_i}(\alpha_i, \alpha_{-i}) P_i^{g_i}(\alpha_i, \alpha_{-i}, s_i)] dG_i(s_i) dF(\alpha_i) \\
&= \pi_i(\underline{\alpha}, \underline{\alpha}) + E_{\alpha} \left\{ \sum_{g_i} A^{g_i}(\alpha_i, \alpha_{-i}) \left[ \int \frac{1-F(\alpha_i)}{f(\alpha_i)} u_1(\alpha_i, s_i) P_i^{g_i}(\alpha_i, \alpha_{-i}, s_i) dG_i(s_i) \right] \right\}.
\end{aligned}$$

The second equality above is due to Fubini's Theorem. Thus

$$\sum_{i=1}^N E\pi_i(\alpha_i, \alpha_i) = \sum_{i=1}^N \pi_i(\underline{\alpha}, \underline{\alpha}) + E_{\alpha} \left\{ \sum_g A^g(\alpha) E_{\mathbf{s}} \left[ \sum_{i \in g} p_i^g(\alpha, \mathbf{s}^g) \frac{1-F(\alpha_i)}{f(\alpha_i)} u_1(\alpha_i, s_i) \right] \right\}.$$

The total expected surplus from the two-stage mechanism is

$$TS = E_{\alpha} \sum_g \left\{ A^g(\alpha) E_{\mathbf{s}} \left[ \sum_{i \in g} p_i^g(\alpha, \mathbf{s}^g) u(\alpha_i, s_i) - |g|c \right] \right\}.$$

The seller's expected revenue is thus given by

$$\begin{aligned}
ER &= TS - \sum_{i=1}^N E\pi_i(\alpha_i, \alpha_i) \\
&= E_{\alpha} \sum_g \left\{ A^g(\alpha) E_{\mathbf{s}} \left[ \sum_{i \in g} p_i^g(\alpha, \mathbf{s}^g) \left( u(\alpha_i, s_i) - \frac{1-F(\alpha_i)}{f(\alpha_i)} u_1(\alpha_i, s_i) \right) - |g|c \right] \right\} - \sum_{i=1}^N \pi_i(\underline{\alpha}, \underline{\alpha}), \quad (7)
\end{aligned}$$

where  $A^g(\alpha)$  is the shortlisting rule and  $p_i^g(\alpha, \mathbf{s}^g)$  is the second-stage allocation rule. To maximize  $ER$  subject to IC and IR (individual rationality), the seller sets  $\pi_i(\underline{\alpha}, \underline{\alpha}) = 0$  for all  $i = 1, 2, \dots, N$ ; i.e., no rent should be given to the buyer with the lowest possible (pre-entry) type.

Define the *virtual value adjusted by the second-stage signal* as follows:

$$w(\alpha_i, s_i) = u(\alpha_i, s_i) - \frac{1-F(\alpha_i)}{f(\alpha_i)} u_1(\alpha_i, s_i). \quad (8)$$

From the expression of the expected revenue, we can derive the optimal allocation rules in both stages

as follows. At the second stage, given the revealed  $\alpha$  and the shortlisted group  $g$ ,  $\forall \mathbf{s}^g$ ,<sup>21</sup>

$$p_i^{*g}(\alpha, \mathbf{s}^g) = \begin{cases} 1 & \text{if } i = \operatorname{argmax}_{j \in g} \{w(\alpha_j, s_j)\} \text{ and } w(\alpha_i, s_i) \geq 0 \\ 0 & \text{otherwise} \end{cases} \quad \forall g, \forall i \in g. \quad (9)$$

So as also identified by Esö and Szentes, the asset should be awarded to the bidder with the highest non-negative virtual value adjusted by the second-stage signal, which is a generalization of the optimal allocation rule in Myerson (1981). Our analysis shows that the generalized Myerson allocation rule is robust to settings with costly entry. By Lemma 3, a buyer's expected payoff does not depend on the entry cost, which implies that the seller bears all the entry costs (indirectly) in equilibrium. As such, costly entry will affect the final allocation only through its effect on the entry right allocation rule.

Define the expected *virtual surplus* (the virtual value less the entry cost) as follows:

$$w^{*g}(\alpha) = E_{\mathbf{s}} \left[ \sum_{i \in g} p_i^{*g}(\alpha, \mathbf{s}^g) w(\alpha_i, s_i) - |g|c \right].$$

Then at the first stage, contingent on the revealed  $\alpha$ , the optimal shortlisting rule is as follows:<sup>22</sup>

$$A^{*g}(\alpha) = \begin{cases} 1 & \text{if } g = \operatorname{argmax}_{\bar{g}} \{w^{*\bar{g}}(\alpha)\} \text{ and } w^{*g}(\alpha) \geq 0 \\ 0 & \text{otherwise} \end{cases} \quad \forall g. \quad (10)$$

The optimal shortlisting rule admits the set of bidders that gives rise to the maximal expected virtual surplus. Alternatively, the optimal shortlisting rule admits the bidders in descending order of their marginal contribution to the expected virtual surplus – the bidder with the highest contribution first, the bidder with the second-highest contribution second, etc. – provided that their marginal contribution is positive.

Similarly to Esö and Szentes, following Assumptions 1 and 2, we can establish the following properties of the optimal second-stage allocation rule:<sup>23</sup>

**Corollary 1.** (i)  $p_i^{*g_i}(\alpha, \mathbf{s}^{g_i})$  increases in both  $\alpha_i$  and  $s_i$ ,  $\forall i \in g_i$ ,  $\forall g_i$ ,  $\alpha_{-i}$ , and  $\mathbf{s}_{-i}^{g_i}$ , which implies that  $P_i^{*g_i}(\alpha_i, \alpha_{-i}, s_i)$  increases in both  $\alpha_i$  and  $s_i$ ,  $\forall g_i$ ,  $\alpha_{-i}$ ; (ii) If  $\alpha_i > \hat{\alpha}_i$ ,  $s_i < \hat{s}_i$  and  $u(\alpha_i, s_i) = u(\hat{\alpha}_i, \hat{s}_i)$ , then  $p_i^{*g_i}(\alpha_i, \alpha_{-i}, s_i, \mathbf{s}_{-i}^{g_i}) \geq p_i^{*g_i}(\hat{\alpha}_i, \alpha_{-i}, \hat{s}_i, \mathbf{s}_{-i}^{g_i})$ , which implies  $P_i^{*g_i}(\alpha_i, \alpha_{-i}, s_i) \geq P_i^{*g_i}(\hat{\alpha}_i, \alpha_{-i}, \hat{s}_i)$ ,  $\forall g_i$ ,  $\alpha_{-i}$ .

Property (ii) above suggests that whenever  $\alpha_i > \hat{\alpha}_i$ ,  $s_i < \hat{s}_i$  and  $u(\alpha_i, s_i) = u(\hat{\alpha}_i, \hat{s}_i)$ , the optimal allocation rule favors the “truth-telling” pair  $(\alpha_i, s_i)$ .

Given  $\alpha_i$ , let  $s(\alpha_i)$  be defined such that  $w(\alpha_i, s(\alpha_i)) = 0$ . To identify properties of the optimal shortlist-

<sup>21</sup>Ties occur with probability zero and are hence ignored.

<sup>22</sup>Again ties occur with probability zero and are hence ignored.

<sup>23</sup>Assumption 2 is used to show property (ii).

ing rule, we define a truncated random variable as follows:

$$w_i^+(\alpha_i, s_i) = \begin{cases} w(\alpha_i, s_i) & \text{if } w(\alpha_i, s_i) \geq 0 \text{ or equivalently } s_i \geq s(\alpha_i) \\ 0 & \text{otherwise} \end{cases} \quad \forall i.$$

Note that conditional on  $\alpha$ ,  $w_i^+$ 's are independent across  $i \in g$ .

Let  $\Delta \tilde{S}^g(\alpha_i; \alpha_{-i})$  denote buyer  $i$ 's marginal contribution to the expected virtual surplus,  $i \in g$ , then

$$\Delta \tilde{S}^g(\alpha_i; \alpha_{-i}) = \tilde{S}(\alpha^g) - \tilde{S}(\alpha_{-i}^g), i \in g, \forall \alpha^g,$$

where  $\alpha_{-i}^g = \alpha^g \setminus \{\alpha_i\}$  and

$$\tilde{S}(\alpha^g) = E_{\mathbf{s}^g} \max_{i \in g} \{w_i^+(\alpha_i, s_i)\}, \forall g, \forall \alpha^g.$$

The following two properties are obvious:

- (1)  $\Delta \tilde{S}^g(\alpha_i; \alpha_{-i})$  increases with  $\alpha_i$ , and decreases with  $\alpha_j, \forall j \neq i, \forall i \in g, \forall g$ .
- (2)  $\Delta \tilde{S}^g(\alpha_i; \alpha_{-i}) \geq \Delta \tilde{S}^{g'}(\alpha_i; \alpha_{-i}), \forall \alpha_{-i}, \forall i \in g, \forall g \subset g'$ .

The revenue-optimal shortlisting rule can be alternatively described as follows. For given  $\alpha$ , we can rank all  $\alpha_i$  from the highest to the lowest. The seller admits bidders one by one in descending order of  $\alpha_i$ 's as long as the bidder's marginal contribution to the expected virtual surplus is nonnegative, i.e.

$$\Delta \tilde{S}^g(\alpha_i; \alpha_{-i}) - c = \tilde{S}(\alpha^g) - \tilde{S}(\alpha_{-i}^g) - c \geq 0,$$

where  $g$  denotes the group of bidders with the highest  $|g|$  types before entry.

Two properties follow immediately from the optimal shortlisting rule  $A^{*g}$ :

**Corollary 2.** (i) Given  $\alpha_{-i}$ , if bidder  $i$  with  $\alpha_i$  is shortlisted, then she would also be shortlisted with a higher type  $\tilde{\alpha}_i (> \alpha_i)$ ; (ii) Suppose  $i$  is shortlisted given  $\alpha_{-i}$ , then bidder  $i$  would remain being shortlisted as long as  $\alpha_i$  is higher than a threshold  $\hat{\alpha}_i(\alpha_{-i})$ . As  $\alpha_i$  increases, the shortlisted group weakly shrinks. As  $\alpha_i$  increases from  $\hat{\alpha}_i(\alpha_{-i})$ , the bidders in  $g^*(\alpha) \setminus \{i\}$  would be excluded one by one (with the lowest type originally shortlisted being excluded first).

We are now ready to show that the optimal final good allocation and entry right allocation rules (9) and (10) are truthfully implementable by some well constructed payment rules in both stages.

**Theorem 1.** Under Assumptions 1 and 2, the optimal final good allocation and entry right allocation rules (9) and (10) are IR and IC implementable.

*Proof.*  $u(\alpha_i, s_i)$  increases with  $s_i$  and by Assumption 1,  $u_1(\alpha_i, s_i)$  (weakly) decreases with  $s_i$ . This implies that  $w(\alpha_i, s_i)$  increases with  $s_i$ . By the final good allocation rule (9), the winning probability  $P_i^{*g}(\alpha, s_i)$  is weakly increasing in  $s_i$ . By Lemma 1, the second-stage mechanism is incentive compatible (given  $\alpha$  and  $g$ ). Thus, given the truthfully revealed  $\alpha$  and shortlisted group  $g$ , a second-stage payment rule,

say,  $t_i^{*g}(\alpha, \mathbf{s}^g), \forall i \in g, \forall g$ , can be constructed to truthfully implement the second-stage allocation rule  $p_i^{*g}(\alpha, \mathbf{s}^g), \forall i \in g, \forall g$  while maintaining the second-stage IR constraints (to participate in the second-stage mechanism), i.e.  $\tilde{\pi}_i^{*g}(\alpha, \alpha_i; s_i, s_i) \geq 0$  on equilibrium path. This resembles the Myerson (1981) setting with asymmetric bidders.

We use  $\tilde{\pi}_i^{*g_i}(\alpha_i, \hat{\alpha}_i; \alpha_{-i})$  to denote the second-stage expected payoff to buyer  $i$  of type  $\alpha_i$  if she announces  $\hat{\alpha}_i$  and is shortlisted in group  $g_i$ , given that everyone else announces  $\alpha_{-i}$  truthfully at the first stage.  $\tilde{\pi}_i^{*g_i}(\alpha_i, \hat{\alpha}_i; \alpha_{-i})$  is well defined given Lemma 2. Therefore, when buyer  $i$  of type  $\alpha_i$  announces  $\hat{\alpha}_i$  while others reveal  $\alpha_{-i}$  truthfully, her first-stage expected payoff can be written as follows:

$$\pi_i^*(\alpha_i, \hat{\alpha}_i) = E_{\alpha_{-i}} \left\{ \sum_{g_i} A^{*g_i}(\hat{\alpha}_i, \alpha_{-i}) [\tilde{\pi}_i^{*g_i}(\alpha_i, \hat{\alpha}_i; \alpha_{-i}) - c] - x_i^*(\hat{\alpha}_i, \alpha_{-i}) \right\},$$

where  $x_i^*$  is the first-stage payment rule.

Next, we will show that the optimal shortlisting rule (10) is truthfully implementable by a properly chosen first-stage payment rule  $x_i^*$ , together with the second-stage payment rules  $t_i^{*g}$  chosen above.

Note that by (5), we have

$$\pi_i^*(\alpha_i, \alpha_i) = E_{\alpha_{-i}} \left\{ \sum_{g_i} A^{*g_i}(\alpha_i, \alpha_{-i}) [\tilde{\pi}_i^{*g_i}(\alpha_i, \alpha_i; \alpha_{-i}) - c] - x_i^*(\alpha_i, \alpha_{-i}) \right\}. \quad (11)$$

Construct the first-stage payment rule as follows:

$$\begin{aligned} x_i^*(\alpha) &= \sum_{g_i} A^{*g_i}(\alpha_i, \alpha_{-i}) [\tilde{\pi}_i^{*g_i}(\alpha_i, \alpha_i; \alpha_{-i}) - c] \\ &\quad - \int_{\underline{\alpha}}^{\alpha_i} \int u_1(y, s_i) \cdot \sum_{g_i} [E_{\alpha_{-i}} A^{*g_i}(y, \alpha_{-i}) P_i^{*g_i}(y, \alpha_{-i}, s_i)] dG_i(s_i) dy \end{aligned} \quad (12)$$

Substituting (12) into (11), we can verify that

$$\pi_i^*(\alpha_i, \alpha_i) = \int_{\underline{\alpha}}^{\alpha_i} \int u_1(y, s_i) \cdot \sum_{g_i} [E_{\alpha_{-i}} A^{*g_i}(y, \alpha_{-i}) P_i^{*g_i}(y, \alpha_{-i}, s_i)] dG_i(s_i) dy,$$

which is precisely equation (6) with  $\pi_i^*(\underline{\alpha}, \underline{\alpha}) = 0$  (the optimality requirement). Note that  $\pi_i^*(\alpha_i, \alpha_i) \geq 0$ , so IR is satisfied in the first stage.

Suppose that all buyers except  $i$  report their types  $\alpha_{-i}$  truthfully. Consider buyer  $i$  with  $\alpha_i$  contemplating to misreport to  $\hat{\alpha}_i < \alpha_i$ . The deviation payoff is

$$\Delta = \pi_i^*(\alpha_i, \hat{\alpha}_i) - \pi_i^*(\alpha_i, \alpha_i) = [\pi_i^*(\alpha_i, \hat{\alpha}_i) - \pi_i^*(\hat{\alpha}_i, \hat{\alpha}_i)] + [\pi_i^*(\hat{\alpha}_i, \hat{\alpha}_i) - \pi_i^*(\alpha_i, \alpha_i)].$$

Since (6) is satisfied by the construction of  $x_i^*(\alpha)$ , we have

$$\pi_i^*(\hat{\alpha}_i, \hat{\alpha}_i) - \pi_i^*(\alpha_i, \alpha_i) = - \int_{\hat{\alpha}_i}^{\alpha_i} \int u_1(y, s_i) \cdot \sum_{g_i} [E_{\alpha_{-i}} A^{*g_i}(y, \alpha_{-i}) P_i^{*g_i}(y, \alpha_{-i}, s_i)] dG_i(s_i) dy.$$

Recall the definitions of  $\pi_i^*(\alpha_i, \hat{\alpha}_i)$  above, we have from Lemma 2 that

$$\pi_i^*(\alpha_i, \hat{\alpha}_i) - \pi_i^*(\hat{\alpha}_i, \hat{\alpha}_i) = \int_{\hat{\alpha}_i}^{\alpha_i} \int u_1(y, s_i) \cdot \sum_{g_i} [E_{\alpha_{-i}} A^{*g_i}(\hat{\alpha}_i, \alpha_{-i}) P_i^{*g_i}(\hat{\alpha}_i, \alpha_{-i}, \sigma_i(y, \hat{\alpha}_i, s_i))] dG_i(s_i) dy.$$

Therefore, we have

$$\begin{aligned} \Delta &= \int_{\hat{\alpha}_i}^{\alpha_i} E_{\alpha_{-i}} \sum_{g_i} A^{*g_i}(y, \alpha_{-i}) \int u_1(y, s_i) [P_i^{*g_i}(\hat{\alpha}_i, \alpha_{-i}, \sigma_i(y, \hat{\alpha}_i, s_i)) - P_i^{*g_i}(y, \alpha_{-i}, s_i)] dG_i(s_i) dy \\ &\quad + \int_{\hat{\alpha}_i}^{\alpha_i} E_{\alpha_{-i}} \sum_{g_i} [A^{*g_i}(\hat{\alpha}_i, \alpha_{-i}) - A^{*g_i}(y, \alpha_{-i})] \int u_1(y, s_i) P_i^{*g_i}(\hat{\alpha}_i, \alpha_{-i}, \sigma_i(y, \hat{\alpha}_i, s_i)) dG_i(s_i) dy. \end{aligned} \quad (13)$$

From Corollary 1 (ii), we have  $P_i^{*g_i}(\hat{\alpha}_i, \alpha_{-i}, \sigma_i(y, \hat{\alpha}_i, s_i)) - P_i^{*g_i}(y, \alpha_{-i}, s_i) \leq 0$ , which implies that the first term in  $\Delta$  is negative.

We now consider the second term in  $\Delta$  when  $y > \hat{\alpha}_i$ . By Corollary 2, the optimal shortlisting rule implies that given  $\alpha_{-i}$ , when buyer  $i$  is admitted with a higher  $\alpha_i$ , she must be admitted to a group with a weakly smaller size. If  $y$  and  $\hat{\alpha}_i$  are admitted in the same group, then  $A^{*g_i}(\hat{\alpha}_i, \alpha_{-i}) = A^{*g_i}(y, \alpha_{-i})$  and this term in  $\Delta$  is zero.

We now turn to the case where  $g^*(\hat{\alpha}_i, \alpha_{-i}) \supset g^*(y, \alpha_{-i}) \supset \{i\}$ . Note that  $A^{*g_i}(\cdot, \alpha_{-i})$  is 1 for the short-listed group, and 0 for all other groups. Therefore,

$$\begin{aligned} &\sum_{g_i} [A^{*g_i}(\hat{\alpha}_i, \alpha_{-i}) - A^{*g_i}(y, \alpha_{-i})] u_1(y, s_i) P_i^{*g_i}(\hat{\alpha}_i, \alpha_{-i}, \sigma_i(y, \hat{\alpha}_i, s_i)) \\ &= u_1(y, s_i) [P_i^{*g^*(\hat{\alpha}_i, \alpha_{-i})}(\hat{\alpha}_i, \alpha_{-i}, \sigma_i(y, \hat{\alpha}_i, s_i)) - P_i^{*g^*(y, \alpha_{-i})}(\hat{\alpha}_i, \alpha_{-i}, \sigma_i(y, \hat{\alpha}_i, s_i))] \\ &\leq 0, \end{aligned}$$

which implies that the second term in  $\Delta$  is negative. Since  $g^*(\hat{\alpha}_i, \alpha_{-i}) \supset g^*(y, \alpha_{-i}) \supset \{i\}$ , we must have  $P_i^{*g^*(\hat{\alpha}_i, \alpha_{-i})}(\hat{\alpha}_i, \alpha_{-i}, \sigma_i(y, \hat{\alpha}_i, s_i)) \leq P_i^{*g^*(y, \alpha_{-i})}(\hat{\alpha}_i, \alpha_{-i}, \sigma_i(y, \hat{\alpha}_i, s_i))$ , i.e. entrant  $i$  wins with a smaller probability if a strictly bigger group is shortlisted.

A similar argument can be used to rule out deviation to  $\hat{\alpha}_i > \alpha_i$ . □

It is worth noting that Assumptions 1 and 2 are sufficient but not necessary for the optimal entry rule to be truthfully implementable: the necessary and sufficient condition is that  $\Delta$  defined in (13) is non-positive, which is also the *integral monotonicity* condition characterized by Pavan, Segal, and Toikka (2014).

**Example 3.** Assumptions 1 and 2 are not necessary for Theorem 1 to hold. One can verify that for the

case with  $H_\alpha(v) = v^\alpha$ , where  $v \in [0, 1]$ , Assumption 1 fails, but Corollary 1 holds. In this case,  $v = s^{1/\alpha} \equiv u(\alpha, s)$ . Thus

$$w(\alpha, s) = s^{1/\alpha} \left[ 1 + \frac{1-F(\alpha)}{f(\alpha)} \frac{1}{\alpha} \log s^{1/\alpha} \right] = u(\alpha, s) \left[ 1 + \frac{1-F(\alpha)}{f(\alpha)} \frac{1}{\alpha} \log u(\alpha, s) \right].$$

When there is only one potential bidder, the second-stage allocation rule can be implemented via a take-it-or-leave-it offer with a price  $P(\alpha) = s(\alpha)^{1/\alpha}$ , where  $s(\alpha)$  is defined by  $w(\alpha, s(\alpha)) = 0$ . Note that  $s(\alpha) = \exp \left\{ -\alpha^2 \frac{f(\alpha)}{1-F(\alpha)} \right\}$ , which decreases with  $\alpha$ . Thus  $P'(\alpha) < 0$ . Define  $w(\alpha) = E_s \max\{0, w(\alpha, s)\}$ . The optimal shortlisting rule is given by  $A(\alpha) = 1$  if and only if  $\alpha \geq \alpha^*$ , where  $w(\alpha^*) = 0$ . We next derive the first-stage payment rule. Note that

$$\pi(\alpha, \hat{\alpha}) = A(\hat{\alpha}) \left[ \int_0^1 \max\{0, s^{1/\alpha} - P(\hat{\alpha})\} ds - c \right] - x(\hat{\alpha}), \quad \forall \alpha, \hat{\alpha} \geq \alpha^*.$$

The FOC requires that  $\frac{\partial \pi(\alpha, \hat{\alpha})}{\partial \hat{\alpha}} \Big|_{\hat{\alpha}=\alpha} = 0$ . Taking  $\hat{\alpha} > \alpha \geq \alpha^*$ , we have  $s(\hat{\alpha}) < s(\alpha) \leq s(\alpha^*)$ . Note  $A(\hat{\alpha}) = A(\alpha) = 1$ . We thus have

$$\pi(\alpha, \hat{\alpha}) - \pi(\alpha, \alpha) = \int_{s(\hat{\alpha})}^{s(\alpha)} [s^{1/\alpha} - P(\hat{\alpha})] ds + \int_{s(\alpha)}^1 [P(\alpha) - P(\hat{\alpha})] ds + [x(\alpha) - x(\hat{\alpha})].$$

Since  $\partial \int_{s(\hat{\alpha})}^{s(\alpha)} [s^{1/\alpha} - P(\hat{\alpha})] ds / \partial \hat{\alpha} \Big|_{\hat{\alpha}=\alpha} = 0$ , we have

$$\frac{\partial \pi(\alpha, \hat{\alpha})}{\partial \hat{\alpha}} \Big|_{\hat{\alpha}=\alpha} = -P'(\alpha)(1 - s(\alpha)) - x'(\alpha) = 0.$$

Thus  $x'(\alpha) = -P'(\alpha)(1 - s(\alpha))$ , together with boundary condition  $x(\alpha^*) = \int_0^1 \max\{0, s^{1/\alpha^*} - P(\alpha^*)\} ds - c$ , jointly determines the payment function  $x(\alpha)$  for the first stage.

### 3.2 Implementation of Optimal Mechanisms

When  $u(\alpha_i, s_i)$  is linear in  $\alpha_i$ , i.e., when  $u(\alpha_i, s_i) = u_1 \alpha_i + r(s_i)$  for some constant  $u_1$  and function  $r$ , we will demonstrate that the optimal mechanism can be implemented via a two-stage auction, with the first stage being an auction for both entry rights and price premia and the second stage being a second-price or English auction for the final good. This two-stage auction can be regarded as a handicap auction introduced in Esö and Szentes, augmented by an additional auction at the entry stage.<sup>24</sup>

More specifically, our two-stage auction works as follows. The first stage is an all-pay auction, where bidders need to pay what they bid, regardless of being awarded entry rights or not. Suppose buyer  $i$ , knowing her type  $\alpha_i$ , bids an amount  $b_i$ ,  $i = 1, 2, \dots, N$ . After all the first-stage bids are collected, underlying types will be recovered based on a recovery function,  $x^{*-1}$ , such that buyer  $i$ 's perceived type  $\hat{\alpha}_i$  is  $x^{*-1}(b_i)$ ,  $i = 1, 2, \dots, N$ . Given the recovered type profile  $\{\hat{\alpha}_i\}_{i=1}^N$ , the entry rights are implemented

<sup>24</sup>The assumption that  $u_1$  is constant is satisfied in both Examples 1 and 2.

according to the optimal entry rule (10), and a “price premium” is determined for each shortlisted buyer according to the following premium schedule:  $p(\hat{\alpha}_i) = u_1(1 - F(\hat{\alpha}_i))/f(\hat{\alpha}_i)$ . Both the recovery function  $x^{*-1}$  and the premium determination rule  $p$  are made public at the outset of the game, which remain common knowledge throughout the auction process. Upon being admitted, each entrant bidder will incur the information acquisition cost and participate in the second-round bidding. The second stage is a traditional second-price or English auction with a zero reserve price, but the winner is required to pay her premium over the price.<sup>25</sup> This mechanism is referred to as the handicap auction in Esö and Szentes, since the buyers compete under unequal conditions: a bidder with a smaller premium has an advantage. In our setting, the handicap auction is modified so that the optimal entry rule is also implemented after the first-round bidding. In Esö and Szentes, buyers pay fees regardless of winning the final good or not; in our setting, buyers pay  $b_i$ 's regardless of being admitted to the final sale or not. For this reason, the first-stage auction is a variant of the all-pay auction.

In the second-stage auction, it is a (weakly) dominant strategy for entrant bidder  $i$  with a price premium  $p_i$  (determined from the first stage) to bid  $u(\alpha_i, s_i) - p_i$ . Assuming that all the entrant buyers follow this weakly dominant strategy in the second stage, the mechanism can be represented by a pair of functions,  $p : [\underline{\alpha}, \bar{\alpha}] \rightarrow \mathbb{R}_+$  and  $x^* : [\underline{\alpha}, \bar{\alpha}] \rightarrow \mathbb{R}$  for  $i = 1, 2, \dots, N$ , where  $p(\alpha_i)$  is the price premium for a buyer who bids an amount of  $b_i = x^*(\alpha_i)$ .

**Theorem 2.** If  $u_1$  is constant then the optimal mechanism of Theorem 1 can be implemented via a two-stage auction described above with the recovery function  $x^{*-1}$  and price premium function  $p$  defined as follows:

$$p(\alpha_i) = \frac{1 - F(\alpha_i)}{f(\alpha_i)} u_1, \quad (14)$$

$$x^*(\alpha_i) = E_{\alpha_{-i}} \sum_{g_i} A^{*g_i}(\alpha_i, \alpha_{-i}) E_{\mathbf{s}^{g_i}} [\max\{w(\alpha_i, s_i) - w_{-i}^{g_i}(\alpha_{-i}, s_{-i}), 0\} - c] \\ - E_{\alpha_{-i}} \int_{\underline{\alpha}}^{\alpha_i} \sum_{g_i} A^{*g_i}(\alpha_i, \alpha_{-i}) E_{\mathbf{s}^{g_i}} \left[ u_1(y, s_i) \mathbf{1}_{\{w(y, s_i) > w_{-i}^{g_i}(\alpha_{-i}, \mathbf{s}_{-i}^{g_i})\}} \right] dy, \quad (15)$$

where  $w(\alpha_i, s_i) = u(\alpha_i, s_i) - p(\alpha_i)$  and  $w_{-i}^{g_i}(\alpha_{-i}, s_{-i}) = \max_{j \neq i, j \in g_i} \{w(\alpha_j, s_j), 0\}$ .

*Proof.* See Appendix. □

The implementation is established by showing that  $x^*(\cdot)$  as defined in (15) constitutes a symmetric (strictly) increasing equilibrium bid function in the (reduced) all-pay auction game, with the second stage being replaced by its associated equilibrium payoffs. A major (and tedious) step in the proof of Theorem 2 is to establish that  $x^*(\alpha_i)$  as defined in (15) is strictly increasing for  $\alpha_i \in [\alpha^*, \bar{\alpha}]$ , where  $\alpha^* \in (\underline{\alpha}, \bar{\alpha})$  is the minimum type that could possibly be allocated with an entry right in equilibrium.<sup>26</sup> Thus the recovery

<sup>25</sup>Should there be only one entrant, the price premium for this sole entrant becomes the effective reserve price.

<sup>26</sup> $\alpha^*$  is defined such that

$$E_{s_i} \max \left\{ u(\alpha^*, s_i) - \frac{1 - F(\alpha^*)}{f(\alpha^*)} u_1, 0 \right\} = c.$$



function  $x^{*-1}(\cdot)$  is well defined over  $[\alpha^*, \bar{\alpha}]$  and a (truncated) profile of pre-entry types can be recovered from their bids.<sup>27</sup> Optimal entry can then be implemented based on the recovered type profile according to (10). Upon being selected in a group, say,  $g$ , everyone will follow the (weakly) dominant strategies in the second round bidding (to bid their value less the price premium), so buyer  $i \in g$  with pre-entry type  $\alpha_i$  will win the asset if and only if

$$u(\alpha_i, s_i) - \frac{1 - F(\alpha_i)}{f(\alpha_i)} u_1 \geq \max \left\{ 0, \max_{j \in g, j \neq i} \left[ u(\alpha_j, s_j) - \frac{1 - F(\alpha_j)}{f(\alpha_j)} u_1 \right] \right\}.$$

Hence the optimal allocation rule (9) can indeed be implemented, provided that bidding according to  $x^*(\cdot)$  constitutes a symmetric equilibrium in the (reduced) first-stage auction game, which is established in the second step of the proof.

When  $u_1$  is not a constant, in particular, if  $u_1$  is a function of  $s_i$ , then one's optimal price premium also depends on her second-stage signal. As such, one's second-stage bid also affects the (total) price to pay should she win the object even under a second-price auction. A direct consequence is that it is no longer a (weakly) dominant strategy for bidder  $i$  to bid  $w(\alpha_i, s_i)$ . To avoid such an inconvenience, as in Esö and Szentés, we also focus on the case in which  $u_1$  is a constant for auction implementation.<sup>28</sup>

Since  $x^*(\cdot)$  is increasing while  $p(\cdot)$  is decreasing, the price premium is decreasing in the first-stage bids. Thus a buyer with a higher pre-entry type bids higher in the first round, which results in a higher probability to be admitted and a lower price premium.

The equilibrium (entry) fee  $x^*$  has an intuitive interpretation. As can be seen from (15), one's entry fee equals her expected profit from entry less her informational rent due to her private information about her type  $\alpha_i$ . So the additional information from the second-stage signals does not contribute to buyers' rents: the seller appropriates all rents from entry by "charging" each buyer an upfront entry fee equal to her "value" of entry (or equivalently, the value of additional information).<sup>29</sup>

Fixing the auction rules in the second round, it is also clear by the envelope theorem that payoff equivalence holds among all the entry mechanisms in which (1) the same entry rule (10) is implemented, and (2) the buyer with type  $\alpha^*$  makes zero expected profit. In light of this equivalence result, based on  $x^*(\alpha_i)$  we can derive the candidate equilibrium bid function under any alternative entry mechanism. If the candidate equilibrium bid function so derived is strictly increasing, then the inverse of the equilibrium bid function can serve as the recovery function for the implementation of optimal entry. For example, we can consider a discriminatory-price auction, where only bidders who are awarded entry rights need to pay, and they pay what they bid. Using the payoff equivalence, the candidate equilibrium bid functions

---

That is,  $\alpha^*$  is the minimal type that one can possibly be shortlisted (as a sole entrant). We assume  $\alpha^* \in (\underline{\alpha}, \bar{\alpha})$ .

<sup>27</sup>We assume that buyers with types below  $\alpha^*$  will stay away from bidding. But that should not affect the implementation of optimal entry as those buyers should not be admitted anyway.

<sup>28</sup>The auction implementation in the environment where  $u_1$  depends on  $s_i$  is an open question.

<sup>29</sup>This seems to be a robust prediction in optimal dynamic mechanism design (see, for example, Courty and Li, 2000 and Pavan, Segal, and Toikka, 2014). As pointed out in Esö and Szentés, the "value" of additional information is not well defined, however, as it depends on the specific rules in the second-round auction.

under the discriminatory-price auction is given by

$$x_D^*(\alpha_i) = x^*(\alpha_i) / \Pr\{\alpha_{-i} | i \in g^*(\alpha_i, \alpha_{-i})\},$$

where  $g^*(\alpha_i, \alpha_{-i})$  is the set of shortlisted bidders determined by the optimal entry rule (10), given the reported type profile  $(\alpha_i, \alpha_{-i})$ . A discriminatory-price auction can implement optimal entry if and only if  $x_D^*(\alpha_i)$  so derived is strictly increasing.

### 3.3 Applications

Our optimal mechanism analysis is general enough to encompass many existing models in the literature on auctions with costly entry. Below we demonstrate how we can apply our general optimal mechanism to special models previously studied.

1. Bidders do not have pre-entry types and only learn about their values after entry (e.g., McAfee and McMillan, 1987; Tan, 1992; and Levin and Smith, 1994). In this case,  $u(\alpha_i, s_i) = s_i$ . Hence  $w(\alpha_i, s_i) = s_i$ , which implies that the optimal auction is *ex post* efficient, and the optimal entry is to select a set of bidders that results in the maximal expected social surplus. Since bidders are identical before entry, optimal entry is entirely characterized by  $n^*$ , the optimal number of bidders to be selected. The implementation is somewhat simple: the second round is a standard auction (first-price, second-price, or English auction – no price premium is involved). The first round (entry stage) is to select exactly  $n^*$  bidders, and whomever selected is required to pay an upfront entry fee  $e^*$ , which is set so that no rent is left for the entrants *ex ante*.
2. Bidders know their values before entry, and entry is merely a bid preparation process (without value updating) (e.g. Samuelson, 1985; Stegeman, 1996; Campbell, 1998; Menezes and Monteiro, 2000; Tan and Yilankaya, 2006; Cao and Tian, 2009; and Lu, 2009). In this setting,  $u(\alpha_i, s_i) = \alpha_i$ , and hence  $w(\alpha_i, s_i) = \alpha_i - (1 - F(\alpha_i))/f(\alpha_i)$ . It is easily verified that according to Theorem 1, the optimal allocation rules can be described as follows: the bidder with the highest “type” ( $\alpha_i$ ) is admitted as the sole entrant to win the item, as long as her contribution to the virtual surplus  $w(\alpha_i, s_i) - c$  is positive. The optimal mechanism can be implemented as follows: each buyer pays what she bids in the first stage (regardless of being admitted or not), and the only entrant wins the item at a price equal to her price premium determined from her first-round bid. For an illustration, below we derive the equilibrium first-stage bid function  $x^*$ .

Consider a bidder with type  $\alpha_i > \alpha^*$ , where  $\alpha^* - (1 - F(\alpha^*))/f(\alpha^*) = c$ . Suppose, in the (reduced) first-stage direct game, bidder  $i$  reports  $\hat{\alpha}_i$ , a sufficiently small deviation from  $\alpha_i$ . Her expected payoff is then given by

$$\pi_i(\alpha_i, \hat{\alpha}_i) = \left[ \alpha_i - \frac{1 - F(\hat{\alpha}_i)}{f(\hat{\alpha}_i)} - c \right] \Pr\left(\alpha_{(1)}^{-i} < \hat{\alpha}_i\right) - x^*(\hat{\alpha}_i)$$

$$= \left[ \alpha_i - \frac{1 - F(\hat{\alpha}_i)}{f(\hat{\alpha}_i)} - c \right] F_{\alpha_{(1)}^{-i}}(\hat{\alpha}_i) - x^*(\hat{\alpha}_i), \quad (16)$$

where  $\alpha_{(1)}^{-i}$  is the highest type among all the buyers other than  $i$ .

Incentive compatibility implies

$$\frac{d\pi_i(\alpha_i, \alpha_i)}{d\alpha_i} = F_{\alpha_{(1)}^{-i}}(\alpha_i).$$

Thus we have

$$\pi_i(\alpha_i, \alpha_i) = \left[ \alpha_i - \frac{1 - F(\alpha_i)}{f(\alpha_i)} - c \right] F_{\alpha_{(1)}^{-i}}(\alpha_i) - x^*(\alpha_i) = \int_{\alpha^*}^{\alpha_i} F_{\alpha_{(1)}^{-i}}(\tau) d\tau. \quad (17)$$

Substituting  $\hat{\alpha}_i = \alpha_i$  into (16) to obtain the equilibrium expected payoff, and then equating that with (17), gives the (symmetric) equilibrium first-stage bid function

$$x^*(\alpha_i) = \left[ \alpha_i - \frac{1 - F(\alpha_i)}{f(\alpha_i)} - c \right] F_{\alpha_{(1)}^{-i}}(\alpha_i) - \int_{\alpha^*}^{\alpha_i} F_{\alpha_{(1)}^{-i}}(\tau) d\tau. \quad (18)$$

It is easily verified that  $x^*(\cdot)$  is strictly increasing so types can be recovered from first-round bids in equilibrium.

If a discriminatory-price auction is conducted instead, by payoff equivalence the candidate equilibrium bid function in the first round is given by

$$x_D^*(\alpha_i) = \frac{x^*(\alpha_i)}{F_{\alpha_{(1)}^{-i}}(\alpha_i)} = \left[ \alpha_i - \frac{1 - F(\alpha_i)}{f(\alpha_i)} - c \right] - \frac{1}{F_{\alpha_{(1)}^{-i}}(\alpha_i)} \int_{\alpha^*}^{\alpha_i} F_{\alpha_{(1)}^{-i}}(\tau) d\tau. \quad (19)$$

It is also easily verified that  $x_D^*(\cdot)$  is strictly increasing. Thus the discriminatory-price auction works to implement optimal entry in this context as well.

3. Each bidder is endowed with pre-entry type  $\alpha_i$ , and learns an additional private value component  $s_i$  (e.g., Ye, 2007; Quint and Hendricks, 2013). The total value is given by  $u(\alpha_i, s_i) = \alpha_i + s_i$ . Hence  $w(\alpha_i, s_i) = \alpha_i + s_i - (1 - F(\alpha_i))/f(\alpha_i)$ . The optimal second-stage allocation rule thus requires that the asset be allocated to the entrant bidder with the highest virtual value  $w(\alpha_i, s_i)$  provided that it is nonnegative. The optimal entry rule requires that bidders be admitted in descending order of their pre-entry types, as long as their contribution to the expected virtual surplus is nonnegative.

To further illustrate the optimal entry rule, we assume that  $\alpha_i$  is distributed uniformly over  $[0, 1]$  and  $s_i$  follows a Bernoulli distribution, taking value 1 (“High”) with probability  $q$  and 0 (“Low”) with probability  $1 - q$ . Then  $w(\alpha_i, s_i) = 2\alpha_i + s_i - 1$ . If only one buyer (the one with the highest type  $\alpha_{(1)}$ ) is admitted, the expected virtual surplus is given by  $w_1 = E(2\alpha_{(1)} + s_1 - 1) - c = 2\alpha_{(1)} + q - 1 - c$ . So the optimal number of entrants  $n^* \geq 1$  if  $2\alpha_{(1)} + q - 1 - c \geq 0$ . For ease of computation we assume

that  $\alpha_{(1)} \geq \alpha_{(2)} \geq .5$  (so that the virtual values from the top two bidders are guaranteed to be non-negative). If two top buyers are admitted, the expected virtual surplus is given by

$$\begin{aligned}
w_2 &= E \left[ \max \{ 2\alpha_{(1)} + s_1 - 1, 2\alpha_{(2)} + s_2 - 1 \} \right] - 2c \\
&= E \left[ \max \{ 2\alpha_{(1)} + s_1, 2\alpha_{(2)} + s_2 \} \right] - 1 - 2c \\
&= \Pr(s_1 = 1) \cdot (2\alpha_{(1)} + 1) + \Pr(s_1 = s_2 = 0) \cdot 2\alpha_{(1)} + \Pr(s_1 = 0, s_2 = 1) \cdot (2\alpha_{(2)} + 1) - 1 - 2c \\
&= q \cdot (2\alpha_{(1)} + 1) + (1 - q)^2 \cdot 2\alpha_{(1)} + (1 - q)q \cdot (2\alpha_{(2)} + 1) - 1 - 2c
\end{aligned}$$

So the optimal number of entrants  $n^* \geq 2$  if the incremental expected virtual surplus  $\Delta w = w_2 - w_1 = q(1 - q) [1 - 2(\alpha_{(1)} - \alpha_{(2)})] - c \geq 0$ . Continuing this procedure of calculation,<sup>30</sup> it can be verified that  $n^* \geq n$  if  $q(1 - q)^{n-1} [1 - 2(\alpha_{(1)} - \alpha_{(n)})] - c \geq 0$ ,<sup>31</sup> or

$$\alpha_{(1)} - \alpha_{(n)} \leq \frac{1}{2} \left[ 1 - \frac{c}{q(1 - q)^{n-1}} \right]. \tag{20}$$

This condition is intuitive: the admission of the  $n$ -th highest buyer is more likely to be justified if (1) the probability that she will turn out to be the winner in the second round is sufficiently high; (2) the entry cost is sufficiently low; or (3) her type is sufficiently close to the highest type. It is thus clear that the optimal number of entrants,  $n^*$ , is determined by the following conditions:

$$\alpha_{(1)} - \alpha_{(n^*)} \leq \frac{1}{2} \left[ 1 - \frac{c}{q(1 - q)^{n^*-1}} \right], \quad \alpha_{(1)} - \alpha_{(n^*+1)} > \frac{1}{2} \left[ 1 - \frac{c}{q(1 - q)^{n^*}} \right].$$

#### 4 DISCUSSION

In our preceding analysis, we have focused on the revelation policy so that the first-stage reports are fully revealed to the shortlisted bidders. Due to this particular revelation policy, one concern is that there might be some loss of generality in identifying optimal mechanisms. To address this concern, we next identify an upper bound for the expected revenue that can be achieved by examining a relaxed setting by dropping the IC and IR constraints for the shortlisted in the second stage so that all shortlisted bidders must incur entry costs to learn their second-stage signals as in our original setup, and regardless of their second-stage signals, they must participate in the second-stage selling mechanism and report truthfully their second stage signals. As a result, regardless of the disclosure policy of the first-stage reports, the highest possible expected revenue achievable in this relaxed setting should impose an upper bound for the expected revenue that can be obtained in our original setup, where the bidders' second-stage IC and IR must both be satisfied. A useful observation is that in the relaxed setting, bidders can only misreport their first-stage signals, and the shortlisted buyers' beliefs on buyers' first-stage type profiles have no

<sup>30</sup>We continue to consider the case  $\alpha_{(1)} > \alpha_{(2)} \geq \dots \geq \alpha_{(n)} \geq .5$  so that the virtual value from these buyers will be positive.

<sup>31</sup>The addition of the  $n$ th highest buyer only contributes to the expected virtual surplus when she turns out to be the only one having a good "shot" in the second stage (i.e.,  $s_n = 1$ , while  $s_1 = \dots = s_{n-1} = 0$ ).

impact on their second-stage decisions (as shortlisted must enter and truthfully report their second-stage signals). This observation implies that the revelation policy of the first-stage reports is not relevant to the mechanism design in the relaxed setting. Consequently, the highest expected revenue attainable in this relaxed setting does not depend on the prevailing disclosure policy of the first-stage signals. We next proceed to identify this bound.

In the relaxed setting, the mechanisms are specified exactly the same as in Section 2. All potential bidders report their types  $\alpha_i$ , giving rise to a reported type profile  $\alpha$ . The mechanism specifies the first-stage shortlisting rule  $A^g(\alpha)$  and payment rule  $x_i(\alpha_i, \alpha_{-i})$ . Every shortlisted bidder  $j$  incurs cost  $c$  to discover her second-stage signal  $s_j$ . The second-stage selling mechanism specifies the winning probability  $p_i^g(\alpha, \mathbf{s}^g)$  and payment rule  $t_i^g(\alpha, \mathbf{s}^g)$ ,  $\forall i \in g, \forall g \in 2^N$ .

Recall that  $P_i^g(\alpha, s_i) = E_{\mathbf{s}_{-i}^g} p_i^g(\alpha, \mathbf{s}^g)$  and  $T_i^g(\alpha, s_i) = E_{\mathbf{s}_{-i}^g} t_i^g(\alpha, \mathbf{s}^g)$ . For shortlisted bidder  $i \in g_i$  with type  $\alpha_i$ , her interim expected payoff when she reports  $\hat{\alpha}_i$  and others report truthfully is given by

$$\pi_i(\alpha_i, \hat{\alpha}_i) = E_{\alpha_{-i}} \left\{ \sum_{g_i} A^{g_i}(\hat{\alpha}_i, \alpha_{-i}) [E_{s_i} ((u(\alpha_i, s_i) P_i^{g_i}(\hat{\alpha}_i, \alpha_{-i}, s_i) - T_i^{g_i}(\hat{\alpha}_i, \alpha_{-i}, s_i)) - c) - x_i(\hat{\alpha}_i, \alpha_{-i}) \right\}. \quad (21)$$

Applying the envelope theorem, the IC condition  $\pi_i(\alpha_i, \alpha_i) \geq \pi_i(\alpha_i, \hat{\alpha}_i)$  leads to the following necessary condition:

$$\frac{d\pi_i(\alpha_i, \alpha_i)}{d\alpha_i} = \frac{\partial \pi_i(\alpha_i, \hat{\alpha}_i)}{\partial \hat{\alpha}_i} \Big|_{\hat{\alpha}_i = \alpha_i} = E_{\alpha_{-i}} \left\{ \sum_{g_i} A^{g_i}(\alpha_i, \alpha_{-i}) E_{s_i} [u_1(\alpha_i, s_i) P_i^{g_i}(\alpha_i, \alpha_{-i}, s_i)] \right\}.$$

Therefore, we have

$$\begin{aligned} \pi_i(\alpha_i, \alpha_i) &= \pi_i(\underline{\alpha}, \underline{\alpha}) + E_{\alpha_{-i}} \int_{\underline{\alpha}}^{\alpha_i} \sum_{g_i} A^{g_i}(y, \alpha_{-i}) E_{s_i} [u_1(y, s_i) P_i^{g_i}(y, \alpha_{-i}, s_i)] dy \\ &= \pi_i(\underline{\alpha}, \underline{\alpha}) + E_{\alpha_{-i}} \int_{\underline{\alpha}}^{\alpha_i} \int u_1(y, s_i) \cdot \sum_{g_i} A^{g_i}(y, \alpha_{-i}) P_i^{g_i}(y, \alpha_{-i}, s_i) dG_i(s_i) dy. \end{aligned}$$

Note that the above expression is exactly the same as (6), which implies that the seller's expected revenue must be the same as in (7); in other words, the expected revenue upper bound in the relaxed setting is achieved in our original setting. In this sense, there is no loss of generality to derive optimal mechanisms by only considering mechanisms that fully reveal the buyers' first-stage reports to all admitted bidders.

Another important aspect in our analysis is that we model information acquisition as entry. An implication is that information acquisition is mandatory, in the sense that a bidder is not allowed to bid without going through the "due diligence" process. This assumption is due to the specific institutional setup we are trying to model. For example, "data rooms" are usually provided by the selling party to disclose a large amount of confidential data to bidders during the due diligence process. A typical data room is a continually monitored space that the bidders and their advisers will visit in order to inspect and report on the various documents and data made available. Often only one bidder at a time will

be allowed to enter a data room. Teams involved in large due diligence processes will typically remain available throughout the process. Such teams often consist of a number of experts in different fields, hence the overall cost of keeping such groups on call near to the data room is often extremely high.<sup>32</sup> In a typical electrical generating asset sale as studied by Ye (2007), before submitting a final bid, each bidder (more precisely, bidding team) usually needs to go through the due diligence process to meet with senior management and personnel, study equipment conditions and operating history, evaluate supply contracts and employment agreements, etc. This process is strictly controlled and closely monitored by the auctioneer (typically an investment banker serving as the financial advisor for the selling party). Given the complexity and high-stakes nature of the sale, it is very unlikely that a seller would be comfortable accepting a bid from someone who did not go through such an important information acquisition process. As such, we believe that it is appropriate to model information acquisition as entry for such an environment. From both theoretical and practical points of view, it would be interesting to identify optimal mechanisms in environments where bidders are allowed to bid without having to go through information acquisition (and information acquisition may not be observable or contractible). Such an analysis would be more involved, however, as we will need to worry that the informed and uninformed buyers may mimic each other.

Finally, we restrict our search for optimal mechanisms to the class of two-stage mechanisms. A consequence is that if some bidders are excluded from entry after the first stage, the seller cannot go back to these bidders after the second-stage bidding. For a more general characterization of optimal mechanisms, we should allow for mechanisms with more than two stages/rounds. For example, the seller may select a single bidder or a subset of bidders to go through due diligence and submit final bids, and if the seller is not satisfied with any offer, he can go back to the unselected first-round bidders and invite another bidder or another subset of bidders to go through due diligence and submit final bids. This process can then repeat itself, until the seller finds a satisfactory offer. Such mechanisms can be quite complicated. For one, the seller will need to determine the order of bidders to invite for conducting due diligence (i.e., who should be invited first and who second, etc.). Given that bidders are heterogenous before entry, it is desirable to make the optimal “ordering” or “sequencing” of entry contingent on their pre-entry types. But this also points to the need for running an IC shortlisting mechanism for entry, which is a central part of our two-stage mechanism. Another concern with such a multiple-round entry mechanism is that the bidders who are contacted in the later rounds will have to be extremely cautious, as the news conveyed from the non-sale with early entrants must be deemed “bad.”<sup>33</sup> This would make the IR constraints much more binding for late entrants. Finally, an obvious drawback of running a multi-stage mechanism is the potential of delay, which would be too costly and therefore favors a more time-efficient two-stage mechanism, as analyzed in this paper. With all these being said, it would be

---

<sup>32</sup>See Vallen and Bullinger (1999) for a detailed description of the due diligence process in a typical electric power plant sale in the US.

<sup>33</sup>This will be a practical concern, although we assume it away with independent signals across bidders in our model.

interesting to explore the potential benefit of running multi-stage mechanisms, beyond the two-stage framework that we have examined. However, given the complexity involved, we feel that it would be more appropriate to explore such mechanisms in a separate project, which is left for future research.

## 5 CONCLUDING REMARKS

Our paper contributes to the literature on two fronts. First, it characterizes optimal two-stage mechanisms for an environment of two-stage auctions, which are commonly employed in sales of complicated and high-valued business assets, procurements, privatization, takeover, and merger and acquisition contests. Our analysis is general enough to nest many existing studies in the literature of auctions with costly entry. Second, our paper contributes to the literature on sequential screening by introducing costly entry into a dynamic auction framework. Entry provides a natural setting for sequential information acquisition; on the other hand, entry also makes the optimal mechanism design more challenging, as now it must balance information acquisition at the entry stage and information elicitation in the final good allocation stage, which are interdependent.

Implementation of the optimal mechanism characterized in this paper may face some practical obstacles. First, the industry may not be comfortable with the idea of paying entry fees whether or not they win the object eventually, and this is the major reason, we believe, that contributes to the common use of non-binding indicative bidding. Second, the optimal mechanism is so complicated that the industry bidders might face great difficulties in developing bidding strategies for both rounds (although such a concern is alleviated to some extent if professional or sophisticated experts are hired to help). For these reasons the nature of our analysis is primarily normative, offering a “market design” approach to guide a potential refinement of an extremely important transaction procedure widely used in the industry. Despite this limitation, our analysis does conform to the “norm” of business in at least two aspects. First, a defining feature of our optimal mechanism is the shortlisting rule, which is also central in the two-stage auction practices. Second, we demonstrate that the optimal number shortlisted is endogenously determined (by the first-stage bids), which is also consistent with the fact that in real sales, the number of finalists is often not pre-determined.<sup>34</sup>

Our analysis offers a theoretical benchmark for evaluating various two-stage auctions currently used in the real world. The information structure modeled in this research has recently received attention not only from theorists but also from econometricians and empiricists. For example, Marmer, Shneyerov, and Xu (2013) and Gentry and Li (2014) have successfully proposed nonparametric specification tests on a so-called affiliated-signal (AS) model with entry, and Roberts and Sweeting (2013) estimate a parametric variant of the AS model using data on California timber auctions. The affiliated-signal models can be

---

<sup>34</sup>For example, in the ongoing sale of PGW (Philadelphia Gas Works), a recent application of two-stage auctions, a “smaller number” of firms were invited to submit final bids after the first round – although this number was neither pre-announced nor disclosed (CBS Philly, November 19, 2013, “Sell-off of Philadelphia’s Natural Gas Utility Goes To Binding Bidding,” by Mike Dunn).

regarded as a special case in the framework studied in our paper, and the optimal mechanism characterized in this paper may potentially serve as a calibration benchmark for counter-factual simulations for related empirical works to come.



6 APPENDIX

**Proof of Lemma 3:** Let  $g_i$  denote any subset that includes  $i$ . By (5) and Lemma 2, we have

$$\begin{aligned} & \pi_i(\alpha_i, \hat{\alpha}_i) \\ &= \pi_i(\hat{\alpha}_i, \hat{\alpha}_i) + E_{\alpha_{-i}} \left\{ \sum_{g_i} A^{g_i}(\hat{\alpha}_i, \alpha_{-i}) [\tilde{\pi}_i^{g_i}(\alpha_i, \hat{\alpha}_i; \alpha_{-i}) - \tilde{\pi}_i^{g_i}(\hat{\alpha}_i, \hat{\alpha}_i; \alpha_{-i})] \right\} \\ &= \pi_i(\hat{\alpha}_i, \hat{\alpha}_i) + E_{\alpha_{-i}} \left\{ \sum_{g_i} A^{g_i}(\hat{\alpha}_i, \alpha_{-i}) \int \int_{\hat{\alpha}_i}^{\alpha_i} u_1(y, s_i) P_i^{g_i}(\hat{\alpha}_i, \alpha_{-i}, \sigma_i(y, \hat{\alpha}_i, s_i)) dy dG_i(s_i) \right\}. \end{aligned}$$

Thus for  $\hat{\alpha}_i < \alpha_i$ ,  $\pi_i(\alpha_i, \hat{\alpha}_i) \leq \pi_i(\alpha_i, \alpha_i)$  implies that

$$\pi_i(\alpha_i, \alpha_i) - \pi_i(\hat{\alpha}_i, \hat{\alpha}_i) \geq E_{\alpha_{-i}} \left\{ \sum_{g_i} A^{g_i}(\hat{\alpha}_i, \alpha_{-i}) \int \int_{\hat{\alpha}_i}^{\alpha_i} u_1(y, s_i) P_i^{g_i}(\hat{\alpha}_i, \alpha_{-i}, \sigma_i(y, \hat{\alpha}_i, s_i)) dy dG_i(s_i) \right\}.$$

Similarly,

$$\begin{aligned} & \pi_i(\hat{\alpha}_i, \alpha_i) \\ &= \pi_i(\alpha_i, \alpha_i) + E_{\alpha_{-i}} \left\{ \sum_{g_i} A^{g_i}(\alpha_i, \alpha_{-i}) [\tilde{\pi}_i^{g_i}(\hat{\alpha}_i, \alpha_i; \alpha_{-i}) - \tilde{\pi}_i^{g_i}(\alpha_i, \alpha_i; \alpha_{-i})] \right\} \\ &= \pi_i(\alpha_i, \alpha_i) - E_{\alpha_{-i}} \left\{ \sum_{g_i} A^{g_i}(\alpha_i, \alpha_{-i}) \int \int_{\hat{\alpha}_i}^{\alpha_i} u_1(y, s_i) P_i^{g_i}(\alpha_i, \alpha_{-i}, \sigma_i(y, \alpha_i, s_i)) dy dG_i(s_i) \right\}. \end{aligned}$$

Thus for  $\hat{\alpha}_i < \alpha_i$ ,  $\pi_i(\hat{\alpha}_i, \alpha_i) \leq \pi_i(\hat{\alpha}_i, \hat{\alpha}_i)$  implies that

$$\pi_i(\alpha_i, \alpha_i) - \pi_i(\hat{\alpha}_i, \hat{\alpha}_i) \leq E_{\alpha_{-i}} \left\{ \sum_{g_i} A^{g_i}(\alpha_i, \alpha_{-i}) \int \int_{\hat{\alpha}_i}^{\alpha_i} u_1(y, s_i) P_i^{g_i}(\alpha_i, \alpha_{-i}, \sigma_i(y, \alpha_i, s_i)) dy dG_i(s_i) \right\}.$$

So

$$\begin{aligned} & E_{\alpha_{-i}} \left\{ \sum_{g_i} A^{g_i}(\hat{\alpha}_i, \alpha_{-i}) \int \frac{\int_{\hat{\alpha}_i}^{\alpha_i} u_1(y, s_i) P_i^{g_i}(\hat{\alpha}_i, \alpha_{-i}, \sigma_i(y, \hat{\alpha}_i, s_i)) dy}{\alpha_i - \hat{\alpha}_i} dG_i(s_i) \right\} \\ &\leq \frac{\pi_i(\alpha_i, \alpha_i) - \pi_i(\hat{\alpha}_i, \hat{\alpha}_i)}{\alpha_i - \hat{\alpha}_i} \leq E_{\alpha_{-i}} \left\{ \sum_{g_i} A^{g_i}(\alpha_i, \alpha_{-i}) \int \frac{\int_{\hat{\alpha}_i}^{\alpha_i} u_1(y, s_i) P_i^{g_i}(\alpha_i, \alpha_{-i}, \sigma_i(y, \alpha_i, s_i)) dy}{\alpha_i - \hat{\alpha}_i} dG_i(s_i) \right\}. \end{aligned}$$

By Fubini's Theorem, we have

$$\begin{aligned} & E_{\alpha_{-i}} \left\{ \sum_{g_i} A^{g_i}(\hat{\alpha}_i, \alpha_{-i}) \int \frac{\int_{\hat{\alpha}_i}^{\alpha_i} u_1(y, s_i) P_i^{g_i}(\hat{\alpha}_i, \alpha_{-i}, \sigma_i(y, \hat{\alpha}_i, s_i)) dy}{\alpha_i - \hat{\alpha}_i} dG_i(s_i) \right\} \\ &= \sum_{g_i} \int \frac{\int_{\hat{\alpha}_i}^{\alpha_i} u_1(y, s_i) E_{\alpha_{-i}} [A^{g_i}(\hat{\alpha}_i, \alpha_{-i}) P_i^{g_i}(\hat{\alpha}_i, \alpha_{-i}, \sigma_i(y, \hat{\alpha}_i, s_i))] dy}{\alpha_i - \hat{\alpha}_i} dG_i(s_i). \end{aligned}$$

Since  $A^{g_i}, P_i^{g_i} \leq 1$ , and  $u$  is concave in  $\alpha_i$ , we have

$$\frac{\int_{\hat{\alpha}_i}^{\alpha_i} u_1(y, s_i) E_{\alpha_{-i}} [A^{g_i}(\hat{\alpha}_i, \alpha_{-i}) P_i^{g_i}(\hat{\alpha}_i, \alpha_{-i}, \sigma_i(y, \hat{\alpha}_i, s_i))] dy}{\alpha_i - \hat{\alpha}_i} \leq \frac{\int_{\hat{\alpha}_i}^{\alpha_i} u_1(y, s_i) dy}{\alpha_i - \hat{\alpha}_i} \leq u_1(\hat{\alpha}_i, s_i).$$

By assumption  $u_1(\hat{\alpha}_i, s_i)$  has a finite expectation with respect to  $s_i$ . Hence, by the Lebesgue convergence theorem,

$$\begin{aligned} & \lim_{\hat{\alpha}_i \rightarrow \alpha_i} E_{\alpha_{-i}} \left\{ \sum_{g_i} A^{g_i}(\hat{\alpha}_i, \alpha_{-i}) \int \frac{\int_{\hat{\alpha}_i}^{\alpha_i} u_1(y, s_i) P_i^{g_i}(\hat{\alpha}_i, \alpha_{-i}, \sigma_i(y, \hat{\alpha}_i, s_i)) dy}{\alpha_i - \hat{\alpha}_i} dG_i(s_i) \right\} \\ &= \sum_{g_i} \int \lim_{\hat{\alpha}_i \rightarrow \alpha_i} \frac{\int_{\hat{\alpha}_i}^{\alpha_i} u_1(y, s_i) E_{\alpha_{-i}} [A^{g_i}(\hat{\alpha}_i, \alpha_{-i}) P_i^{g_i}(\hat{\alpha}_i, \alpha_{-i}, \sigma_i(y, \hat{\alpha}_i, s_i))] dy}{\alpha_i - \hat{\alpha}_i} dG_i(s_i) \\ &= \sum_{g_i} \int \lim_{\hat{\alpha}_i \rightarrow \alpha_i} \{u_1(\hat{\alpha}_i, s_i) E_{\alpha_{-i}} [A^{g_i}(\hat{\alpha}_i, \alpha_{-i}) P_i^{g_i}(\hat{\alpha}_i, \alpha_{-i}, s_i)]\} dG_i(s_i) \\ &= \sum_{g_i} \int \{u_1(\alpha_i, s_i) E_{\alpha_{-i}} [A^{g_i}(\alpha_i, \alpha_{-i}) P_i^{g_i}(\alpha_i, \alpha_{-i}, s_i)]\} dG_i(s_i) \\ &= E_{\alpha_{-i}} \left\{ \sum_{g_i} A^{g_i}(\alpha_i, \alpha_{-i}) \int u_1(\alpha_i, s_i) P_i^{g_i}(\alpha_i, \alpha_{-i}, s_i) dG_i(s_i) \right\}. \end{aligned}$$

The third equality above is due to the assumption that  $E_{\alpha_{-i}} [A^{g_i}(\hat{\alpha}_i, \alpha_{-i}) P_i^{g_i}(\hat{\alpha}_i, \alpha_{-i}, s_i)]$  is continuous in  $\hat{\alpha}_i$  (which is guaranteed as long as both  $A^{g_i}$  and  $P_i^{g_i}$  are continuous a.e. in  $[\underline{\alpha}, \bar{\alpha}]^{|\mathcal{G}_i|}$ ).

Analogously, we can show that

$$\begin{aligned} & \lim_{\hat{\alpha}_i \rightarrow \alpha_i} E_{\alpha_{-i}} \left\{ \sum_{g_i} A^{g_i}(\alpha_i, \alpha_{-i}) \int \frac{\int_{\hat{\alpha}_i}^{\alpha_i} u_1(y, s_i) P_i^{g_i}(\alpha_i, \alpha_{-i}, \sigma_i(y, \alpha_i, s_i)) dy}{\alpha_i - \hat{\alpha}_i} dG_i(s_i) \right\} \\ &= E_{\alpha_{-i}} \left\{ \sum_{g_i} A^{g_i}(\alpha_i, \alpha_{-i}) \int u_1(\alpha_i, s_i) P_i^{g_i}(\alpha_i, \alpha_{-i}, s_i) dG_i(s_i) \right\}. \end{aligned}$$

Thus the left derivative of  $\pi_i(\alpha_i, \alpha_i)$  is given by

$$\frac{d\pi_i^-(\alpha_i, \alpha_i)}{d\alpha_i} = E_{\alpha_{-i}} \left\{ \sum_{g_i} A^{g_i}(\alpha_i, \alpha_{-i}) \int u_1(\alpha_i, s_i) P_i^{g_i}(\alpha_i, \alpha_{-i}, s_i) dG_i(s_i) \right\}.$$

Working with the case  $\hat{\alpha}_i > \alpha_i$ , we can obtain the right derivative of  $\pi_i(\alpha_i, \alpha_i)$ , which is given by

$$\frac{d\pi_i^+(\alpha_i, \alpha_i)}{d\alpha_i} = E_{\alpha_{-i}} \left\{ \sum_{g_i} A^{g_i}(\alpha_i, \alpha_{-i}) \int u_1(\alpha_i, s_i) P_i^{g_i}(\alpha_i, \alpha_{-i}, s_i) dG_i(s_i) \right\}.$$

Therefore, we conclude that  $\pi_i(\alpha_i) = \pi_i(\alpha_i, \alpha_i)$  is differentiable everywhere, and

$$\pi_i'(\alpha_i) = E_{\alpha_{-i}} \left\{ \sum_{g_i} A^{g_i}(\alpha_i, \alpha_{-i}) \int u_1(\alpha_i, s_i) P_i^{g_i}(\alpha_i, \alpha_{-i}, s_i) dG_i(s_i) \right\}$$

$$= \int u_1(\alpha_i, s_i) \cdot \sum_{g_i} [E_{\alpha_{-i}} A^{g_i}(\alpha_i, \alpha_{-i}) P_i^{g_i}(\alpha_i, \alpha_{-i}, s_i)] dG_i(s_i)$$

Since  $\pi'_i(\alpha_i)$  is bounded over  $[\underline{\alpha}, \bar{\alpha}]$ ,  $\pi_i$  satisfies a Lipschitz condition and hence it can be recovered from its derivative, which gives rise to (6).

**Proof of Theorem 2:** Since the second stage is a second-price auction, the contingent expected payoff  $\tilde{\pi}_i^{*g}(\alpha_i, \alpha_i; \alpha_{-i})$  for bidder  $i$  in the shortlisted group  $g$  is basically her marginal contribution to the virtual surplus:

$$\tilde{\pi}_i^{*g}(\alpha_i, \alpha_i; \alpha_{-i}) = \tilde{S}(\alpha^g) - \tilde{S}(\alpha_{-i}^g), i \in g, \forall \alpha^g.$$

We will proceed in two steps.

**Step 1:** We will show that  $x^*$  as defined in (15) is strictly increasing over  $[\alpha^*, \bar{\alpha}]$ , where  $\alpha^*$  is the minimum type that one can possibly be shortlisted in optimal entry.

Recall that  $s(\alpha_i)$  is defined such that  $w(\alpha_i, s(\alpha_i)) = 0$ . Define  $t = \max_{j \neq i, j \in g_i} w_j^+(\alpha_j, s_j)$  and let  $\Psi(t)$  denote the cdf of  $t$ . We have

$$\begin{aligned} & \tilde{\pi}_i^{*g_i}(y, y; \alpha_{-i}) \\ &= \int_{s(y)}^1 \int_0^{w(y, s_i)} [w(y, s_i) - t] d\Psi(t) dG_i(s_i) + \int_{s(y)}^1 w(y, s_i) \Psi(0) dG_i(s_i) \\ &= \int_{s(y)}^1 w(y, s_i) \int_0^{w(y, s_i)} d\Psi(t) dG_i(s_i) - \int_{s(y)}^1 \int_0^{w(y, s_i)} t d\Psi(t) dG_i(s_i) + \int_{s(y)}^1 w(y, s_i) \Psi(0) dG_i(s_i) \\ &= \int_{s(y)}^1 w(y, s_i) \Psi(w(y, s_i)) dG_i(s_i) - \int_{s(y)}^1 \int_0^{w(y, s_i)} t d\Psi(t) dG_i(s_i) \\ &= \int_{s(y)}^1 \int_0^{w(y, s_i)} \Psi(t) dt dG_i(s_i). \end{aligned}$$

As  $w(y, s(y)) = 0$ , we have

$$\begin{aligned} & \frac{d\tilde{\pi}_i^{*g_i}(y, y; \alpha_{-i})}{dy} \\ &= \int_{s(y)}^1 \frac{\partial w(y, s_i)}{\partial y} \Psi(w(y, s_i)) dG_i(s_i) \\ &= \int_{s(y)}^1 \left[ u_1(y, s_i) - u_1(y, s_i) \left( \frac{1 - F(y)}{f(y)} \right)' \right] \Psi(w(y, s_i)) dG_i(s_i) \\ &\geq \int_{s(y)}^1 u_1(y, s_i) \Psi(w(y, s_i)) dG_i(s_i). \end{aligned}$$

Since

$$\int u_1(y, s_i) P_i^{*g_i}(y, \alpha_{-i}, s_i) dG_i(s_i) = \int_{s(y)}^1 u_1(y, s_i) \Psi(w(y, s_i)) dG_i(s_i),$$

we have

$$\frac{d\tilde{\pi}_i^{*g_i}(y, y; \alpha_{-i})}{dy} \geq \int u_1(y, s_i) P_i^{*g_i}(y, \alpha_{-i}, s_i) dG_i(s_i) \geq 0. \quad (22)$$

As in the proof of Theorem 1, we construct a first-stage payment function as given by (12):

$$x_i^*(\alpha) = \sum_{g_i} A^{*g_i}(\alpha_i, \alpha_{-i}) [\tilde{\pi}_i^{*g_i}(\alpha_i, \alpha_i; \alpha_{-i}) - c] - \int_{\alpha^*}^{\alpha_i} \int u_1(y, s_i) \cdot \sum_{g_i} A^{*g_i}(y, \alpha_{-i}) P_i^{*g_i}(y, \alpha_{-i}, s_i) dG_i(s_i) dy. \quad (23)$$

By construction, it satisfies the envelope formula (6).

Below we will apply the inequality (22) to show that  $x_i^*(\alpha) \geq 0$  and that  $x^*(\alpha_i) \equiv E_{\alpha_{-i}} x_i^*(\alpha)$  is strictly increasing in  $\alpha_i \geq \alpha^*$ .

Let  $\alpha_i^l(g_i; \alpha_{-i})$  be the lower bound for  $i$  to be included in  $g_i$  and  $\alpha_i^h(g_i; \alpha_{-i})$  be the upper bound for  $i$  to be included in  $g_i$ . Recall that  $\hat{\alpha}_i(\alpha_{-i})$  denotes the minimum  $\alpha_i$  that can be included given  $\alpha_{-i}$ .

Then (23) can be rewritten as

$$\begin{aligned} & x_i^*(\alpha_i; \alpha_{-i}) \\ = & \sum_{g_i} A^{*g_i}(\alpha_i, \alpha_{-i}) [\tilde{\pi}_i^{*g_i}(\alpha_i, \alpha_i; \alpha_{-i}) - c] \\ & - \sum_{\substack{g^*(\alpha_i, \alpha_{-i}) \subseteq g_i \subseteq g^*(\hat{\alpha}_i(\alpha_{-i}), \alpha_{-i}) \\ \text{s.t.}: i \in g_i \text{ and } \alpha \geq \tilde{\alpha}, \forall \alpha \in g_i, \forall \tilde{\alpha} \notin g_i}} \int_{\alpha_i^l(g_i; \alpha_{-i})}^{\min\{\alpha_i, \alpha_i^h(g_i; \alpha_{-i})\}} \left[ \int u_1(y, s_i) P_i^{*g_i}(y, \alpha_{-i}, s_i) dG_i(s_i) \right] dy \\ \geq & \sum_{g_i} A^{*g_i}(\alpha_i, \alpha_{-i}) [\tilde{\pi}_i^{*g_i}(\alpha_i, \alpha_i; \alpha_{-i}) - c] \\ & - \sum_{\substack{g^*(\alpha_i, \alpha_{-i}) \subseteq g_i \subseteq g^*(\hat{\alpha}_i(\alpha_{-i}), \alpha_{-i}) \\ \text{s.t.}: i \in g_i \text{ and } \alpha \geq \tilde{\alpha}, \forall \alpha \in g_i, \forall \tilde{\alpha} \notin g_i}} \int_{\alpha_i^l(g_i; \alpha_{-i})}^{\min\{\alpha_i, \alpha_i^h(g_i; \alpha_{-i})\}} \frac{d\tilde{\pi}_i^{*g_i}(y, y; \alpha_{-i})}{dy} dy \quad (\text{by (22)}) \\ = & \sum_{g_i} A^{*g_i}(\alpha_i, \alpha_{-i}) [\tilde{\pi}_i^{*g_i}(\alpha_i, \alpha_i; \alpha_{-i}) - c] \\ & - \sum_{\substack{g^*(\alpha_i, \alpha_{-i}) \subseteq g_i \subseteq g^*(\hat{\alpha}_i(\alpha_{-i}), \alpha_{-i}) \\ \text{s.t.}: i \in g_i \text{ and } \alpha \geq \tilde{\alpha}, \forall \alpha \in g_i, \forall \tilde{\alpha} \notin g_i}} \tilde{\pi}_i^{*g_i}(\tilde{\alpha}_i, \tilde{\alpha}_i; \alpha_{-i}) \Big|_{\alpha_i^l(g_i; \alpha_{-i})}^{\min\{\alpha_i, \alpha_i^h(g_i; \alpha_{-i})\}}. \end{aligned}$$

To simplify notation, we use  $\tilde{\pi}_i^{*g_i}(\alpha_i; \alpha_{-i})$  to denote  $\tilde{\pi}_i^{*g_i}(\alpha_i, \alpha_i; \alpha_{-i})$ .

Recall the following two facts: (i)  $\tilde{\pi}_i^{*g}(\alpha) \geq \tilde{\pi}_i^{*\tilde{g}}(\alpha), \forall i \in g \subset \tilde{g}$ , and (ii) by the definition of  $\hat{\alpha}_i(\alpha_{-i})$ ,<sup>35</sup>

$$\tilde{\pi}_i^{*g^*(\hat{\alpha}_i(\alpha_{-i}), \alpha_{-i})}(\alpha_i^l(g^*(\hat{\alpha}_i(\alpha_{-i}), \alpha_{-i}); \alpha_{-i}); \alpha_{-i}) \geq c.$$

Thus when  $i \in g^*(\alpha)$ ,

$$\begin{aligned} x_i^*(\alpha_i; \alpha_{-i}) & \geq [\tilde{\pi}_i^{*g^*(\alpha_i, \alpha_{-i})}(\alpha_i^l(g^*(\alpha_i, \alpha_{-i}); \alpha_{-i}); \alpha_{-i}) - c] \\ & - \sum_{\substack{g^*(\alpha_i, \alpha_{-i}) \subseteq g_i \subseteq g^*(\hat{\alpha}_i(\alpha_{-i}), \alpha_{-i}) \\ \text{s.t.}: i \in g_i \text{ and } \alpha \geq \tilde{\alpha}, \forall \alpha \in g_i, \forall \tilde{\alpha} \notin g_i}} \tilde{\pi}_i^{*g_i}(\tilde{\alpha}_i; \alpha_{-i}) \Big|_{\alpha_i^l(g_i; \alpha_{-i})}^{\min\{\alpha_i, \alpha_i^h(g_i; \alpha_{-i})\}}. \end{aligned}$$

<sup>35</sup>Note  $\alpha_i^l(g^*(\hat{\alpha}_i(\alpha_{-i}), \alpha_{-i}), \alpha_{-i}) = \hat{\alpha}_i(\alpha_{-i})$ .

Note that for two neighboring sets  $g_i$  and  $g'_i$  such that  $g_i \subset g'_i$  and only the lowest  $\alpha_j$  in  $g'_i$  is excluded from  $g_i$ , we have  $\alpha_i^l(g_i) = \alpha_i^h(g'_i)$ . Thus  $x_i^*(\alpha_i; \alpha_{-i})$  can be written alternatively as follows:

$$\begin{aligned}
& x_i^*(\alpha_i; \alpha_{-i}) \\
\geq & [\tilde{\pi}_i^{g^*(\hat{\alpha}_i(\alpha_{-i}), \alpha_{-i})}(\alpha_i^l(g^*(\hat{\alpha}_i(\alpha_{-i}), \alpha_{-i}); \alpha_{-i}); \alpha_{-i}) - c] \\
& + \sum_{\substack{\forall g_i, g'_i, |g_i| = |g'_i| - 1 \\ \text{s.t. } i \in g \text{ and } \alpha \geq \bar{\alpha}, \forall \alpha \in g, \forall \bar{\alpha} \notin g, \text{ for } g = g_i, g'_i, \\ \text{and } g^*(\alpha_i, \alpha_{-i}) \subseteq g_i \subset g^*(\hat{\alpha}_i(\alpha_{-i}), \alpha_{-i})}} [\tilde{\pi}_i^{g_i}(\alpha_i^l(g_i; \alpha_{-i}); \alpha_{-i}) - \tilde{\pi}_i^{g'_i}(\alpha_i^l(g_i; \alpha_{-i}); \alpha_{-i})]. \quad (24)
\end{aligned}$$

Therefore  $x_i^*(\alpha_i; \alpha_{-i}) \geq 0$  because

$$\tilde{\pi}_i^{g_i}(\alpha_i^l(g_i; \alpha_{-i}); \alpha_{-i}) - \tilde{\pi}_i^{g'_i}(\alpha_i^l(g_i; \alpha_{-i}); \alpha_{-i}) > 0,$$

and

$$\tilde{\pi}_i^{g^*(\hat{\alpha}_i(\alpha_{-i}), \alpha_{-i})}(\alpha_i^l(g^*(\hat{\alpha}_i(\alpha_{-i}), \alpha_{-i}); \alpha_{-i}); \alpha_{-i}) - c \geq 0.$$

Recall that  $\alpha^*$  is the minimum type that can be possibly admitted, i.e.,

$$\alpha^* = \min_{\forall \alpha_{-i} \text{ s.t. } |g^*(\hat{\alpha}_i(\alpha_{-i}), \alpha_{-i})| = 1} \hat{\alpha}_i(\alpha_{-i}),$$

we thus have  $\tilde{\pi}_i^{g^*(\hat{\alpha}_i(\alpha_{-i}), \alpha_{-i})}(\alpha^*; \alpha_{-i}) = c$  when  $|g^*(\hat{\alpha}_i(\alpha_{-i}), \alpha_{-i})| = 1$  or  $g^*(\hat{\alpha}_i(\alpha_{-i}), \alpha_{-i}) = \{i\}$ .

Take any two representative bidders, say, bidders 1 and 2, and define

$$\tilde{\alpha}_2(\alpha_1) = \begin{cases} \arg \min_{\alpha_2} \{\tilde{\pi}_2^{g^{\{1,2\}}}(\alpha_1, \alpha_2) = c\} & \text{if } \tilde{\pi}_2^{g^{\{1,2\}}}(\alpha_1, \bar{\alpha}) \geq c, \\ +\infty & \text{if } \tilde{\pi}_2^{g^{\{1,2\}}}(\alpha_1, \bar{\alpha}) < c. \end{cases}$$

Clearly,  $\tilde{\alpha}_2(\alpha_1)$  strictly increases with  $\alpha_1$  before it reaches the upper bound  $\bar{\alpha}$ . Define  $\alpha_2^*(\alpha_1) = \min\{\tilde{\alpha}_2(\alpha_1), \alpha_1\} \leq \alpha_1$ . Clearly,  $\alpha_2^*(\alpha_1)$  strictly increases with  $\alpha_1$ . Thus, only bidder 1 is admitted given  $\alpha_{-1}$  iff  $\alpha_1 \geq \alpha^*$  and  $\max_{j \neq i} \{\alpha_j\} \leq \alpha_2^*(\alpha_1)$ .

We next show that  $\forall \alpha_{-i}$ , we have  $x_i^*(\alpha'_i; \alpha_{-i}) \geq x_i^*(\alpha_i; \alpha_{-i})$  for  $\alpha'_i > \alpha_i$ . This is clear if  $i \notin g^*(\alpha_i; \alpha_{-i})$ . If  $i \in g^*(\alpha_i; \alpha_{-i})$ , we must have  $g^*(\alpha'_i; \alpha_{-i}) \subseteq g^*(\alpha_i, \alpha_{-i})$ . Consider the difference of the two payments:

$$\begin{aligned}
& x_i^*(\alpha'_i; \alpha_{-i}) - x_i^*(\alpha_i; \alpha_{-i}) \\
= & \sum_{g_i} A^{g_i}(\alpha'_i, \alpha_{-i}) [\tilde{\pi}_i^{g_i}(\alpha'_i; \alpha_{-i}) - c] \\
& - \sum_{\substack{g^*(\alpha'_i, \alpha_{-i}) \subseteq g_i \subseteq g^*(\hat{\alpha}_i(\alpha_{-i}), \alpha_{-i}) \\ \text{s.t. } i \in g_i \text{ and } \alpha \geq \bar{\alpha}, \forall \alpha \in g_i, \forall \bar{\alpha} \notin g_i}} \int_{\alpha_i^l(g_i; \alpha_{-i})}^{\min\{\alpha'_i, \alpha_i^h(g_i; \alpha_{-i})\}} \left[ \int u_1(y, s_i) P_i^{g_i}(y, \alpha_{-i}, s_i) dG_i(s_i) \right] dy
\end{aligned}$$

$$\begin{aligned}
& - \sum_{g_i} A^{*g_i}(\alpha_i, \alpha_{-i}) [\tilde{\pi}_i^{*g_i}(\alpha_i; \alpha_{-i}) - c] \\
& + \sum_{\substack{g^*(\alpha_i, \alpha_{-i}) \subseteq g_i \subseteq g^*(\hat{\alpha}_i(\alpha_{-i}), \alpha_{-i}) \\ \text{s.t.}: i \in g_i \text{ and } \alpha \geq \tilde{\alpha}, \forall \alpha \in g_i, \forall \tilde{\alpha} \notin g_i}} \int_{\alpha_i^l(g_i; \alpha_{-i})}^{\min\{\alpha_i, \alpha_i^h(g_i; \alpha_{-i})\}} \left[ \int u_1(y, s_i) P_i^{*g_i}(y, \alpha_{-i}, s_i) dG_i(s_i) \right] dy \\
= & \sum_{g_i} A^{*g_i}(\alpha'_i, \alpha_{-i}) [\tilde{\pi}_i^{*g_i}(\alpha'_i; \alpha_{-i}) - c] \\
& - \sum_{\substack{g^*(\alpha'_i, \alpha_{-i}) \subseteq g_i \subseteq g^*(\alpha_i, \alpha_{-i}) \\ \text{s.t.}: i \in g_i \text{ and } \alpha \geq \tilde{\alpha}, \forall \alpha \in g_i, \forall \tilde{\alpha} \notin g_i}} \int_{\alpha_i^l(g_i; \alpha_{-i})}^{\min\{\alpha'_i, \alpha_i^h(g_i; \alpha_{-i})\}} \left[ \int u_1(y, s_i) P_i^{*g_i}(y, \alpha_{-i}, s_i) dG_i(s_i) \right] dy \\
& - \sum_{g_i} A^{*g_i}(\alpha_i, \alpha_{-i}) [\tilde{\pi}_i^{*g_i}(\alpha_i; \alpha_{-i}) - c] \\
\geq & \sum_{g_i} A^{*g_i}(\alpha'_i, \alpha_{-i}) [\tilde{\pi}_i^{*g_i}(\alpha'_i, \alpha'_i; \alpha_{-i}) - c] \\
& - \sum_{\substack{g^*(\alpha'_i, \alpha_{-i}) \subseteq g_i \subseteq g^*(\alpha_i, \alpha_{-i}) \\ \text{s.t.}: i \in g_i \text{ and } \alpha \geq \tilde{\alpha}, \forall \alpha \in g_i, \forall \tilde{\alpha} \notin g_i}} \int_{\alpha_i^l(g_i; \alpha_{-i})}^{\min\{\alpha'_i, \alpha_i^h(g_i; \alpha_{-i})\}} \frac{d\tilde{\pi}_i^{*g_i}(y, y; \alpha_{-i})}{dy} dy \\
& - \sum_{g_i} A^{*g_i}(\alpha_i, \alpha_{-i}) [\tilde{\pi}_i^{*g_i}(\alpha_i, \alpha_i; \alpha_{-i}) - c] \\
= & \sum_{g_i} A^{*g_i}(\alpha'_i, \alpha_{-i}) [\tilde{\pi}_i^{*g_i}(\alpha'_i; \alpha_{-i}) - c] \\
& - \sum_{\substack{g^*(\alpha'_i, \alpha_{-i}) \subseteq g_i \subseteq g^*(\alpha_i, \alpha_{-i}) \\ \text{s.t.}: i \in g_i \text{ and } \alpha \geq \tilde{\alpha}, \forall \alpha \in g_i, \forall \tilde{\alpha} \notin g_i}} \tilde{\pi}_i^{*g_i}(\tilde{\alpha}_i; \alpha_{-i}) \Big|_{\alpha_i^l(g_i; \alpha_{-i})}^{\min\{\alpha'_i, \alpha_i^h(g_i; \alpha_{-i})\}} \\
& - \sum_{g_i} A^{*g_i}(\alpha_i, \alpha_{-i}) [\tilde{\pi}_i^{*g_i}(\alpha_i; \alpha_{-i}) - c], \\
\geq & \sum_{\substack{\forall g_i, g'_i, |g_i| = |g'_i| - 1 \\ \text{s.t.}: i \in g \text{ and } \alpha \geq \tilde{\alpha}, \forall \alpha \in g, \forall \tilde{\alpha} \notin g, \text{ for } g = g_i, g'_i, \\ \text{and } g^*(\alpha'_i, \alpha_{-i}) \subseteq g'_i \subseteq g^*(\alpha_i, \alpha_{-i})}} [\tilde{\pi}_i^{*g_i}(\alpha_i^l(g_i; \alpha_{-i}); \alpha_{-i}) - \tilde{\pi}_i^{*g'_i}(\alpha_i^l(g'_i; \alpha_{-i}); \alpha_{-i})] \\
& + [\tilde{\pi}_i^{*g^*(\alpha_i, \alpha_{-i})}(\alpha_i^h(g^*(\alpha_i, \alpha_{-i}); \alpha_{-i}); \alpha_{-i}) - \tilde{\pi}_i^{*g^*(\alpha_i, \alpha_{-i})}(\alpha_i; \alpha_{-i})] \tag{25} \\
\geq & 0.
\end{aligned}$$

Thus for  $\alpha_{-i}$  such that both  $\alpha'_i$  and  $\alpha_i$  are admitted, we must have  $x_i^*(\alpha'_i; \alpha_{-i}) \geq x_i^*(\alpha_i; \alpha_{-i})$ . In addition, there is a positive measure of  $\alpha_{-i}$  such that  $\alpha_i$  cannot be admitted as a sole entrant, but  $\alpha'_i$  can be admitted at least as a sole entrant. These  $\alpha_{-i}$  types cover the set  $\Omega(\alpha'_i; \alpha_i) = \{\alpha_{-i} | \alpha'_i \geq \alpha_2^*(\alpha'_i) > \max_{j \neq i} \{\alpha_j\} \geq \alpha_2^*(\alpha_i) \text{ for } \alpha'_i > \alpha_i \geq \alpha^*\}$ . Note that  $\alpha'_1 \geq \alpha_2^*(\alpha'_1)$ . For any  $\alpha_{-i}$  in  $\Omega(\alpha'_i; \alpha_i)$ , we must have  $x_i^*(\alpha'_i; \alpha_{-i}) > x_i^*(\alpha_i; \alpha_{-i})$  as there are two possibilities spelled out below.

First, consider  $\alpha_{-i}$  in  $\Omega(\alpha'_i; \alpha_i)$  such that type  $\alpha_i$  is not admitted. In this case,  $x_i^*(\alpha_i; \alpha_{-i}) = 0$ , but  $x_i^*(\alpha'_i; \alpha_{-i}) > 0$  must hold for the following reasons. There are two subcases. Case I: If for such an  $\alpha_{-i}$  in  $\Omega(\alpha'_i; \alpha_i)$ , only bidder  $i$  is admitted when her type changes from  $\hat{\alpha}_i(\alpha_{-i})$  to  $\alpha'_i$ , then  $|g^*(\hat{\alpha}_i(\alpha_{-i}), \alpha_{-i})| = 1$

and  $\hat{\alpha}_i(\alpha_{-i}) > \alpha^*$ . We have  $\hat{\alpha}_i(\alpha_{-i}) > \alpha^*$ , because  $\hat{\alpha}_i(\alpha_{-i}) \geq \max_{j \neq i} \{\alpha_j\}$  ( $\hat{\alpha}_i(\alpha_{-i})$  is admitted as a sole entrant) and  $\max_{j \neq i} \{\alpha_j\} \geq \alpha_i > \alpha^*$  ( $\alpha_i$  is not even admitted as a sole entrant). Since  $\hat{\alpha}_i(\alpha_{-i}) > \alpha^*$ , we have  $\tilde{\pi}_i^{*g^*(\hat{\alpha}_i(\alpha_{-i}), \alpha_{-i})}(\hat{\alpha}_i(\alpha_{-i}); \alpha_{-i}) > \tilde{\pi}_i^{*g^*(\hat{\alpha}_i(\alpha_{-i}), \alpha_{-i})}(\alpha^*; \alpha_{-i}) = c$ . Case II: If for such an  $\alpha_{-i}$  in  $\Omega(\alpha'_i; \alpha_i)$ , there can be more than two bidders to be admitted when her type changes from  $\hat{\alpha}_i(\alpha_{-i})$  to  $\alpha'_i$ , then by (25)  $x_i^*(\alpha'_i; \alpha_{-i})$  is larger than strictly positive term  $\tilde{\pi}_i^{*g_i}(\alpha'_i(g_i; \alpha_{-i}); \alpha_{-i}) - \tilde{\pi}_i^{*g_i}(\alpha'_i(g_i; \alpha_{-i}); \alpha_{-i})$  where  $g_i = g^*(\alpha'_i, \alpha_{-i})$ . This means  $x_i^*(\alpha'_i; \alpha_{-i}) > 0$ .

Second, consider  $\alpha_{-i}$  in  $\Omega(\alpha'_i; \alpha_i)$  such that type  $\alpha_i$  is admitted with at least two bidders. Note that  $\alpha_i$  cannot be admitted as a sole entrant by construction of  $\Omega(\alpha'_i; \alpha_i)$ . In this case, we have  $x_i^*(\alpha_i; \alpha_{-i}) \geq 0$ , but  $x_i^*(\alpha'_i; \alpha_{-i}) > x_i^*(\alpha_i; \alpha_{-i})$  for the following reason. For such an  $\alpha_{-i}$  in  $\Omega(\alpha'_i; \alpha_i)$ , there can be more than two bidders admitted when bidder  $i$ 's type changes from  $\hat{\alpha}_i(\alpha_{-i})$  to  $\alpha_i$  and further to  $\alpha'_i$ , but eventually bidder  $i$  is only admitted as the sole entrant when her type reaches  $\alpha'_i$ . Thus by (25)  $x_i^*(\alpha'_i; \alpha_{-i})$  is larger than  $x_i^*(\alpha_i; \alpha_{-i})$  by at least one strictly positive term  $\tilde{\pi}_i^{*g_i}(\alpha'_i(g_i; \alpha_{-i}); \alpha_{-i}) - \tilde{\pi}_i^{*g_i}(\alpha'_i(g_i; \alpha_{-i}); \alpha_{-i})$  where  $g_i = \{i\}$ . This means  $x_i^*(\alpha'_i; \alpha_{-i}) > x_i^*(\alpha_i; \alpha_{-i})$ .

We thus have proven that given  $\alpha_{-i}$ ,  $x_i^*(\alpha_i, \alpha_{-i})$  must be strictly increasing in  $\alpha_i$  for  $\alpha_i \geq \alpha^*$ . Since  $A^{*g_i}(\alpha_i, \alpha_{-i})$ ,  $\tilde{\pi}_i^{*g_i}(\alpha_i, \alpha_i; \alpha_{-i})$ , and  $P_i^{*g_i}(y, \alpha_{-i}, s_i)$  are all symmetric across bidders,  $x_i^*(\alpha)$  is also a symmetric function by inspecting (23). Thus  $x^*(\alpha_i) \equiv E_{\alpha_{-i}} x_i^*(\alpha)$  is strictly increasing in  $\alpha_i$  for  $\alpha_i \geq \alpha^*$ . Under the handicap auction rules,  $x^*(\alpha_i)$  can be rewritten as (15).

**Step 2:** We will show that, given (14) and (15),  $x^*$  constitutes a symmetric (strictly) increasing equilibrium in the (reduced) first-stage all-pay auction game, with the second stage being replaced by its correlated equilibrium payoffs.

To that end, we consider the associated direct mechanism described by the entry allocation rule (10) and payment rules (14) and (15). By construction of (15), the envelope formula (6) is satisfied (with  $\pi_i(\underline{\alpha}, \underline{\alpha}) = 0$ ). The second-round allocation rule is dominant strategy implementable, so (4) is also satisfied. Thus we can follow exactly the same procedure as in the proof for Theorem 1 to demonstrate that

$$\Delta = \pi_i^*(\alpha_i, \hat{\alpha}_i) - \pi_i^*(\alpha_i, \alpha_i) \leq 0, \text{ for any } \alpha_i \text{ and } \hat{\alpha}_i \in [\underline{\alpha}, \bar{\alpha}].$$

That is, the associated direct mechanism is incentive compatible, which in turn implies that in the original (reduced) first-stage all-pay auction game,  $x^*$  constitutes a symmetric (strictly) increasing equilibrium.

Hence the two-stage auction can implement the optimal mechanism described in Theorem 1.

## REFERENCES

- [1] Baron, D. and Besanko, D. (1984), Regulation and Information in a Continuing Relationship, *Information Economics and Policy* 1, 267-302.
- [2] Bergemann, D. and J. Välimäki. “Information in Mechanism Design,” in *Proceedings of the 9th World Congress of the Econometric Society*, ed. by R. Blundell, W. Newey, and T. Persson, 2006, pp. 186-221. Cambridge University Press, Cambridge.
- [3] Bergemann, Dirk and Achim Wambach (2014), Sequential Information Disclosure in Auctions, forthcoming *Journal of Economic Theory*.
- [4] Boone, Jan and Jacob Goeree (2009), Optimal Privatization Under Asymmetric Information, *Economic Journal* 119, 277-297.
- [5] Campbell, Colin (1998), Coordination in Auctions with Entry, *Journal of Economic Theory* 82, 425-450.
- [6] Cao, X., Tian, G. (2010), Equilibria in First Price Auctions with Participation Costs, *Games and Economic Behavior* 69, 258–273.
- [7] Compte, Olivier, and Philippe Jehiel (2007), Auctions and Information Acquisition: Sealed Bid or Dynamic Formats? *RAND Journal of Economics* 38(2), 355-372.
- [8] Courty, Pascal and Li, Hao (2000), Sequential Screening, *Review of Economic Studies* 67, pp. 697-717.
- [9] Crémer, J. and McLean, R.P. (1988), Full Extraction of the Surplus in Bayesian and Dominant Strategy Auctions, *Econometrica* 53, 1988, pp. 345-361.
- [10] R. Engelbrecht-Wiggans (1993), Optimal Auctions Revisited, *Games and Economic Behavior* 5, 227-239.
- [11] Esö, Péter and Balázs Szentes (2007), Optimal Information Disclosure in Auctions and the Handicap Auction, *Review of Economic Studies* 74, pp. 705-731.
- [12] Fullerton, Richard and Preston McAfee (1999), Auctioning Entry into Tournaments, *Journal of Political Economy* 107, 573-605.
- [13] Foley, A. (2003), Multi-round Auctions for Institutional Real Estate Assets: Theory and Practice. M.S.Thesis in Real Estate Development, MIT.
- [14] Gentry, Matthew and Tong Li (2014), Identification in Auctions with Selective Entry, *Econometrica* 82(1), 315-344.



- [15] Levin, Dan and James Smith (1994), Equilibrium in Auctions with Entry, *American Economic Review* 84, 585-599.
- [16] Li, Hao and Shi, Xianwen (2013), Discriminatory Information Disclosure, Working paper.
- [17] Lu, Jingfeng (2009), Auction design with opportunity cost, *Economic Theory* 38(1), 73-103.
- [18] Lu, Jingfeng (2010), Entry Coordination and Auction Design with Private Costs of Information-Acquisition, *Economic Inquiry* 48(1), 274-289.
- [19] Lu, Jingfeng and Lixin Ye (2013), Efficient and Optimal Mechanisms with Private Information Acquisition Costs, *Journal of Economic Theory* 148(1), 393-408.
- [20] McAdams, David (2007), Adjustable Supply in Uniform Price Auctions: Non-Commitment as a Strategic Tool, *Economic Letters* 95(1), 48-53.
- [21] Marmer, Vadim, Artyom Shneyerov, and Pai Xu (2013), What Model for Entry in First-price Auctions? A nonparametric approach, *Journal of Econometrics* 176 (1), 46-58.
- [22] McAfee, Preston and John McMillan (1987), Auctions with Entry, *Economic Letters* 23, 343-347.
- [23] Menezes, F. M. and P. K. Monteiro (2000), Auctions with Endogenous Participation, *Review of Economic Design* 5: 71-89.
- [24] Milgrom, Paul. *Putting Auction Theory to Work*, Cambridge University Press, 2004.
- [25] Myerson, Roger (1981), Optimal Auction Design, *Mathematics of Operations Research* 6, 58-73.
- [26] Myerson, Roger (1986), Multistage Games with Communication, *Econometrica* 54 (2), 323-358.
- [27] Pavan, Alessandro, Ilya Segal, and Juuso Toikka (2014), Dynamic Mechanism Design: A Myersonian Approach, *Econometrica* 82 (2), 601-653.
- [28] Persico, Nicola (2000), Information Acquisition in Auctions, *Econometrica* 68, 135-148.
- [29] Quint, Daniel and Kenneth Hendricks (2013), Indicative Bidding in Auctions with Costly Entry, Working paper.
- [30] Rezedo, Leonardo (2013), Mid-Auction Information Acquisition, Working paper.
- [31] Riordan, M. and Sappington, D. (1987), Awarding Monopoly Franchises, *American Economic Review* 77, 375-387.
- [32] Roberts, James and Andrew Sweeting (2013), When Should Sellers Use Auctions? *American Economic Review* 103 (5), 1830-61.

- [33] Samuelson, William (1985), Competitive Bidding with Entry Costs, *Economic Letters* 17, 53-57.
- [34] Stegeman, Mark (1996), Participation Costs and Efficient Auctions, *Journal of Economic Theory* 71, 228-259.
- [35] Tan, Guofu (1992), Entry and R&D in Procurement Contracting, *Journal of Economic Theory* 58, 41-60.
- [36] Tan, Guofu and Okan Yilankaya (2006), Equilibria in Second Price Auctions with Participation Costs, *Journal of Economic Theory* 130, 205-219.
- [37] M. Vallen, C. Bullinger (1999), The Due Diligence Process for Acquiring and Building Power Plants, *The Electricity Journal* 12(8), 28-37.
- [38] Ye, Lixin (2004), Optimal Auctions with Endogenous Entry, *The B.E. Journals: Contributions to Theoretical Economics* 4, 1-27.
- [39] Ye, Lixin (2007), Indicative Bidding and a Theory of Two-Stage Auctions, *Games and Economic Behavior* 58, 181-207.