Optimal Contest Design with Incomplete Information*

Bin Liu[†] Jingfeng Lu[‡] Ruqu Wang[§] Jun Zhang[¶]

January 13, 2015

Abstract

In this paper, we characterize the optimal contest mechanism with independent private abilities of the contestants. The contest designer has a fixed prize budget to extract effort from the contestants, and both positive and negative prizes are allowed. We find that there exists no optimal contest mechanism maximizing the total effort from the contestants. Nevertheless, by invoking exploding negative prizes, the designer can extract effort approaching the utmost level (i.e., highest possible effort inducible when all contestants are of the maximum ability with certainty). When a bound (i.e., K) on the negative prizes is imposed, we fully characterize the optimal contest mechanism, which can be implemented by a modified all-pay auction with a minimum bid and an entry fee of K per contestant.

JEL Classification Numbers: D72, D82, C73

Keywords: optimal contest, mechanism design, negative prize, incomplete information, cross-type transfer, leverage

*We are grateful to James Atsu Amegashie, Yaron Azrieli, Tilman Börgers, Yi-Chun Chen, Soo Hong Chew, Jeff Ely, Drew Fudenberg, Robert Hammond, Paul Healy, Johannes Hörner, Yuan Ju, Jinwoo Kim, Peter Klibanoff, Charles Knoeber, Fuhito Kojima, Georgia Kosmopoulou, Takashi Kunimoto, Roger Lagunoff, Stephan Lauermann, Dan Levin, Xiao Luo, George Mailath, Thomas Mariotti, Preston McAfee, Diego Moreno, Thayer Morrill, Wojciech Olszewski, Alessandro Pavan, James Peck, Bob Roberts, Ariel Rubinstein, Yuval Salant, Mark Satterthwaite, Klaus Schmidt, Paul Schweinzer, Xianwen Shi, Ron Siegel, Ning Sun, Yeneng Sun, Satoru Takahashi, Flavio Toxvaerd, Rakesh Vorah, Quan Wen, Asher Wolinsky, Huanxing Yang, Zaifu Yang, Lixin Ye, Xiaoyong Zheng and participants at various conferences for very helpful comments and suggestions. Lu gratefully acknowledges the financial support from MOE of Singapore. Wang gratefully acknowledges the financial support from SSHRC of Canada. Zhang thanks University of Technology Sydney for financial support under Business Research Grant. The usual disclaimer applies.

[†]Bin Liu: Department of Economics, National University of Singapore, Singapore 117570. Email: binliu29@u.nus.edu.

[‡]Jingfeng Lu: Department of Economics, National University of Singapore, Singapore 117570. Email: ecsljf@nus.edu.sg.

[§]Ruqu Wang: Department of Economics, Queen's University, Ontario K7L 3N6, Canada. Email: wangr@queensu.ca.

[¶]Jun Zhang: Economics Discipline Group, School of Business, University of Technology Sydney; Tel: +61 2 95143203, Fax: +61 2 95147722, E-mail: zhangjunqueens@gmail.com.

1 Introduction

Many activities and events, such as promotions within organizations, school admissions, political elections, R&D races and sports, can be viewed as contests. Contests, as a well established institution, essentially incentivize contestants to exert costly and irreversible effort by awarding prizes to the winners. A contestant may possess private information, such as his own ability, talent, strength, or valuation for the prize. A contest designer desires an optimal prize architecture to deal with such private information to achieve her objective. This objective is total effort maximization in our paper, as is usually the case in the contest literature such as Moldovanu and Sela [13].

This optimal contest design problem is very similar to the optimal auction design problem. The abilities of the contestants in a contest can be mathematically translated into the valuations of the bidders in an auction. The prizes in the contest can be reinterpreted as the winning probabilities in the auction. Meanwhile, effort from the contestants in the contest can be treated as payments from the bidders in the auction. Therefore, maximizing the total effort from the contestants by optimally allocating the prizes is equivalent to maximizing the total payments from the bidders by optimally allocating the winning probabilities. As such, we make use of the well developed techniques on optimal auction design pioneered by Myerson [17] to analyze the optimal contest design problem.

The above two problems, however, are not the same. In particular, we allow for negative prizes in our contest design problem. This relaxation in the prize structure differentiates our paper from existing studies in the auction design literature, as well as in the contest design literature. Negative prizes are technically equivalent to negative winning probabilities in auctions. Without allowing for negative prizes, the optimal contest design problem is mathematically equivalent to the optimal auction design problem in Myerson [17]. Once negative prizes are allowed, Myerson's [17] technique is no longer adequate. In this paper, we characterize the optimal contest mechanism using the techniques of continuous linear programming.

In an optimal auction, the bidder with the highest virtual valuation (if it is higher than the seller's reservation value) should have a winning probability equal to one and all other bidders should have zero probability of winning. Allowing for negative prizes in contests is equivalent to allowing for negative winning probabilities in auctions. It creates an additional venue to increase the seller's revenue in auctions (or, equivalently, the contestants' effort in contests). Allocating a negative prize to a contestant with a lower (including negative) virtual ability would increase the positive prize to a contestant with a higher virtual ability by the same amount while keeping the total prize budget unchanged.[1] This kind of leveraging on the contestants' virtual abilities relaxes the original optimization constraints and increases the total level of effort elicited. Of course, we still need to ensure that the interim individual rationality constraints for the lower ability contestants

[1] "Virtual ability" in this paper is parallel to "virtual value" in Myerson [17].

are satisfied by awarding them with positive prizes at appropriate occasions. The optimal contest mechanism finds the optimal balance between these incentives.

We obtain two surprising results. First, we find that an optimal contest mechanism does not exist. Second, any level of total effort arbitrarily close to the **utmost total effort** (i.e., the highest total effort that can be induced when all contestants are of the maximum ability with certainty) can be achieved by an incentive compatible contest mechanism. This level of total effort is much more than the level from the full surplus extraction using the optimal auctions techniques, as all contestants having the maximum ability is an unlikely event. It is worth noting that this result is derived under the assumption that types are independently distributed.[2]

Negative prizes play a crucial role in the above results. Certain features in real-life contests could lead to negative prizes for the contestants. In the FCC-organized contest to set the standard for high-definition television, any firm can enter the contest but with a $200,000$ entry fee (cf. Taylor [21]). A firm that enters but not winning the contest thus gets a negative prize. Similarly, professional tournaments, such as golf, sailing, chess and horse racing, often charge significant membership fees, registration fees, nomination fees and/or starting fees, which constitute a significant portion of the prizes. In certain poker tournaments, as another example, in order to compete, a player needs to pay a "buy-in," which is an upfront payment that goes towards the winning prize pool. In this case, this "buy-in" becomes the negative prize if a player loses in the game.

There must be a limit on the amount of negative prizes that one can set. Unfortunately, in order to induce effort levels closer and closer to the utmost total effort, the contest designer needs to impose larger and larger negative prizes on the contestants. This is obviously not always feasible. In this paper, we address this issue by analyzing a bounded negative prize problem. We assume that any negative prize cannot exceed a bound K, which is common to all contestants. We characterize the optimal contest mechanism given K. This mechanism features a threshold level of ability that depends on K. When K is small, contestants cannot be punished too severely, and the threshold level is exactly the same as the cut-off level in the optimal auction mechanism in Myerson [17]. When K is large, however, the threshold is strictly higher than the cut-off in Myerson [17]. This threshold is important. When the abilities of all contestants are lower than the threshold, all contestants share equally either a portion (in the former case) or the entirety (in the latter case) of the original prize budget. Equivalently, a contestant is randomly selected as the winner of the relevant prize. When the highest ability of the contestants is above the threshold, this highest ability contestant will receive an enhanced prize, which is equal to the original prize budget plus the extra money collected from the negative prizes imposed on other contestants.

[2](Almost) full surplus extraction in auctions requires the bidders' types to be correlated. In our model, the abilities of the contestants are distributed independently, making it even more difficult to extract full surplus. See the discussions after Proposition 1 for more details.

This optimal contest mechanism can be implemented by a modified all-pay auction with the following features: an entry fee equal to K (i.e., the bound on the negative prize), a minimum bid, and a grand prize equal to the original prize budget plus all of the entry fees. In a traditional auction, a bidder gets zero if he does not bid. In our modified all-pay auction, contestants share a prize or they are randomly selected as a single winner of the prize when no one bids. This feature of the mechanism serves two purposes. First, as required by the efficiency consideration, contestants with low abilities are never asked to exert effort as it is less efficient. Second, it is to ensure the participation from the low ability contestants since they pay an entry fee.[3]

Our paper belongs to the literature on optimal contest design with incomplete information. Fullerton and McAfee [5] study the optimal shortlisting in a procurement environment. They allow the sponsor to use entry fees to screen the firms and offset costs of the procurement. Optimal prize allocation has been studied in various all-pay auction frameworks starting from the seminal work of Moldovanu and Sela [13]. They establish the winner-take-all principle in contest under linearity assumptions. Minor [12] reexamines this principle in cases where contestants have convex costs of effort and where the contest designer has concave benefit of effort. Moldovanu and Sela [14] generalize their own investigation to a two-stage all-pay auction framework. Meanwhile, Moldovanu et al [15] analyze the environment where contestants care about their relative status. They further allow for negative prizes in Moldovanu et al [16].

There are two major differences between our paper and the above works on optimal prize allocation. The first difference is that our analysis accommodates any contest rule that allocates the budget contingent on the contestants' reported type profile, while theirs focus on all-pay auctions. The second difference is that we model negative prize differently. In Moldovanu et al [16], a negative prize, which they call punishment, is costly for the organizer to implement. In our paper, when negative prizes are imposed on some contestants, similarly to Fullerton and McAfee [5], the money collected can be utilized by the contest designer to strengthen the incentives to other contestants.

The mechanism design approach utilized in our analysis has also been employed by Polishchuk and Tonis [19] to examine the optimal contest design, but they focus on nonnegative prizes. Kirkegaard [8] uses a similar approach to study the optimal favoritism in contests with asymmetric players. Our paper differentiates from these existing studies in providing a first investigation on the role of negative prizes using a mechanism design approach. Our analysis sheds light on the necessity of negative prizes in the optimal mechanism and the leverage of prizes on contestants with different virtual abilities.

The rest of this paper is organized as follows. In Section 2, we present the model. In Section

[3]We may not observe this exactly in reality. However, most real life contests have subsequent ones, usually to be held, say a year later. Therefore, accumulating the unawarded prize money for the prize pool of next year's contest (as the US Lotto) would have similar effects as distributing the money right away.

3, we carry out our analysis on the optimal contest design. We show that there exists no optimal mechanism, and then construct a sequence of mechanisms which can achieve almost the utmost total effort. These mechanisms require large negative prizes to be imposed with positive probability. Section 4 considers the case where negative prizes are bounded. In a sequence of analytical steps, we fully characterize the optimal contest under the regularity condition of increasing virtual ability function. In Section 5, we provide some concluding remarks. An appendix collects long proofs.

2 The model

A risk neutral contest designer has a total prize budget of $V > 0$ to elicit effort from $N \geq 2$ risk neutral contestants in a contest. This budget V does not need to be all in cash; it could be the value to the contestants of non-monetary and sometimes indivisible rewards, such as honors, recognitions and gifts. Each contestant has an ability for the contest. The cost for contestant i with ability t_i to exert effort $e_i \geq 0$ is given by $c(e_i, t_i) = e_i/t_i$. This ability or type t_i,[4] is the private information of contestant i, and it follows an independent and identical distribution with cumulative distribution function $F(\cdot)$, and probability density function $f(\cdot)$ which is strictly positive on support $[a, b]$ with $a > 0$. The parameter b is referred as the maximum ability. Similarly to Myerson [17], we define $J(t_i) = t_i - \frac{1-F(t_i)}{f(t_i)}$ as the virtual ability function.

The payoff of a contestant in the contest is equal to the prize he receives minus his cost of effort. The contest designer uses the prize budget V to induce effort from the contestants. At the same time, if there is money left in the budget, she values that money as well. To simplify the notation, assume that there is a linear relationship between effort and money for the contest designer. Let t_0 denote the money to effort ratio; 1 dollar is equivalent to t_0 units of effort. If the contest designer's objective is to maximize the total effort, then $t_0 = 0$. Assume that t_0 is common knowledge.[5] Note that the cost of 1 unit of effort for the maximum ability (b) contestant is $1/b$, which needs to be less than $1/t_0$, the value of 1 unit of effort to the contest designer; otherwise, it is obviously optimal for the designer not to spend any of the prize budget. Therefore, we assume that $t_0 < b$.

According to the revelation principle, we can focus our analysis on direct mechanisms. Let $\tilde{t}_i \in [a, b]$ be the report of contestant i regarding his own ability. Then we can define contestant i's prize and effort as functions of the profile of reports $\tilde{\mathbf{t}} = (\tilde{t}_1, \cdots, \tilde{t}_N)$ by $v_i(\tilde{\mathbf{t}})$ and $e_i(\tilde{\mathbf{t}})$, respectively.[6] Given the profile of reports $\tilde{\mathbf{t}} = (\tilde{t}_1, \cdots, \tilde{t}_N)$, the contest designer gives a prize of $v_i(\tilde{\mathbf{t}})$ to contestant i and demands an effort of $e_i(\tilde{\mathbf{t}})$ from him.[7] A direct contest mechanism can thus be denoted by

[4] In Moldovanu and Sela [13], a contestant's ability is defined as $c_i = \frac{1}{t_i}$, which is mathematically equivalent.

[5] As we shall see later, this t_0 is similar to the reservation value of a seller in the auction design problem.

[6] This is similar to auction analysis. Given the profile of type reports from the bidders, the auctioneer determines the probability of winning and and the payment of each bidder.

[7] $v_i(\tilde{\mathbf{t}})$ and $e_i(\tilde{\mathbf{t}})$ can be interpreted as expected prize and effort in a stochastic mechanism. As everyone's payoff

$(\mathbf{v}(\tilde{\mathbf{t}}), \mathbf{e}(\tilde{\mathbf{t}}))$, where $\mathbf{v}(\tilde{\mathbf{t}}) = (v_1(\tilde{\mathbf{t}}), \cdots, v_N(\tilde{\mathbf{t}}))$ and $\mathbf{e}(\tilde{\mathbf{t}}) = (e_1(\tilde{\mathbf{t}}), \cdots, e_N(\tilde{\mathbf{t}}))$.

In the following section, we will examine the existence of the optimal mechanism and offer remedies when it does not exist.

3 Optimal contest design and utmost total effort

The model setup in the previous section hints on the connections between an optimal auction problem and an optimal contest problem. In this section, in the process of analyzing the optimal contest design, we will illustrate how the two distinguish themselves from each other.

3.1 Positive and Negative Prizes

Define the expected prize of contestant i with report $\tilde{t}_i$ as

$$V_i(\tilde{t}_i) = \int_{\mathbf{t}_{-i}} v_i(\tilde{t}_i, \mathbf{t}_{-i})\mathbf{f}_{-i}(\mathbf{t}_{-i})d\mathbf{t}_{-i}, \tag{1}$$

where $\mathbf{t}_{-i} = (t_1, ...t_{i-1}, t_{i+1}, ...t_N)$ and $\mathbf{f}_{-i}(\mathbf{t}_{-i})$ denotes the density of $\mathbf{t}_{-i}$.

Given that other contestants truthfully report their abilities, contestant i's expected payoff when reporting $\tilde{t}_i$ is

$$\begin{aligned} u_i(\tilde{t}_i, t_i) &= \int_{\mathbf{t}_{-i}} v_i(\tilde{t}_i, t_{-i})\mathbf{f}_{-i}(\mathbf{t}_{-i})d\mathbf{t}_{-i} - \frac{\int_{\mathbf{t}_{-i}} e_i(\tilde{t}_i, \mathbf{t}_{-i})\mathbf{f}_{-i}(\mathbf{t}_{-i})d\mathbf{t}_{-i}}{t_i} \\ &= V_i(\tilde{t}_i) - \frac{\int_{\mathbf{t}_{-i}} e_i(\tilde{t}_i, \mathbf{t}_{-i})\mathbf{f}_{-i}(\mathbf{t}_{-i})d\mathbf{t}_{-i}}{t_i}. \end{aligned} \tag{2}$$

The contest designer maximizes the expected total effort from the contestants plus the effort equivalent of any money left in the prize budget. In the rest of the analysis, this designer is to maximize this total effort equivalent (or total effort for short), which is given by the following:

$$\max_{(\mathbf{v}(\cdot), \mathbf{e}(\cdot))} R = \int_{\mathbf{t}} \left[\sum_i e_i(\mathbf{t}) + t_0(V - \sum_i v_i(\mathbf{t})) \right] \mathbf{f}(\mathbf{t})d\mathbf{t} \tag{3}$$

subject to the following feasibility constraints:

$$u_i(t_i, t_i) \geq u_i(\tilde{t}_i, t_i), \forall \tilde{t}_i, t_i, \forall i, \tag{4}$$

is linear in effort and prizes, the divisibility of the prizes is not an issue; the probabilities of winning can be used to get around the divisibility problem and accommodate the stochastic mechanisms.

$$u_i(t_i, t_i) \geq 0, \forall t_i, \forall i, \tag{5}$$

$$\sum_i v_i(\mathbf{t}) \leq V, \forall \mathbf{t}, \tag{6}$$

$$e_i(\mathbf{t}) \geq 0, \forall \mathbf{t}, \forall i. \tag{7}$$

The feasibility constraints consist of four parts: (4) is the incentive compatibility constraint, (5) is the participation constraint, (6) is the designer's budget constraint, and (7) is the nonnegative effort constraint.

We are now ready to compare our optimal contest design problem with the optimal auction design problem in Myerson [17]. There are three similarities. First, the prize allocations here correspond to the object winning probabilities there. The sum of the winning probabilities in an auction must not exceed 1, while the sum of the prizes awarded in a contest must not exceed V. (Note that the total prize budget V can be normalized to 1.) Second, the effort here resembles the transfer payments there. Third, the contest designer's objective function here is equivalent to the auction designer's revenue there (with t_0 being the seller's reservation value).

Despite these similarities, there are two main differences, both lie in the restrictions on the choice variables. First, in an optimal auction design problem, the winning probabilities must be nonnegative. In our optimal contest design problem, the prizes for the contestants can be positive or negative, and this enlarges the set of feasible mechanisms. Negative prizes provide a venue for further enhancing the contest design by leveraging the differences in the contestants' virtual abilities. For a given ability profile $\mathbf{t}$, the negative prizes to lower ability contestants can be used to increase the positive prizes to higher ability contestants while still balancing the prize budget. This would improve the total effort exerted by the contestants even if the participation constraints for the lower types still need to be satisfied. Second, the monetary transfers in the optimal auction design problem can be positive or negative. However, in our optimal contest design problem, effort must be non-negative. This shrinks the set of feasible mechanisms and thus strictly reduces the amount of effort that can be induced.[8]

Putting these two conflicting effects together, it is not clear whether the optimal contest can do better than the optimal auction. Allowing for negative prizes is seemingly a small deviation from the conventional auction design literature. But it creates significant technical challenges in the analysis. By resolving these challenges, we obtain significantly different results and insights.

In the analysis for the optimal contest design, there is no obvious way to optimally leverage on the prizes assigned to different abilities of the contestants. In the rest of this section, we devote

[8]One can show that allowing negative prizes and negative effort in contest setting (or, equivalently, allowing negative winning probabilities and negative monetary transfers in auction setting) would lead to feasible mechanisms generating arbitrarily high expected total effort (or, equivalently, arbitrarily high expected revenue).

our analysis to solving this optimal contest design problem through the leveraging exercise.

Define $\tilde{u}_i(\tilde{t}_i, t_i) = t_i \cdot u_i(\tilde{t}_i, t_i)$. Then

$$\tilde{u}_i(\tilde{t}_i, t_i) = t_i V_i(\tilde{t}_i) - \int_{\mathbf{t}_{-i}} e_i(\tilde{t}_i, \mathbf{t}_{-i})\mathbf{f}_{-i}(\mathbf{t}_{-i})d\mathbf{t}_{-i}. \tag{8}$$

Constraints (4) and (5) can be rewritten in terms of $\tilde{u}_i(\cdot,\cdot)$. From (4) and the Envelope Theorem, we have

$$\frac{d\tilde{u}_i(t_i, t_i)}{dt_i} = \frac{\partial \tilde{u}_i(\tilde{t}_i, t_i)}{\partial t_i}|_{\tilde{t}_i = t_i} = V_i(t_i),$$

which leads to

$$\tilde{u}_i(t_i, t_i) - \tilde{u}_i(a, a) = \int_a^{t_i} V_i(s)ds.$$

Standard derivations such as those in Myerson [17] lead to the following lemma. The proof is omitted here.

Lemma 1 *Mechanism* $(\mathbf{v}(\cdot), \mathbf{e}(\cdot))$ *is feasible if and only if the following conditions hold together with (6) and (7):*

$$\int_{\mathbf{t}_{-i}} e_i(t_i, \mathbf{t}_{-i})\mathbf{f}_{-i}(\mathbf{t}_{-i})d\mathbf{t}_{-i} = t_i V_i(t_i) - \int_a^{t_i} V_i(s)ds - a \cdot u_i(a, a), \forall t_i, \forall i, \tag{9}$$

$$V_i(t_i') \geq V_i(t_i),\ \forall t_i' > t_i, \forall i, \tag{10}$$

$$u_i(a, a) \geq 0, \forall i. \tag{11}$$

Condition (9) is a direct implication of the incentive compatibility constraint. It implies that the contestants' expected effort levels $\mathbf{e}(\cdot)$ can be fully pinned down by the prize structure $\mathbf{v}(\cdot)$. In other words, two mechanisms with the same prize functions would generate the same total expected effort. Apparently, this result is parallel to the Revenue Equivalence Theorem in the auction design literature. Condition (10) requires that the expected prize must be increasing in a contestant's ability, and it is parallel to the increasing expected winning probability condition in the auction design. Condition (11) implies that a contestant with the lowest ability must be willing to participate, same as in the auction design. We are now ready to investigate the existence of an optimal contest mechanism.

3.2 Problem (P)

Note that in the optimal contest, $u_i(a, a) = 0$, i.e., the lowest ability contestant must earn zero informational rent. Otherwise, the contest designer can simply decrease the informational rent for every ability and yield a higher level of expected total effort. Given (1) and (9), we can replace effort $\mathbf{e}(\cdot)$ by the prize function $\mathbf{v}(\cdot)$ and rewrite the contest designer's objective function as

$$\max \int_{\mathbf{t}} \sum_i \left[J(t_i) - t_0\right] v_i(\mathbf{t})\mathbf{f}(\mathbf{t})d\mathbf{t} + t_0 V. \tag{12}$$

Therefore, the contest designer's optimization problem can be restated as maximizing (12), subject to (6), (7), (9) and (10). We denote this maximization problem as **problem (P)** and the resulting mechanism as the optimal mechanism.

A useful benchmark effort level is the **utmost total effort**, which is the highest amount of total effort inducible given budget V. This level of effort is achieved when all contestants are of the maximum ability b for sure, and it is equal to bV. We can see this from the following arguments. First, the total effort induced cannot be higher than bV from the participation constraints of the contestants. Second, the effort level bV can be obtained by asking each of the N contestants to exert effort $e = bV/N$ and awarding each of them a prize of V/N. In the rest of the analysis, we will mostly compare the total effort generated from a mechanism with this (upper bound) utmost total effort.

We will approach problem (P) using the following method. We first relax the monotonicity condition in expected prize (condition (10)) and establish an upper bound for the expected total effort level in this relaxed problem (denoted as P-Relax). Second, we show that there exists no optimal mechanism in this relaxed problem. Third, we construct a mechanism that is feasible for both the relaxed problem (P-Relax) and the original problem (P), and show that the expected total effort level in this mechanism can approach arbitrarily close to the upper bound (i.e., the utmost total effort). Since the upper bound can only be approached (but not reached) in both problems, these results imply that there exists no optimal mechanism in the original problem (P).

3.3 Problem (P-Relax)

We start by considering the following optimization problem, denoted as (P-Relax):

$$\max \int_{\mathbf{t}} \sum_i \left[J(t_i) - t_0\right] v_i(\mathbf{t})\mathbf{f}(\mathbf{t})d\mathbf{t} + t_0 V \tag{13}$$

subject to

$$\sum_i v_i(\mathbf{t}) \le V, \ \forall \mathbf{t}, \tag{14}$$

$$V_i(t_i) = \int_{\mathbf{t}_{-i}} v_i(t_i, \mathbf{t}_{-i})\mathbf{f}_{-i}(\mathbf{t}_{-i})d\mathbf{t}_{-i} \ge 0, \ \forall t_i, \forall i. \tag{15}$$

This is a relaxed problem of (P): the objective function is the same but the feasibility constraints are less restrictive than the original ones. To see this, first note that constraint (14) is the same as (6). Second, constraint (15) is implied by the feasibility constraints in (P). This is because from the monotonicity condition (10), it suffices to show that $V_i(a) \geq 0, \forall i$. From (9), evaluating at $v_i = a$, we obtain $aV_i(a) = \int_{\mathbf{t}_{-i}} e_i(a, \mathbf{t}_{-i})\mathbf{f}_{-i}(\mathbf{t}_{-i})d\mathbf{t}_{-i}$, which is non-negative from the non-negative effort constraint (7).

The relaxed problem (P-Relax) is a continuous linear programming problem. We construct the Lagrangian by applying multiplier $\lambda(\mathbf{t})$ to constraint (14) and $\mu_i(t_i)$ to constraint (15) before integrating them and adding them to the objective function:

$$\begin{aligned} L &= \int_{\mathbf{t}} \sum_i \left[J(t_i) - t_0\right] v_i(\mathbf{t})\mathbf{f}(\mathbf{t})d\mathbf{t} + t_0 V + \int_{\mathbf{t}} \lambda(\mathbf{t}) \left[V - \sum_i v_i(\mathbf{t})\right] \mathbf{f}(\mathbf{t})d\mathbf{t} \\ &\quad + \sum_i \int_{t_i} \mu_i(t_i) \left(\int_{\mathbf{t}_{-i}} v_i(t_i, \mathbf{t}_{-i})\mathbf{f}_{-i}(\mathbf{t}_{-i})d\mathbf{t}_{-i} \right) f(t_i)dt_i. \end{aligned} \tag{16}$$

Suppose that an optimal solution exists. Then it must satisfy the following Kuhn-Tucker conditions:

$$[J(t_i) - t_0] - \lambda(\mathbf{t}) + \mu_i(t_i) = 0, \forall \mathbf{t}, \forall i, \tag{17}$$

$$\lambda(\mathbf{t}) \geq 0,\ V - \sum_i v_i(\mathbf{t}) \geq 0,\ \text{and } \lambda(\mathbf{t}) \left[V - \sum_i v_i(\mathbf{t})\right] = 0, \forall \mathbf{t}, \tag{18}$$

$$\mu_i(t_i) \geq 0,\ \int_{\mathbf{t}_{-i}} v_i(t_i, \mathbf{t}_{-i})\mathbf{f}_{-i}(\mathbf{t}_{-i})d\mathbf{t}_{-i} \geq 0,\ \text{and } \mu_i(t_i) \int_{\mathbf{t}_{-i}} v_i(t_i, \mathbf{t}_{-i})\mathbf{f}_{-i}(\mathbf{t}_{-i})d\mathbf{t}_{-i} = 0, \forall t_i, \forall i. \tag{19}$$

We show that there is no $v_i(\mathbf{t})$ which can satisfy all of these conditions. We have the following lemma.

Lemma 2 *An optimal solution does not exist for problem (P-Relax).*

Although the above lemma shows that an optimal contest mechanism does not exist for the relaxed problem (P-Relax), we can nevertheless establish that the utmost total effort bV is an upper bound for the total effort in all feasible mechanisms in problem (P-Relax). In problem (P-Relax), the abilities of the contestants are usually less than b, and the total effort elicited in any feasible mechanism must be less than bV. We have the following lemma.

Lemma 3 *The expected total effort elicited in problem (P-Relax) is strictly less than bV.*

The above lemma is very intuitive. The total effort level of bV should be achievable only when all (or at least one) contestants have the maximum ability b with certainty. In our setting, every contestant's ability is strictly lower than b almost for sure. It is thus well expected that the expected effort level of bV is an unreachable upper bound for problem (P-Relax).

In the following analysis, we will show, however, that for problem (P-Relax), there exists a sequence of mechanisms inducing total effort levels approaching bV arbitrarily closely, even though an optimal mechanism does not exist. Therefore, this sequence of mechanisms is asymptotically optimal.

The sequence of mechanisms that we will construct is closely related to K, a parameter which we later on define as the maximum negative prize that the contest designer can impose on a contestant. It can also be interpreted as the entry fee in a contest.

For any $K \geq 0$, define $\hat{t}(K) = F^{-1}((\frac{NK}{V+NK})^{\frac{1}{N-1}})$, $t^*(K) = \max\{J^{-1}(t_0), \hat{t}(K)\}$ and $\Lambda(K) = \frac{K}{F^{N-1}(t^*(K))} - K$. Note that $\Lambda(K) \in (0, \frac{V}{N}]$. Let $S^*(K) = \{j : t_j > t^*(K)\}$, i.e., the set of contestants with abilities higher than the cutoff, and let $t^{(1)}$ denote the first order statistics of $\mathbf{t}$. We define the following sequence of mechanisms $(\mathbf{v}^*(\cdot; K), \mathbf{e}^*(\cdot; K))$ associated with K.

Definition 1 (K-mechanism) *The prize allocation function is given by*

$$v_i^*(\mathbf{t}; K) = \begin{cases} \Lambda(K), & \text{if } S^*(K) = \emptyset, \\ -K, & \text{if } S^*(K) \neq \emptyset \text{ and } t_i < t^{(1)}, \\ V + (N-1)K, & \text{if } S^*(K) \neq \emptyset \text{ and } t_i = t^{(1)}. \end{cases} \tag{20}$$

The effort function is given by

$$e_i^*(\mathbf{t}; K) = \varepsilon(t_i; K) \equiv \begin{cases} 0, & \text{if } t_i \leq t^*(K), \\ t_i V_i^*(t_i; K) - \int_{t^*(K)}^{t_i} V_i^*(s; K) ds, & \text{if } t_i > t^*(K), \end{cases} \tag{21}$$

where $V_i^*(t_i; K) = \int_{\mathbf{t}_{-i}} v_i^*(\mathbf{t}; K) \mathbf{f}_{-i}(\mathbf{t}_{-i}) d\mathbf{t}_{-i}$ is the expected prize function.

Note that the above effort function $e_i^*(\mathbf{t}; K)$ depends on contestant i's type t_i only, and it is symmetric among all contestants. (So we denote it as $\varepsilon(t_i; K)$.) Furthermore, it is strictly increasing in t_i for $t_i > t^*(K)$. We have the following lemma.

Lemma 4 *Each K-mechanism $(\mathbf{v}^*(\cdot; K), \mathbf{e}^*(\cdot; K))$ is feasible in both problem (P-Relax) and the original problem (P). The expected total effort is given by*

$$R(K) = N \int_{t^*(K)}^{b} [J(t_i) - t_0] [(V + NK) F^{N-1}(t_i) - K] dF(t_i) + t_0 V.$$

When K goes to $+\infty$, the expected total effort elicited by the mechanism goes to bV and the expected *payoffs of the contestants go to zero. In the limit, the contestant with the maximum ability b enjoys a positive but finite informational rent $\frac{V}{Nbf(b)}$; all other types enjoy zero information rent.*

This lemma shows that even when the abilities of the contestants are distributed randomly (and therefore their abilities are most likely to be less than the maximum b), the contest designer can still elicit almost bV, the total effort elicitable when all contestants are of ability b, through the above mechanisms. In each of these mechanisms, there is a cut-off ability. If none of the contestants has an ability higher than this cut-off, then every contestant gets a prize $\Lambda(K)$, which is no bigger than V/N; equivalently, each player wins a prize of $N\Lambda(K)$ with probability $1/N$. But if at least one of the contestants has an ability higher than the cut-off, then every contestant (except the highest ability contestant) will be punished by a negative prize $-K$. This highest ability contestant gets the prize budget V plus the extra money generated from the negative prizes from other contestants. The interim incentive to participate for the lower ability contestants is maintained by the positive prize $\Lambda(K)$ when no contestant has an ability above the cut-off. In the equilibrium of this mechanism, those contestants with abilities below the cut-off exert zero effort. Meanwhile, contestants with abilities higher than the cut-off exert a large amount of effort. The level of total expected effort converges to the utmost total effort.

These mechanisms are contest mechanisms as they rely on the relative ranking of efforts of the contestants. Not all contest mechanisms allowed in our analysis resemble real-life contests, similarly to the analysis of auction mechanisms considered in Myerson [17]. As we shall see in Proposition 3, our optimal contest mechanism can be implemented by an all-pay auction, which is an extreme form of contest.

At this point, it is not obvious how we came up with the above mechanisms. In the next section, we shall show that $(\mathbf{v}^*(\cdot; K), \mathbf{e}^*(\cdot; K))$ is in fact an optimal contest mechanism if K is the maximum negative prize we can impose on a contestant.

The nonnegativity of $V_i(\cdot)$ turns out to be a *binding* constraint for the upper bound effort level in Lemma 3. The discussion following Proposition 1 will reveal that allowing $V_i(\cdot)$ to be negative (implying that effort is sometimes negative) could lead to a total effort level infinitely larger than bV.

Lemma 4 implies that for original contest design problem (P), the contest designer can obtain an expected total effort level arbitrarily close to the utmost total effort bV. It also implies that there is no optimal solution for problem (P). This can be seen from the following reasoning. Since problem (P-Relax) is a relaxed problem of problem (P), the upper bound effort level bV in Lemma 3 must apply to problem (P) as well. From Lemma 4, this upper bound is a supremum for both problems. This supremum is not reachable in problem (P-Relax), and therefore it is not reachable in problem (P). Hence, problem (P) has no solution. These results are summarized in the following proposition.

Proposition 1 *There exists no optimal contest mechanism in problem (P). Nevertheless, the contest designer can obtain an expected total effort level that is arbitrarily close to the utmost total effort bV; meanwhile, the expected surplus of a contestant is arbitrarily close to zero.*

The intuitions behind these results can be illustrated as follows. The higher ability contestants are willing to expend more effort for a given prize than the lower ones because of lower cost of effort. Therefore, there is some scope for moving the effort from the lower ability contestants to the higher ones, as this will save money for the contest designer.

To accomplish this maneuver, first suppose that effort can be negative. Then a lower ability contestant is willing to accept a negative prize for exerting a negative effort to maintain his individual rationality constraint. The contest designer can use the additional budget as a result of this negative prize to incentivize the higher ability contestant to exert more effort. Due to the ability difference, the increase in the effort of the higher ability contestant must dominate the decrease in the effort of the lower ability contestant. Therefore, the total effort must increase. As we let the prizes of the lower ability contestants go to negative infinity, the additional budget created for incentivizing the higher ability ones goes to positive infinity. By doing this, an infinite amount of total effort can be generated. Now effort must be non-negative. This limits the amount of negative prizes that can be imposed on the lower ability contestants to satisfy their individual rationality constraints. This in turn limits the amount of additional budget that can be generated to incentivize the high ability ones. Therefore, the amount of effort that can be moved from the lower ability contestants to the higher ability ones becomes restricted. We obtain a tight upper bound bV for the total effort in Lemma 3.

In those K-mechanisms (Definition 1), the individual rationality constraint of a lower ability contestant can be maintained by awarding him with a positive prize most of the time but punishing him with a large negative prize with a small probability. The punishment can increase unboundedly as the probability of punishment shrinks to zero. There is a cut-off in the support of a contestant's ability. A contestant with an ability below the cut-off pays an entry fee K. He exerts no effort, and when all contestants have abilities lower than this cut-off, he wins a positive prize $K + \Lambda(K)$. A contestant with an ability higher than the cut-off exerts positive effort. He also pays K, but he wins $V + NK$ if he has the highest ability among all contestants. A higher K strengthens the incentive for a higher ability contestant to exert more effort. But if a lower ability contestant is asked to pay a higher fee, the probability of the event that he ends up with purely paying the fee must be decreased, implying that the cut-off in ability must move higher. We show that the contest designer always benefits from such trade off and can induce more effort with a larger K. When K goes to infinity, the utmost level of effort bV is asymptotically achievable.

Note that this result is related to the literature on full surplus extraction in auctions. Crémer and McLean [4] and McAfee and Reny [11] establish that the full surplus from the bidders can be extracted when their types are correlated. (Heifetz and Neeman [7] and Chen and Xiong [3] further examine the generality and robustness of the full surplus extraction result.) However, when the types of the bidders are independently distributed and when the interim individual rationality constraints must be satisfied, it is believed that full surplus extraction cannot be achieved. In this paper, we show that if negative prizes are allowed, then the upper bound total effort can almost be achieved in the contest environment when the types of the contestants are distributed independently and when the interim individual rationality constraints are satisfied. This upper bound total effort is, in some sense, more than the full surplus extraction in auctions. It is the total effort achievable when all (or at least one) contestants have the maximum ability b with certainty. The full surplus in auctions is much lower than what the seller can receive when all bidders have the maximum valuation with certainty. From this, we can see that an optimal contest problem can be very different from an optimal auction problem.[9]

4 Optimal contest design with bounded negative prize

The previous section establishes a surprising finding that the utmost total effort bV can be achieved asymptotically when we use increasing negative prizes. It is not difficult to imagine that large negative prizes are not practical. Contestants do not have an infinite amount of wealth to pay for the negative prizes. Furthermore, large negative prizes may not be lawful. In this section, we investigate the optimal contest design problem when there is a bound on the negative prizes. Following the mechanism design literature, we assume that a contestant's ability distribution satisfies the following regularity condition to simplify the characterization of the optimal contest.[10]

Assumption 1. The virtual ability function $J(t) = t - \frac{1-F(t)}{f(t)}$ is strictly increasing in $t \in [a, b]$.

In the auction literature, a bidder's virtual valuation can be interpreted as the marginal revenue that can be elicited from a bidder (cf. Bulow and Roberts [1]). In our contest setting, a contestant's virtual ability can be similarly interpreted as the marginal effort that can be elicited from a contestant using one unit of prize.

In the following subsections, we will first set up the optimal contest design problem, and then

[9]While we obtain the utmost total effort under the condition of independent private abilities, we still assume risk neutrality, unlimited liability, no collusion among the contestants, and no competing designers. Therefore, the critics on full rent extraction by Robert [20], Laffont and Martimort [9], Che and Kim [2], and Peters [18] still apply.

[10]See Myerson [17]. Note that the results in the previous section does not require this assumption. This regularity condition will be discussed further at the end of this section.

carry out the analysis through a few steps.

4.1 Problem (P-K)

Let K be the bound on the negative prizes, where $K > 0$.[11] We add this restriction to the original Problem (P),

$$v_i(\mathbf{t}) \geq -K, \forall \mathbf{t}, \forall i, \tag{22}$$

and we call this the bounded negative prize optimization problem (P-K). We will focus our analysis on this problem in this section. Note that when $K = \infty$, the analysis in the previous section applies.

We adopt a multi-step procedure to solve optimization problem (P-K). Here is our road map for solving this problem. First, we consider a relaxed problem of problem (P-K), denoted by problem (P-K-Relax), and establish some necessary conditions for the optimization. Second, we add these necessary conditions to the constraints of problem (P-K-Relax) and obtain an equivalent problem of the relaxed problem, denoted by problem (P-K-Relax-Equivalent). Note that the optimal solutions of problems (P-K-Relax) and (P-K-Relax-Equivalent) are the same. Third, we further relax problem (P-K-Relax-Equivalent) and examine problem (P-K-Relax-Equivalent-Relax). Fourth, we fully characterize the solution to problem (P-K-Relax-Equivalent-Relax). Finally, we construct a feasible mechanism of the original problem (P-K) that achieves the maximal effort level of problem (P-K-Relax-Equivalent-Relax). This feasible mechanism becomes an optimal mechanism.

In problem (P-K), any feasible $V_i(t_i)$ is nonnegative and increasing.[12] Thus, we define $\hat{t}_i = \sup\{t_i | V_i(t_i) = 0\}$. Without loss of generality, assume that $V_i(t_i)$ is left-continuous at $\hat{t}_i$. Therefore, $V_i(t_i) = 0$ for $t_i \leq \hat{t}_i$ and $V_i(t_i) > 0$ for $t_i > \hat{t}_i$.

4.2 Problem (P-K-Relax)

We start our analysis by considering the following relaxed optimization problem (P-K-Relax) of problem (P-K):

$$\max_{\{v_i(\mathbf{t}), \hat{t}_i, \forall i\}} \int_{\mathbf{t}} \sum_i \left[J(t_i) - t_0\right] v_i(\mathbf{t}) \mathbf{f}(\mathbf{t}) d\mathbf{t} + t_0 V \tag{23}$$

subject to

$$\int_{\mathbf{t}} \sum_i v_i(\mathbf{t}) \mathbf{f}(\mathbf{t}) d\mathbf{t} \quad \leq \quad V, \tag{24}$$

[11] The case for $K = 0$ is trivial. It is equivalent to optimal auction design and Myerson's result applies. We obtain this degenerate case when we let K go to zero in our analysis.

[12] The nonnegativity follows from (15).

$$V_i(t_i) = \int_{\mathbf{t}_{-i}} v_i(t_i, \mathbf{t}_{-i})\mathbf{f}_{-i}(\mathbf{t}_{-i})d\mathbf{t}_{-i} = 0,\ \forall t_i \le \hat{t}_i, \forall i, \tag{25}$$

$$V_i(t_i) = \int_{\mathbf{t}_{-i}} v_i(t_i, \mathbf{t}_{-i})\mathbf{f}_{-i}(\mathbf{t}_{-i})d\mathbf{t}_{-i} > 0,\ \forall t_i > \hat{t}_i, \forall i, \tag{26}$$

$$v_i(\mathbf{t}) \ge -K,\ \forall \mathbf{t}, \forall i, \tag{27}$$

$$a \le \hat{t}_i \le b, \forall i. \tag{28}$$

This is a relaxed problem of problem (P-K). This is because the objective functions are the same in both problems, and the constraints are less restrictive than those in problem (P-K). To see this, constraint (24) follows from (6) by integrating over $\mathbf{t}$. Constraints (25) and (26) directly follow the definition of $\hat{t}_i$. Constraint (27) is the same as (22) in (P-K). Constraint (28) allows for all possible threshold values of $\hat{t}_i$.

We next characterize a key property for the optimal solutions $\{v_i^{\circledast}(\mathbf{t}), \hat{t}_i^{\circledast}, \forall i\}$ of problem (P-K-Relax). Consider problem (P-K-Relax) for a fixed $\hat{t}_i = \hat{t}_i^{\circledast}$. We construct the Lagrangian by introducing multipliers λ for constraint (24), $\mu_i(t_i)$ for constraints (25) and (26), and $\xi_i(\mathbf{t})$ for constraint (27):

$$\begin{aligned} L = & \int_{\mathbf{t}} \sum_i [J(t_i) - t_0]\, v_i(\mathbf{t})\mathbf{f}(\mathbf{t})d\mathbf{t} + t_0 V + \lambda \int_{\mathbf{t}} \left[V - \sum_i v_i(\mathbf{t})\right] \mathbf{f}(\mathbf{t})d\mathbf{t} \\ & + \sum_i \int_{t_i} \mu_i(t_i) \left(\int_{\mathbf{t}_{-i}} v_i(t_i, \mathbf{t}_{-i})\mathbf{f}_{-i}(\mathbf{t}_{-i})d\mathbf{t}_{-i} \right) f(t_i)dt_i \\ & + \sum_i \int_{\mathbf{t}} \xi_i(\mathbf{t})\, [v_i(\mathbf{t}) + K]\, \mathbf{f}(\mathbf{t})d\mathbf{t}. \end{aligned}$$

The Kuhn-Tucker conditions for the optimization are:

$$\varsigma_i(\mathbf{t}) = [J(t_i) - t_0] - \lambda + \mu_i(t_i) + \xi_i(\mathbf{t}) = 0, \forall \mathbf{t}, \forall i,$$

$$\lambda \ge 0,\ V - \int_{\mathbf{t}} \sum_i v_i(\mathbf{t})\mathbf{f}(\mathbf{t})d\mathbf{t} \ge 0,\ \text{and}\ \lambda \left[V - \int_{\mathbf{t}} \sum_i v_i(\mathbf{t})\mathbf{f}(\mathbf{t})d\mathbf{t}\right] = 0, \forall \mathbf{t},$$

$$\mu_i(t_i) \ge 0,\ \int_{\mathbf{t}_{-i}} v_i(\mathbf{t})\mathbf{f}_{-i}(\mathbf{t}_{-i})d\mathbf{t}_{-i} \ge 0,\ \text{and}\ \mu_i(t_i) \int_{\mathbf{t}_{-i}} v_i(\mathbf{t})\mathbf{f}_{-i}(\mathbf{t}_{-i})d\mathbf{t}_{-i} = 0, \forall t_i, \forall i,$$

$$\xi_i(\mathbf{t}) \ge 0,\ v_i(\mathbf{t}) + K \ge 0,\ \text{and}\ \xi_i(\mathbf{t})\, [v_i(\mathbf{t}) + K] = 0, \forall \mathbf{t}, \forall i.$$

These Kuhn-Tucker conditions lead to the following important necessary conditions for the optimal solutions $\{v_i^{\circledast}(\mathbf{t}), \hat{t}_i^{\circledast}, \forall i\}$ for problem (P-K-Relax).

Lemma 5 *(i)* $\hat{t}_i^{\circledast} \geq F^{-1}((\frac{K}{V+NK})^{\frac{1}{N-1}})$;

(ii) For $t_i > \hat{t}_i^{\circledast}$*, we must have* $0 < V_i^{\circledast}(t_i) \leq (V+NK)F^{N-1}(t_i) - K$.

4.3 Problem (P-K-Relax-Equivalent)

Lemma 5 provides a set of necessary conditions for the optimal solution of (P-K-Relax). If we add these necessary conditions to the constraints in (P-K-Relax), we obtain a revised optimization problem (P-K-Relax-Equivalent). The solutions to these two problems are the same. This is because the optimal solution of (P-K-Relax) must satisfy all of the constraints (the original feasibility constraints and the additional necessary conditions) in problem (P-K-Relax-Equivalent). Thus the solution to problem (P-K-Relax-Equivalent) cannot be worse than problem (P-K-Relax). Meanwhile, problem (P-K-Relax-Equivalent) is more restrictive and therefore its solution cannot be better than problem (P-K-Relax).

The equivalent problem (P-K-Relax-Equivalent) can be rewritten as follows:

$$\max_{\{v_i(\mathbf{t}),\, \hat{t}_i, \forall i\}} \sum_{i=1}^{N} \int_a^b [J(t_i) - t_0]\, V_i(t_i) f(t_i) dt_i + t_0 V \tag{29}$$

subject to

$$\sum_{i=1}^{N} \int_a^b V_i(t_i) f(t_i) dt_i \leq V, \tag{30}$$

$$V_i(t_i) = 0, \text{ if } t_i \leq \hat{t}_i, \forall i, \tag{31}$$

$$v_i(\mathbf{t}) \geq -K, \ \forall \mathbf{t}, \forall i, \tag{32}$$

$$F^{-1}((\frac{K}{V+NK})^{\frac{1}{N-1}}) \leq \hat{t}_i \leq b, \forall i, \tag{33}$$

$$0 < V_i(t_i) \leq (V+NK)F^{N-1}(t_i) - K, \text{ if } t_i > \hat{t}_i\,, \forall i. \tag{34}$$

Note that (30) simply rewrites (24).

4.4 Problem (P-K-Relax-Equivalent-Relax)

Problem (P-K-Relax-Equivalent) can be relaxed to problem (P-K-Relax-Equivalent-Relax) by dropping constraint (32). As a result, the entire optimization problem now depends only on the expected prizes $V_i(\cdot)$ instead of the detailed prize structure $v_i(\cdot)$:

$$\max_{\{V_i(\cdot),\,\hat{t}_i,\forall i\}} \sum_{i=1}^{N} \int_a^b [J(t_i)-t_0]\, V_i(t_i) f(t_i) dt_i + t_0 V \tag{35}$$

subject to

$$\sum_{i=1}^{N} \int_a^b V_i(t_i) f(t_i) dt_i \leq V, \tag{36}$$

$$0 < V_i(t_i) \leq (V+NK)F^{N-1}(t_i) - K, \text{ if } t_i > \hat{t}_i\,, \forall i, \tag{37}$$

$$V_i(t_i) = 0, \text{ if } t_i \leq \hat{t}_i, \forall i, \tag{38}$$

$$F^{-1}((\frac{K}{V+NK})^{\frac{1}{N-1}}) \leq \hat{t}_i \leq b, \forall i. \tag{39}$$

Note that in problem (P-K-Relax-Equivalent-Relax), the choice variables are merely $\{V_i(\cdot), \hat{t}_i, \forall i\}$. Recall that $t^*(K) = \max\{J^{-1}(t_0), \hat{t}(K)\}$ where $\hat{t}(K) = F^{-1}((\frac{NK}{V+NK})^{\frac{1}{N-1}})$ are defined in Subsection 3.3. We are now ready to present the following lemma, which is the key to the analysis of the optimal contest mechanism with bounded negative prize.

Lemma 6 *$\forall i$, let* $\bar{V}_i(t_i) = \begin{cases} (V+NK)F^{N-1}(t_i) - K, & \text{if } t_i > t^*(K), \\ 0, & \text{if } t_i \leq t^*(K). \end{cases}$ *Then $\{\bar{V}_i(t_i), \forall i\}$ is an optimal solution to problem (P-K-Relax-Equivalent-Relax).*

Note that (P-K-Relax-Equivalent-Relax) is a relaxed problem of (P-K). Suppose that we find a $\bar{v}_i(\mathbf{t})$ such that it generates $\bar{V}_i(t_i)$ and satisfies the constraints in the original problem (P-K). Then $\{v_i^*(\mathbf{t}), i = 1, 2, ..., N\}$ and the effort functions $\{e_i^*(\mathbf{t}), i = 1, 2, ..., N\}$ that support $\bar{v}_i(\mathbf{t})$ and satisfy $u_i(a,a) = 0, \forall i$ would constitute an optimal solution to problem (P-K). The supporting effort functions $\{e_i^*(\mathbf{t}), i = 1, 2, ..., N\}$ can be constructed based on (9) with $u_i(a,a) = 0$.

The mechanism described in Definition 1 and Lemma 4 in Subsection 3.3, $(\mathbf{v}^*(\cdot;K), \mathbf{e}^*(\cdot;K))$, meets the above requirements. The following proposition illustrates that this mechanism is in fact an optimal contest mechanism.

Proposition 2 *Suppose that Assumption 1 holds. Then the mechanism $(\mathbf{v}^*(\cdot;K), \mathbf{e}^*(\cdot;K))$ of Definition 1 is an optimal contest mechanism for problem (P-K).*

In this optimal contest mechanism, the contest designer gives the highest reward to the highest ability contestant. If negative prizes are not allowed, the optimal prize structure is to allocate the entire prize to the highest ability contestant provided that his ability is higher than $J^{-1}(t_0)$. When negative prizes are allowed, the contest designer tries to make prize transfers across the contestants.

The marginal benefit (in terms of effort generated) of giving one extra dollar to the contestant with the highest ability is higher than the marginal cost (in terms of effort lost) of charging one dollar from the lower ability contestants (i.e., negative prize), since the marginal cost of effort is lower for a contestant with a higher ability. Therefore, the highest ability contestant exerts more effort and the lower ability contestants exert less. Since these marginal benefit and marginal cost are both constant, the contest designer is willing to perform this transfer as long as it is feasible, until the negative prize hits its bound K. Meanwhile, the contest designer is constrained by the participation constraints of the lower ability contestants and needs to compensate them for the negative prizes charged. The optimal way to achieve this balancing is to reward the contestants when they are all of low abilities, i.e., when their abilities are all lower than the cut-off $t^*(K)$. When K becomes larger, the low ability contestants need to be rewarded with positive prizes more often to satisfy their participation constraints, and therefore $t^*(K)$ becomes larger. When K goes to infinity, the contestants who exert positive effort are only those with abilities converging to the upper bound b. In this way, the utmost total effort can be achieved. In this case, even though the contestant with the highest ability which is also higher than the cutoff $t^*(K)$ earns a positive informational rent (Lemma 4), it is more and more difficult to be qualified, and thus the ex ante expected total surplus that each contestant will earn converges to zero.

There are a few distinctive features associated with the prize allocation rule in this optimal contest mechanism. First, a maximal negative prize is imposed on all contestants except the one with the highest ability, unless all of them have abilities lower than the cut-off.

Second, when all of the contestants have abilities lower than the cut-off, they are all treated equally, regardless of their ability rankings. In this case, they all obtain an equal positive prize (or equivalently, they have equal probability of winning one indivisible prize). These two features together ensure that the maximum incentive is granted to the highest ability contestant to exert effort while the lower ability contestants are still willing to participate (and provide the necessary cross-ability subsidies to the highest ability contestant).

Third, similarly to the cut-off type in Myerson [17], a cut-off value $t^*(K)$ is determined in our analysis, but in a somewhat different way. The cut-off value in our optimal contest mechanism is always weakly higher than the one in Myerson [17], and strictly higher if the maximum negative prize is sufficiently high. In particular, our cut-off value would approach the upper limit of the support of the ability distribution when the bound for negative prize becomes higher and higher.

Fourth, when the bound of negative prize is low and the negative prizes are not allowed to be large, our cut-off value coincides with the one in Myerson [17]. In this case, only a portion of the original prize budget is awarded to the contestants when every contestant's ability is below the threshold. This is necessary in the maintenance of the incentive constraints for those contestants with abilities above the cut-off. This partial award scheme in our optimal contest design resembles the no-sale situation in an optimal auction when all bidders' virtual values are below the seller's valuation, even when some of their true valuations are higher than the seller's valuation.

The following corollary specifies the exact condition for the prize budget V to be always completely awarded to the contestants in the optimal contest.

Corollary 1 *The prize budget V is always completely awarded to the contestants of every type profile when and only when $K > 0$ and $J(\hat{t}(K)) \geq t_0$.*

When $K = 0$, or when $K > 0$ but $J(\hat{t}(K)) < t_0$, the budget constraint is not binding in the case where every t_i is lower than $t^*(K)$. Note that this result may hold even when the contest designer does not derive any benefit from any unspent prize budget, i.e. when $t_0 = 0$.

The optimal contest design discussed above can resemble some more familiar form of contests. It can be implemented by a modified all-pay auction with entry fees and minimum bids of effort. Define $\underline{e}(K) = \lim_{t_i \to t^*_+} \varepsilon(t_i; K)$, where $\varepsilon(t_i; K)$ is given by (21) and is the expected effort of type t_i in the optimal mechanism. We have the following proposition.

Proposition 3 *The optimal contest mechanism $(\mathbf{v}^*(\cdot;K), \mathbf{e}^*(\cdot;K))$ can be implemented by a modified all-pay auction with an entry fee K and a minimum bid $\underline{e}(K)$. Every participant pays the entry fee no matter he bids or not. The highest bidder wins V plus the entry fees collected from all of the participants. When no one bids, a participant is randomly selected as the winner of the prize $N\Lambda(K)$ $(\leq V)$ plus the collected entry fees. Every contestant of every type participates in this all-pay auction.*

Proof: It is straightforward to verify that every type of contestants will pay K and participate. For abilities above $t^*(K)$, the optimal bid (in terms of effort) is given by the above expected effort function $\varepsilon(\cdot; K)$. Contestants with abilities below $t^*(K)$ participate but do not bid. $\square$

Entry fees are used in both auctions and contests. There is one difference, however, that sets them apart. In auctions, entry fees are often adopted to screen bidders and contribute directly to the seller's revenue. In contests, however, entry fees are often collected and added to the prize budget, boosting contestants' effort indirectly through the enhanced incentive schemes.[13]

It is worthwhile to emphasize that the optimal contest in Proposition 3 does not require the prize V to be divisible. If the prize is indivisible, winning a proportion of a grand prize for sure is equivalent to winning the whole grand prize with a smaller probability.

[13] Entry fees in auctions have been analyzed by Fullerton and McAfee [5] as an instrument of shortlisting contestants in an R&D tournament. In their paper, the entry fees collected are used by the sponsor to offset the cost of the tournament winner's prize. Recently, Ghosh and McAfee [6] find that free entry is dominated by taxing entry (which is a form of entry fees) in crowdsourcing tournaments, a conclusion echoed by our paper, though in a different environment.

An optimal mechanism for a smaller K must also be feasible when K becomes larger. We can therefore conclude that the expected total effort elicited from problem (P-K) must be increasing in K. When $K = 0$, no negative prizes are allowed in the contest. In this case, the constraints in the contest design problem (P-K) are more restrictive than those in the optimal auction design, as negative effort is not allowed in contests but negative payments are allowed in auctions. However, the contest designer can do equally well in this case simply because the Myerson optimal auction does not involve negative monetary payments. The above results are formalized in the following corollary.

Corollary 2 *(i) The expected total effort elicited in the optimal contest mechanism of problem (P-K) increases in K.*

(ii) When $K = 0$, the optimal contest mechanism resembles the Myerson optimal auction. In particular, when $J(a) \geq t_0$, the optimal contest mechanism can be implemented by a standard all-pay auction with a single prize V for the winner.

Note that the all-pay auction in *(ii)* of the above corollary is the same as the optimal mechanism obtained by Moldovanu and Sela [13].

As a final remark, we provide an illustration for why the monotonicity of the virtual ability function is required for the case of bounded K but not for the case of unbounded K. For the case of bounded K, the optimal mechanism of $(\mathbf{v}^*(\cdot; K), \mathbf{e}^*(\cdot; K))$ in Definition 1 rewards the highest ability contestant with the entire (enhanced) budget if his ability is above the cut-off $t^*(K)$. The optimality of the mechanism thus requires the virtual ability function to be monotone increasing in a contestant's ability. If this monotonicity requirement is dropped, we expect that the ironing technique in Myerson [17] can be utilized in characterizing the optimal contest mechanism.

For the case of unbounded K, the optimal contest mechanism does not exist, and we show that mechanism $(\mathbf{v}^*(\cdot; K), \mathbf{e}^*(\cdot; K))$ can achieve a total effort that is arbitrarily close the the utmost effort bV with no monotonicity requirement. This is because the virtual ability function $J(\cdot)$ is continuous and always reaches its maximum at the maximum ability b. As K goes to infinity, the cut-off type t^* goes to b. This guarantees the winner's virtual ability asymptotically approaching the highest virtual ability even without monotonicity in the virtual ability function.

5 Concluding remarks

In this paper, we examine the design of optimal contests in an environment with multiple contestants under private ability information. In the case of bounded negative prizes, we completely characterize the optimal contest mechanism under a regularity condition on the virtual ability function. In the case of unbounded negative prizes, the optimality can only be approached but not reached, and therefore an optimal contest does not exist. We construct a sequence of "almost" optimal contests, indexed by the magnitude of the allowable negative prizes.

We adopt a mechanism design approach to accommodate all possible prize allocation rules. Our analysis allows for both positive and negative prizes within a fixed prize budget. We find that the utmost total effort can be achieved in the limit when the size of negative prizes becomes larger and larger. In the limit, the utmost total effort is induced and all surpluses from the contestants are extracted. It is noteworthy that this (almost) full extraction result and the (almost) obtainability of the utmost effort are derived in an environment of independent private information, in contrast to related results in the auction literature which requires correlated private information.

Compared to an optimal auction mechanism, one distinct feature of the optimal contest mechanism is the cross-type transfers as the prizes assigned to contestants of different virtual abilities are being leveraged to achieve the efficiency of the contest. The assumption of linear effort cost function plays an important role in these transfers. This assumption is often adopted for tractability in the literature, e.g., Moldovanu and Sela [14], Polishchuk and Tonis [19] and Kirkegaard [8], among many others. If effort cost function is convex as in Minor [12], then such transfers will become more and more costly, and only a level lower than the utmost effort can be achieved.

To some extent, the negative prizes in our model have to be in the form of advanced payments, such as entry fees. If no advanced payment is allowed, then the appropriate individual rationality constraints should be the ex post ones. This would render negative prizes infeasible. In that case, every contestant of every type must earn a nonnegative payoff in every profile, implying that the prizes cannot be negative. Therefore, the optimal contest becomes Myerson's optimal auction.

In the analysis, we focus on deriving the optimal mechanism that maximizes the total expected effort from all players. Alternative objective functions for the contest designer can be examined. For example, the contest designer may value only the highest effort among the contestants, such as in innovation contests. In our optimal mechanism, only the highest ability player may exert a positive effort. It follows immediately that our mechanism is also the optimal mechanism when the contest designer values only the highest effort from the contestants.

Our current study provides a few directions for future research. First, in the analysis, we consider only symmetric contestants. It is worthwhile to investigate how to generalize the analysis to accommodate asymmetric contestants. Second, in the optimal contest with bounded negative prizes, we assume a common and fixed bound for the negative prizes. In some situations, this bound could be heterogenous among the contestants. Furthermore, this bound could even be a contestant's private information. It would be of interest to investigate how the optimal leveraging on different virtual abilities should be arranged. Third, similarly to the optimal auctions with risk averse bidders (cf. Maskin and Riley [10]), one can investigate the optimal contest design problem with risk averse contestants. Fourth, our current analysis focuses on an environment with pure adverse selection. Extending the analysis to a setting of adverse selection and moral hazard may yield additional insights.

6 Appendix: Proofs

Proof of Lemma 2: From (17), $\lambda(\mathbf{t})$ should not depend on any $\mathbf{t}_{-i}$, $\forall i$. Thus, $\lambda(\mathbf{t})$ must be a constant and does not depend on $\mathbf{t}$. Since $\lambda(\mathbf{t}) = \lambda$ is a constant, and since $\mu_i(t_i) \geq 0$, from (17) we must have $J(t_i) - t_0 \leq \lambda$ for every t_i. This means that $\lambda \geq b - t_0 > 0$, as $J(t_i) \leq b$. Note $[J(t_i) - t_0] < b - t_0$ for all $t_i < b$. Then $\mu_i(t_i) = \lambda - [J(t_i) - t_0] > 0$ for all $t_i < b, \forall i$. Thus, (15) must be binding for all $t_i < b$ at the optimal solution. This implies that $V_i(t_i) = 0,\ \forall t_i < b, \forall i$.

Since $\lambda > 0$, (14) must be binding for every $\mathbf{t}$, which implies that $\int_{\mathbf{t}} \sum_i v_i(\mathbf{t})\mathbf{f}(\mathbf{t})d\mathbf{t} = V$. We thus have $\sum_i \int_a^b V_i(t_i)f(t_i)dt_i = V$. This means that at least one $V_i(b)$ should be infinity. However, $V_i(b) = \infty$ is impossible. Thus an optimal solution does not exist for the relaxed problem (P-Relax). □

Proof of Lemma 3: We will establish that bV is an upper bound of the expected total effort in problem (P-Relax). First note that

$$\int_{\mathbf{t}} \sum_i [J(t_i) - t_0]\, v_i(\mathbf{t})\mathbf{f}(\mathbf{t})d\mathbf{t} = \sum_i \int_{t_i} [J(t_i) - t_0]\, V_i(t_i)f(t_i)dt_i.$$

Since $V_i(t_i) \geq 0$, $J(t_i) \leq J(b) = b$ and $t_0 < b = J(b)$, we have

$$\begin{aligned} & \sum_i \int_{t_i} [J(t_i) - t_0]\, V_i(t_i)f(t_i)dt_i + t_0 V \leq [J(b) - t_0] \sum_i \int_{t_i} V_i(t_i)f(t_i)dt_i + t_0 V \\ = \ & (b - t_0) \int_{\mathbf{t}} [\sum_i v_i(\mathbf{t})]\mathbf{f}(\mathbf{t})d\mathbf{t} + t_0 V \leq (b - t_0) \int_{\mathbf{t}} V\mathbf{f}(\mathbf{t})d\mathbf{t} + t_0 V \\ = \ & bV. \end{aligned}$$

This upper bound bV cannot be reached by any mechanism as Lemma 2 showed that an optimal solution does not exist for problem (P-Relax). □

Proof of Lemma 4: One can verify that

$$V_i(t_i;K) = \begin{cases} 0, & \text{if } t_i \leq t^*(K); \\ (V+NK)F^{N-1}(t_i) - K > 0, & \text{if } t_i > t^*(K). \end{cases}$$

and it is increasing in t_i.

Note that the construction of the effort functions corresponds to (9) with $u_i(a,a) = 0$. We have $t_i V_i(t_i;K) - \int_a^{t_i} V_i(s;K)ds > 0$ for $t_i > t^*(K)$, as $V_i(s;K)$ is strictly increasing given $t_i > t^*(K)$ and $a > 0$. Therefore, $e_i^*(\mathbf{t};K) \geq 0$ for every $\mathbf{t}$. Finally, it is obvious that the prize allocation function $v_i^*(\mathbf{t};K)$ satisfies the budget constraint. Thus, all constraints in the original problem (P) are satisfied and therefore, $(\mathbf{v}(\cdot;K), \mathbf{e}(\cdot;K))$ is feasible for every $K \geq 0$. Since problem (P-Relax) is a relaxed problem of problem (P), $(\mathbf{v}(\cdot;K), \mathbf{e}(\cdot;K))$ is also feasible in problem (P-Relax) for every $K \geq 0$.

The expected total effort induced by mechanism $(\mathbf{v}(\cdot;K), \mathbf{e}(\cdot;K))$ is given by

$$\begin{aligned} R(K) &= N\int_{t^*(K)}^b [J(t_i) - t_0]\, V_i(t_i;K)dF(t_i) + t_0 V \\ &= N\int_{t^*(K)}^b [J(t_i) - t_0]\, [(V+NK)F^{N-1}(t_i) - K]dF(t_i) + t_0 V \\ &= N\int_{t^*(K)}^b [J(t_i) - t_0]\, VF^{N-1}(t_i)dF(t_i) + NK\int_{t^*(K)}^b [J(t_i) - t_0]\, [NF^{N-1}(t_i) - 1]dF(t_i) + t_0 V. \end{aligned}$$

When $K \to +\infty$, $t^*(K)$ goes to b. Therefore, the first part in the last expression goes to zero. The third part is a constant. To show that the expected total effort converges to bV when $K \to +\infty$, it suffices to show that the second part converges to $(b - t_0)V$ when $K \to +\infty$. For the second part, note that when K is large enough, $t^*(K) = \hat{t}(K)$. Thus, $F^{N-1}(t^*(K)) = \frac{NK}{V+NK}$. That leads to

$$\frac{dt^*(K)}{dK} = \frac{NV}{(V+NK)^2(N-1)F^{N-2}(t^*(K))f(t^*(K))}.$$

Therefore,

$$\begin{aligned} &\lim_{K\to+\infty} NK\int_{t^*(K)}^b [J(t_i) - t_0]\, [NF^{N-1}(t_i) - 1]dF(t_i) \\ =\ &\lim_{K\to+\infty} N\frac{\int_{t^*(K)}^b [J(t_i) - t_0]\, [NF^{N-1}(t_i) - 1]dF(t_i)}{\frac{1}{K}} \end{aligned}$$

$$= \lim_{K\to+\infty} N\frac{-[J(t^*(K))-t_0][NF^{N-1}(t^*(K))-1]f(t^*(K))\frac{dt^*}{dK}}{-\frac{1}{K^2}} \text{ (by L'Hospital's rule)}$$

$$= \lim_{K\to+\infty} N\frac{[J(t^*(K))-t_0][NF^{N-1}(t^*(K))-1]\frac{NV}{(V+NK)^2(N-1)F^{N-2}(t^*(K))}}{\frac{1}{K^2}}$$

$$= \lim_{K\to+\infty} N\frac{[J(b)-t_0][NF^{N-1}(b)-1]\frac{NV}{(V+NK)^2(N-1)F^{N-2}(b)}}{\frac{1}{K^2}}$$

$$= \lim_{K\to+\infty} N(b-t_0)(N-1)\frac{NVK^2}{(V+NK)^2(N-1)}$$

$$= (b-t_0)V.$$

Hence, the expected total effort $R(K)$ converges to bV.

We now turn to the contestants' expected payoffs. Recall that the contestants' expected total payoffs are at most the difference between V and the expected total effort costs.[14] Since expected total effort converges to bV, we must have the total effort costs converge to $bV/b = V$ since only those types within a small neighborhood of b would exert positive effort. It follows that the contestants' expected total payoffs must converge to zero.

As $K \to +\infty$, almost all types except b obtain zero expected prize and exert zero effort. What remains interesting is how much informational rent the type b can enjoy at the limit.

Note that $F^{N-1}(t^*(K)) \leq F^{N-1}(s) \leq 1, \forall s \in [t^*(K), b]$. Thus, $(N-1)K \leq (V+NK)F^{N-1}(s) - K \leq (N-1)K + V$. Therefore,

$$\lim_{K\to+\infty} \tilde{u}_i(b,b) = \lim_{K\to+\infty}\int_{t^*(K)}^{b} V_i(s;K)ds = \lim_{K\to+\infty}\int_{t^*(K)}^{b}\left[(V+NK)F^{N-1}(s)-K\right]ds$$

$$= \lim_{K\to+\infty}[b-t^*(K)](N-1)K = \lim_{K\to+\infty}(N-1)\frac{b-t^*(K)}{1/K}$$

$$= \lim_{K\to+\infty}(N-1)\frac{dt^*(K)/dK}{1/K^2} = \lim_{K\to+\infty}\frac{N(N-1)VK^2}{(V+NK)^2(N-1)F^{N-2}(t^*(K))f(t^*(K))}$$

$$= \frac{V}{Nf(b)}.$$

□

[14] When the budget constraint is binding, the contestants' expected total payoffs are equal to the difference between V and the expected total effort costs.

Proof of Lemma 5: We claim $v_i^{\circledast}(t_i, \mathbf{t}_{-i}) = -K$ for $t_i > \hat{t}_i^{\circledast}$ if there exists some contestant $j \neq i$ such that $t_j > t_i$. Suppose not, then $v_i^{\circledast}(t_i, \mathbf{t}_{-i}) > -K$, which means $\xi_i(\mathbf{t}) = 0$. In addition, we have $\mu_i(t_i) = 0$ from the fact $V_i^{\circledast}(t_i) > 0$. Thus $[J(t_i) - t_0] - \lambda = 0$. Note $[J(t_j) - t_0] - \lambda + \mu_j(t_j) + \xi_j(t_i, t_j, \mathbf{t}_{-ij}) = 0$. Thus $J(t_j) - t_0 = \lambda - \mu_j(t_j) - \xi_j(t_i, t_j, \mathbf{t}_{-ij}) \leq \lambda = [J(t_i) - t_0]$, which contradicts the assumption that $J(\cdot)$ is a strictly increasing function.

When t_i is the highest among all contestants, contestant i can at most collect $V + (N-1)K$; when $t_i > \hat{t}_i^{\circledast}$ is not the highest, $v_i^{\circledast}(t_i, \mathbf{t}_{-i}) = -K$. For contestant i, when $t_i > \hat{t}_i^{\circledast}$, we must have

$$0 < V_i^{\circledast}(t_i) \leq [V + (N-1)K] F^{N-1}(t_i) - K \cdot (1 - F^{N-1}(t_i)) = (V + NK) F^{N-1}(t_i) - K. \quad (40)$$

(40) implies that $\hat{t}_i^{\circledast} \geq \tilde{t}_0 = F^{-1}((\frac{K}{V+NK})^{\frac{1}{N-1}})$. $\square$

Proof of Lemma 6: It is straight-forward to verify that $\bar{V}_i(t_i)$ satisfies all the conditions in the maximization problem (P-K-Relax-Equivalent-Relax). To simplify notations, we hereafter use $\hat{t}$ and t^* to denote $\hat{t}(K)$ and $t^*(K)$, respectively. Define $t^M = J^{-1}(t_0)$. We next consider two cases to show the optimality of $\bar{V}_i(t_i)$. Case 1: $t^* = \hat{t}$, i.e., $J(\hat{t}) \geq t_0$. Case 2: $t^* = t^M$, i.e., $J(\hat{t}) \leq t_0$.

First, we consider Case 1 where $J(\hat{t}) \geq t_0$, i.e., $t^* = \hat{t} \geq t^M$. We shall show that for any functions $V_i(t_i)$ satisfying (36) to (39), we have

$$\sum_{i=1}^{N} \int_a^b [J(t_i) - t_0] V_i(t_i) f(t_i) dt_i \leq \sum_{i=1}^{N} \int_a^b [J(t_i) - t_0] \bar{V}_i(t_i) f(t_i) dt_i = \sum_{i=1}^{N} \int_{t^*}^b [J(t_i) - t_0] \bar{V}_i(t_i) f(t_i) dt_i. \quad (41)$$

This is equivalent to

$$\sum_{i=1}^{N} \int_a^{t^*} [J(t_i) - t_0] V_i(t_i) f(t_i) dt_i \leq \sum_{i=1}^{N} \int_{t^*}^b [J(t_i) - t_0] (\bar{V}_i(t_i) - V_i(t_i)) f(t_i) dt_i. \quad (42)$$

Note that $V_i(t_i) \geq 0$. In addition, when $t_i > t^* = \hat{t}$, $\bar{V}_i(t_i) = (V + NK) F^{N-1}(t_i) - K$. So $\bar{V}_i(t_i) - V_i(t_i) \geq 0$ for $t_i > t^* = \hat{t}$.

Since $J(\cdot)$ is strictly increasing and $J(t^*) \geq t_0$, we have

$$\sum_{i=1}^{N} \int_a^{t^*} [J(t_i) - t_0] V_i(t_i) f(t_i) dt_i \leq [J(t^*) - t_0] \sum_{i=1}^{N} \int_a^{t^*} V_i(t_i) f(t_i) dt_i,$$

and

$$[J(t^*)-t_0]\sum_{i=1}^{N}\int_{t^*}^{b}(\bar{V}_i(t_i)-V_i(t_i))f(t_i)dt_i\leq\sum_{i=1}^{N}\int_{t^*}^{b}[J(t_i)-t_0]\,(\bar{V}_i(t_i)-V_i(t_i))f(t_i)dt_i.$$

Thus, in order for (42) to hold, we only need to show that

$$\sum_{i=1}^{N}\int_{a}^{t^*}V_i(t_i)f(t_i)dt_i\leq\sum_{i=1}^{N}\int_{t^*}^{b}(\bar{V}_i(t_i)-V_i(t_i))f(t_i)dt_i,$$

which is equivalent to

$$\sum_{i=1}^{N}\int_{a}^{b}V_i(t_i)f(t_i)dt_i\leq\sum_{i=1}^{N}\int_{t^*}^{b}\bar{V}_i(t_i)f(t_i)dt_i. \tag{43}$$

According to constraint (36), the LHS of (43) must be bounded by V. For the RHS of (43), we have

$$\begin{aligned}
&\sum_{i=1}^{N}\int_{t^*}^{b}\bar{V}_i(t_i)f(t_i)dt_i= N\int_{t^*}^{b}[(V+NK)F^{N-1}(t_i)-K]f(t_i)dt_i\\
=\;& (V+NK)F^{N}(t_i)|_{t_i=t^*}^{b}-NK(1-F(t^*))=(V+NK)(1-F^N(t^*))-NK+NKF(t^*)\\
=\;& V-(V+NK)F^{N-1}(t^*)F(t^*)+NKF(t^*)=V-(V+NK)\cdot\frac{NK}{V+NK}\cdot F(t^*)+NKF(t^*)\\
=\;& V.
\end{aligned}$$

Hence (43) holds.

Second, we consider Case 2: $J(\hat{t})\leq t_0$, i.e., $\hat{t}\leq t^*=t^M$. We shall show that for any $V_i(t_i)$ that satisfies (36) to (39), we have

$$\sum_{i=1}^{N}\int_{a}^{b}[J(t_i)-t_0]\,V_i(t_i)f(t_i)dt_i\leq\sum_{i=1}^{N}\int_{a}^{b}[J(t_i)-t_0]\,\bar{V}_i(t_i)f(t_i)dt_i,$$

which is equivalent to

$$\sum_{i=1}^{N}\int_{\hat{t}_i}^{b}[J(t_i)-t_0]\,V_i(t_i)f(t_i)dt_i\leq\sum_{i=1}^{N}\int_{t^M}^{b}[J(t_i)-t_0]\,\bar{V}_i(t_i)f(t_i)dt_i.$$

Consider contestant i. Suppose that $\hat{t}_i\geq t^M$. Note that when $t_i>t^M$, we have $V_i(t_i)\leq$

$(V+NK)F^{N-1}(t_i) - K = \bar{V}_i(t_i)$ and $[J(t_i) - t_0] > 0$. Therefore,

$$\int_{\hat{t}_i}^{b} [J(t_i) - t_0]\, V_i(t_i) f(t_i) dt_i = \int_{t^M}^{b} [J(t_i) - t_0]\, V_i(t_i) f(t_i) dt_i \leq \int_{t^M}^{b} [J(t_i) - t_0]\, \bar{V}_i(t_i) f(t_i) dt_i.$$

Now suppose that $\hat{t}_i < t^M$. Note that $[J(t_i) - t_0] < 0$ when $t_i < t^M$. We have

$$\int_{\hat{t}_i}^{b} [J(t_i) - t_0]\, V_i(t_i) f(t_i) dt_i = \int_{\hat{t}_i}^{t^M} [J(t_i) - t_0]\, V_i(t_i) f(t_i) dt_i + \int_{t^M}^{b} [J(t_i) - t_0]\, V_i(t_i) f(t_i) dt_i$$

$$\leq \int_{t^M}^{b} [J(t_i) - t_0]\, V_i(t_i) f(t_i) dt_i \leq \int_{t^M}^{b} [J(t_i) - t_0]\, \bar{V}_i(t_i) f(t_i) dt_i.$$

The last inequality holds because when $t_i > t^M$, $V_i(t_i) \leq (V+NK)F^{N-1}(t_i) - K = \bar{V}_i(t_i)$ and $[J(t_i) - t_0] > 0$.

To conclude, either when $\hat{t}_i \geq t^M$ or when $\hat{t}_i < t^M$, we always have

$$\int_{\hat{t}_i}^{b} [J(t_i) - t_0]\, V_i(t_i) f(t_i) dt_i \leq \int_{t^M}^{b} [J(t_i) - t_0]\, \bar{V}_i(t_i) f(t_i) dt_i.$$

Thus,

$$\sum_{i=1}^{N} \int_{a}^{b} [J(t_i) - t_0]\, V_i(t_i) f(t_i) dt_i \leq \sum_{i=1}^{N} \int_{a}^{b} [J(t_i) - t_0]\, \bar{V}_i(t_i) f(t_i) dt_i.$$

This completes the proof. $\square$

Proof of Proposition 2: It is easy to verify that the prize allocation function generates

$$V_i^*(t_i) = \begin{cases} 0, & \text{if } t_i \leq t^*(K); \\ (V+NK)F^{N-1}(t_i) - K > 0, & \text{if } t_i > t^*(K). \end{cases} \tag{44}$$

which is the optimal $\bar{V}_i(t_i)$ of Lemma 6. Note that $V_i^*(t_i)$ is discontinuous at $t_i = t^*(K)$.

As we noted earlier, the supporting effort functions $\{e_i^*(\mathbf{t}), i = 1, 2, ..., N\}$ can be constructed using (9) with $u_i(a,a) = 0$, which gives us $e_i^*(\mathbf{t}) = \varepsilon(t_i; K)$. One can verify that $\varepsilon(t_i; K)$ is strictly positive and increasing in t_i for $t_i > t^*(K)$. Note also that because $V_i^*(t_i)$ is discontinuous at $t_i = t^*(K)$, $\varepsilon(t_i; K)$ is also discontinuous at $t_i = t^*(K)$.

We can easily verify that all constraints in problem (P-K) are satisfied in mechanism $(\mathbf{v}^*(\cdot;K), \mathbf{e}^*(\cdot;K))$. We thus have established that $(\mathbf{v}^*(\cdot;K), \mathbf{e}^*(\cdot;K))$ is indeed an optimal solution for problem (P-K). $\square$

References

[1] Bulow, J., and Roberts, J., The simple economics of optimal auctions, *Journal of Political Economy*, 97, 1989, 1060-1090.

[2] Che, Y.-K., and Kim, J., Robustly collusion-proof mechanisms, *Econometrica*, 74, 2006, 1063-1107.

[3] Chen, Y.C., and Xiong, S.Y., Genericity and robustness of full surplus extraction, *Econometrica*, 81(2), 2013, 825-847.

[4] Crémer, J., and McLean, R., Full extraction of the surplus in Bayesian and dominant strategy auctions, *Econometrica*, 56, 1988, 1247-1257.

[5] Fullerton, R., and McAfee, P., Auctioning entry into tournaments, *Journal of Political Economy,* 107(3), 1999, 573-605.

[6] Ghosh, A., and McAfee, P., Crowdsourcing with endogenous entry, *Proceedings of the 21st International Conference on World Wide Web,* 2012, 999-1008.

[7] Heifetz, A., and Neeman, Z., On the generic (im)possibility of full surplus extraction in mechanism design, *Econometrica*, 74, 2006, 213-234.

[8] Kirkegaard, R., Favoritism in asymmetric contests: head starts and handicaps. *Games and Economic Behavior*, 76, 2012, 226-248.

[9] Laffont, J.J., and Martimort, D., Mechanism design with collusion and correlation, *Econometrica*, 68, 2000, 309-342.

[10] Maskin, E.S., and Riley, J., Optimal auctions with risk averse buyers, *Econometrica*, 52(6), 1984, 1473-1518.

[11] McAfee, R.P., and Reny, P.J., Correlated information and mechanism design, *Econometrica*, 60, 1992, 395-421.

[12] D. Minor, Increasing effort through rewarding the best less (or not at all), working paper, Northwestern University, 2012.

[13] Moldovanu, B., and Sela, A., The optimal allocation of prizes in contests. *American Economic Review* 91, 3 2001, 542-558.

[14] Moldovanu, B., and Sela, A., Contest architecture. *Journal of Economic Theory* 126(1), 2006, 70-97.

[15] Moldovanu, B., Sela, A., and Shi, X., Contests for status, *Journal of Political Economy*, 115, 2007, 338-363.

[16] Moldovanu, B., Sela, A. and Shi, X., Carrots and sticks: prizes and punishments in contests, *Economic Inquiry*, 50(2), 2012, 453-462.

[17] Myerson, R.B., Optimal auction design. *Mathematics of Operations Research* 6, 1981, 58-73.

[18] Peters, M., Surplus extraction and competition, *Review of Economic Studies,* 68, 2001, 613-631.

[19] Polishchuk, L., and Tonis, A., Endogenous contest success functions: a mechanism design approach, *Economic Theory*, 52(1), 2013, 271-297.

[20] Robert, J., Continuity in auction design, *Journal of Economic Theory* 55, 1991, 169-179.

[21] Taylor, C., Digging for golden carrots: an analysis of research tournaments, *American Economic Review*, 85, 1995, 872–890.