Orchestrating Information Acquisition*

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Abstract

We study how to orchestrate information acquisition in an environment where bidders endowed with original estimates ("types") about their private values can acquire further information by incurring a cost. We consider both single-round and fully sequential shortlisting rules. The optimal single-round shortlisting rule admits the set of most efficient bidders that maximizes expected virtual surplus adjusted by the second-stage signal and information acquisition cost. When shortlisting is fully sequential, at each round, the most efficient remaining bidder is admitted provided that her conditional expected contribution to the virtual surplus is positive.

Keywords: Two-stage auctions, information acquisition, entry, sequential screening, optimal shortlisting, optimal mechanisms.
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1 Introduction

In high-valued asset sales, buyers often need to go through a due diligence process before developing final bids. Due diligence is usually a process to update or acquire information about the value of

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the asset for sale or to prepare for the bidding process (e.g., to establish qualifications to bid). This process is costly and is usually modeled as entry as it is closely monitored by the auctioneer. For a sale of an asset worth billions of dollars, the entry cost can run from tens of thousands to millions of dollars.[1]

Given the substantial entry cost, how to coordinate agents' costly information acquisition becomes one central issue in optimal mechanism design. The importance of coordinating bidders' entry for the purpose of enhancing seller revenue as well as total surplus has been revealed as early as in Levin and Smith (1994), who find that seller revenue and total surplus can decrease with the number of potential bidders when bidder entry is symmetric. Clearly, the success of a sale very much depends on whether the adopted mechanism attracts the most qualified bidders conducting the due diligence process and participating in the final sale. Mainly motivated by the need for entry screening and coordination, variants of two-stage selling mechanisms have emerged in the real world. A leading example of the two-stage auction procedure is known as indicative bidding, which is commonly used in sales of complicated business assets with very high values. It works as follows: the auctioneer actively markets the assets to a large group of potentially interested buyers. The potential buyers are then asked to submit non-binding bids, based on which a final set of bidders is shortlisted to advance to the second stage. The auctioneer then communicates only with these final bidders, providing them with extensive access to information about the assets, and finally runs the auction (typically using binding sealed bids). The use of this two-stage auction procedure is quite widespread. For example, in response to the restructuring of the electric power industry in the U.S. – which was designed to separate power generation from transmission and distribution – billions of dollars of electrical generating assets were divested through this two-stage auction procedure over the last two decades.[2] This two-stage auction procedure is also commonly used in privatization, takeover, and merger and acquisition contests.[3] Finally, it is commonly used in the institutional real estate market, which has an annual sales volume in the order of \$60 to \$100 billion.[4]

Ye (2007) studies this two-stage auctions based on the assumption of costly information acquisition.[5] Ye's analysis suggests that the current design of indicative bidding cannot reliably select the most qualified bidders for the final sale, as there does not exist a symmetric, strictly increasing equilibrium bid function in the indicative bidding stage. In a more recent paper, by restricting indicative bids to a finite discrete domain, Quint and Hendricks (2018) show that a symmetric equilibrium exists

[1] According to Kleiberg, Waggoner and Weyl (2018), due diligence costs for the acquisition by a large technology company of a start-up are typically 20-40% of the size of a deal.

[2] A list of industry examples using this two-stage auction design can be found in Ye (2007).

[3] Leading examples include the privatization of the Italian Oil and Energy Corporation (ENI), the acquisition of Ireland's largest cable television provider Cablelink Limited, and the takeover contest for South Korea's second largest conglomerate Daewoo Motors.

[4] See Foley (2003) for a detailed account.

[5] Boone and Goeree (2009) provide an analysis of pre-qualifying auctions, which are similar to indicative bidding.

in weakly-monotone strategies. But again, the highest-value bidders are not always selected, as bidder types "pool" over a finite number of bids. Without safely selecting the most qualified bidders for the final sale, the mechanism is unlikely optimal in maximizing expected revenue. What the optimal mechanism is in this two-stage auction environment remains an open question in the literature, and this paper seeks to provide an answer.

We model the situation as follows. Before entry, each potential bidder is endowed with a private signal, α_i, which can be regarded as her pre-entry "type." After entry (by incurring a common entry cost, c), each bidder i fully observes her (private) value v_i, which is positively correlated with her pre-entry type. Given costly entry, it is not optimal for all potential bidders to be included in the final sale. As such, a general mechanism must consist of an entry-right allocation stage to shortlist bidders into the sale and the final stage to allocate the asset. The entry-right allocation stage may potentially consist of multiple rounds, depending on whether the shortlisting is conducted in one single round or multiple rounds. The shortlisting rule in a subsequent stage would depend on all information revealed in previous stages. Despite the potential complication due to both sequential screening and endogenous information acquisition, we are able to characterize the optimal revenue-maximizing selling mechanism with sequential information acquisition. Our analysis benefits greatly from recent developments in the literature of sequential screening (e.g., Courty and Li (2000); Esö and Szentes (2007); Pavan, Segal, and Toikka (2014); and Bergemann and Wambach (2015).[6] In particular, our model resembles that of Esö and Szentes and we follow their main approaches including the orthogonalization technique in characterizing optimal dynamic mechanisms. Our paper differs from theirs in that buyer information acquisition is costly and endogenous in our model, and our analysis focuses on identifying the shortlisting mechanism that optimally orchestrates information acquisition of buyers.

Given the widespread use of two-stage auctions, we start our analysis with the case where shortlisting is completed simultaneously in one single round. In effect, we restrict our search of optimal mechanisms to the class of two-stage mechanisms, with the first stage allocating entry rights (shortlisting) and the second stage allocating the asset. We show that the optimal allocation rule of the asset requires that the asset be allocated to the bidder with the highest *virtual value* adjusted by the second-stage signal, same finding as identified by Esö and Szentes. Our analysis thus suggests that the optimality of the generalized Myerson optimal allocation rule (adjusted by second-round signals) is robust to the dynamic auction setting with costly and endogenous entry. The first-stage shortlisting mechanism is new to the original Esö-Szentes framework, and we show that the optimal entry right allocation rule is to shortlist the set of bidders that gives rise to the maximum expected *virtual surplus* (adjusted by both the second-stage signal and entry cost). Alternatively, given the

[6] Early contributions on dynamic contracting with a single agent are due to Baron and Besanko (1984) and Riordan and Sappington (1987).

regularity assumption and that buyers are *ex ante* symmetric in our model, the optimal entry rule is to admit the bidders in descending order of their pre-entry "types", the highest type first, the second highest type second, etc., provided that their marginal contribution to the expected virtual surplus is positive. Therefore, the optimal number of shortlisted bidders typically depends on the reported type profile from the potential bidders, which is endogenously determined. We then show that specific payment rules can be constructed in each stage to implement both optimal entry and allocation rules truthfully.

In Section 4, we relax the restriction of single-round shortlisting and consider the case with potentially multiple-round sequential shortlisting. The seller may now select a single bidder or any subset of bidders at each round to go through due diligence and submit final bids, and if the seller is not satisfied with any offer, he can go back to the unselected bidders and invite another bidder or another subset of bidders to go through due diligence and submit final bids. This process can then repeat itself, until the seller finds a satisfactory offer. Such mechanisms can be more complicated. First of all, the seller will need to determine the order of bidders to invite for conducting due diligence (i.e., who should be invited first and who second, etc.). Given that bidders are heterogenous before entry, it is desirable to make the optimal "ordering" or "sequencing" of entry contingent on their pre-entry types. Our main results with such a general analysis are as follows. First, the optimal final good allocation rule is the same as characterized previously, that is, the object is allocated to the shortlisted bidder with the highest virtual value $w(\alpha_i, s_i)$, provided that it is positive. The optimal shortlisting rule should be modified, however, in a way that at each round, at most one bidder (the one with the highest pre-entry type among all the bidders outside the auction) is shortlisted, and a new bidder is shortlisted at a given round if and only if conditional on all the information revealed up to this round, her expected contribution to the virtual surplus is positive. The bidders are approached sequentially in the order of their first stage types, starting from the highest type.

Other than the connection with sequential screening and dynamic auctions mentioned above, our paper is related to the literature on auctions and mechanism design with information acquisition.[7] Papers in this literature either study bidders' incentives to acquire information in different specific auction formats or consider single-stage optimal mechanism design. Our paper differs from theirs in that we follow the normative approach to identify optimal dynamic mechanisms with information acquisition. Our paper is also closely related to Krähmer and Strausz (2011), who study procurement contracts with pre-project information acquisition, and Halac, Kartik, and Liu (2016), who consider optimal dynamic contracts with experimentation. Unlike in our model, information acquisition in their models is unobservable and thus not contractible, so they have to deal with both adverse selection and moral hazard in their analysis. In our model, since information acquisition is modeled

[7] See, for example, Persico (2000), Compte and Jehiel (2001), Shi (2012), Rezende (2013), Li (2019), Gershkov, Moldovanu and Strack (2018), and Zhang (2018).

as entry, moral hazard is absent from our analysis. Our paper differs from theirs also in that we work with multiple agents/bidders, while there is only one agent in their models.

To the extent that information acquisition is modeled as entry, our paper is closely related to the growing literature on auctions with costly entry.[8] This literature can be summarized into three branches. In the first branch, bidders are assumed to possess no private information before entry and they learn their private values or signals only after entry.[9] In the second branch, it is assumed that bidders are endowed with private information about their values but have to incur entry costs to participate in an auction.[10] Finally, in the third branch, bidders are endowed with some private information before entry, and are able to acquire additional private information after entry (Ye, 2007; Quint and Hendricks, 2018). The framework in this current paper nests all the models mentioned above as special cases. Our paper thus characterizes optimal mechanisms for a very general framework in the literature on auctions with costly entry.

Our paper is related to a literature on search, which is originally inspired by Weitzman (1979) who studies the so-called Pandora's problem of infinite sequential search with recall and establishes the well-known Pandora rule. Crémer, Spiegel, and Zheng (2009) extends this model to an auction context. In the environment where bidders are not endowed with pre-entry private information, they find that an ex post efficient auction with sequential entry is both ex ante efficient and revenue maximizing. In our setting, bidders are endowed with both pre-entry and post-entry private information, which dramatically complicates the analysis due to the additional incentive compatibility conditions.[11]

Our research is also related to a small literature on auctions of entry rights. Fullerton and McAfee (1999) introduce auctions for entry rights to shortlist contestants for a final tournament. Ye (2007) extends their approach to the setting of two-stage auctions described above. Our current approach differs from theirs in the way the set of finalists is determined: while in their approach the number of finalists to be selected is fixed and pre-announced, in our entry right allocation mechanism the selection of shortlisted bidders is contingent on the reported bid profile, making the number of finalists endogenously determined. For this reason the entry right allocation mechanism examined in this research is more general.[12]

In another relevant paper, Lu and Ye (2013) explore optimal two-stage mechanisms in an environment where bidders are characterized by heterogenous and private information acquisition costs

[8] See Bergemann and Välimäki (2006) for a thoughtful survey of this literature.

[9] See, for example, McAfee and McMillan (1987), Engelbrecht-Wiggans (1993), Tan (1992), Levin and Smith (1994), Ye (2004), and Jehiel and Lamy (2015).

[10] See, for example, Samuelson (1985), Stegeman (1996), Campbell (1998), Menezes and Monteiro (2000), Tan and Yilankaya (2006), Cao and Tian (2009), and Lu (2009).

[11] Other related papers in the search literature include, e.g., Szech (2011), Lee and Li (2018), Olszewski and Weber (2015), Doval (2018), and Kleiberg, Waggoner, and Weyl (2018).

[12] In fact, it resembles multi-unit auctions with endogenously determined supply (see, e.g., McAdams, 2007).

before entry. In that setting the pre-entry "type" is the entry cost, which is neither correlated to nor part of the value of the asset for sale. As such, there is no benefit to make the second-stage mechanism contingent on the reports of the pre-entry types, resulting in a much simpler characterization of optimal mechanisms. In our current setting, the optimal allocation and payment rules in a subsequent stage depend on report(s) from the previous stage(s). Therefore the characterization of optimal mechanisms is more demanding, and the implementation of the optimal mechanism is also more sophisticated.

The rest of the paper is organized as follows. Section 2 presents the model. Section 3 characterizes the optimal two-stage mechanism when shortlisting is restricted to one single round. Section 4 characterizes the optimal multi-stage mechanisms when multi-stage sequential shortlisting is allowed. Section 5 extends our analysis to allow sale in the first stage and provide some numerical examples. Section 6 concludes.

2 The Model

A single indivisible asset is offered for sale to N potentially interested buyers. The seller and bidders are assumed to be risk neutral. The seller's own valuation for the asset is normalized to 0. Buyer i's true valuation for the asset is v_i. However, initially she only observes a noisy signal of it, α_i, which is her private information and can be interpreted as her original "type." After incurring a common information acquisition cost (or entry cost) of $c(> 0)$, bidder i fully observes her ex post value, v_i. The pairs (α_i, v_i) are assumed to be independent across i.[13]

Ex ante, α_i follows distribution $F(\cdot)$ with its associated density $f(\cdot)$ on support $[\underline{\alpha}, \overline{\alpha}]$.We assume that f is positive on the interval $[\underline{\alpha}, \overline{\alpha}]$ and satisfies the monotone hazard rate condition; that is, $f/(1-F)$ is weakly increasing (the regularity assumption). Given α_i, the ex post value v_i follows distribution $H_{\alpha_i} \equiv H(\cdot|\alpha_i)$ with its density $h_{\alpha_i} \equiv h(\cdot|\alpha_i)$ over support $[\underline{v}(\alpha_i), \overline{v}(\alpha_i)] \subset \mathbb{R}$.[14] The values N and c and distributions F and H_{α_i} are all common knowledge. We assume $H_{\alpha_i}(\cdot)$ decreases with α_i, i.e. $H_{\alpha_i}(\cdot)$ first order dominates $H_{\alpha_i'}(\cdot)$ if $\alpha_i > \alpha_i'$.

Following the signal orthogonalization technique introduced by Esö and Szentes (2007),[15] there exist functions u and s_i, such that $u(\alpha_i, s_i) \equiv v_i$, where u is strictly increasing in both arguments, and s_i is independent of α_i. In particular, s_i can be constructed as follows:

$$s_i = H(v_i|\alpha_i),$$

[13] As in Esö and Szentes (2007) and Pavan, Segal, and Toikka (2014), this assumption rules out the possibility of full rent extraction (Crémer and McLean, 1988).

[14] The support of v_i can depend on the first-stage signal α_i; that is v_i may have a moving support.

[15] The use of this technique has become standard in the literature (see, e.g., Pavan, Segal, and Toikka, 2014, and Bergemann and Wambach, 2015).

which is the percentile of the value realization to bidder i.[16] Thus given type α_i and signal s_i, the valuation can be computed as

$$v_i = H_{\alpha_i}^{-1}(s_i) \equiv u(\alpha_i, s_i).$$

We will denote the c.d.f. of s_i by G_i.[17] Note $u(\alpha_i, s_i)$ increases in both α_i and s_i in our model.

We maintain the following assumptions that are adopted in Esö and Szentes (2007):

Assumption 1. $(\partial H_\alpha(v)/\partial\alpha)/h_\alpha(v)$ is increasing in v.

Assumption 2. $(\partial H_\alpha(v)/\partial\alpha)/h_\alpha(v)$ is increasing in α.

Esö and Szentes show that Assumption 1 is equivalent to $u_{12} \leq 0$ and Assumption 2 is equivalent to $u_{11}/u_1 \leq u_{12}/u_2$. Assumption 1 thus states that the marginal impact of the new information on buyer i's value is decreasing in her type α_i. Assumption 2 implies that an increase in α_i, holding $u(\alpha_i, s_i)$ constant, weakly decreases the marginal value of α_i. Assumptions 1 and 2 can thus be interpreted as a kind of substitutability in buyer i's posterior valuation between α_i and s_i.

Since information acquisition is modeled as entry, we consider a mechanism design framework in which the seller exercises entry control. In Section 3, we restrict our analysis to two-stage mechanisms: the first stage is the entry right allocation mechanism, and the second stage is the private good provision mechanism. In Section 4, we will extend our analysis to multi-stage mechanisms allowing for sequential shortlisting.

We restrict our analysis to direct mechanisms where agents report their types truthfully at each stage on the equilibrium path. We assume that all shortlisted bidders are disclosed and the first-stage reported profile $\boldsymbol{\alpha}$ is revealed to all admitted bidders so that the first-stage entry allocation and payments are immediately verifiable.[18] This revelation policy turns out to be "optimal," in the sense that no other revelation policy (e.g., not revealing or partially revealing $\boldsymbol{\alpha}$) can generate a higher expected revenue to the seller. For this reason, our restriction to fully revealing $\boldsymbol{\alpha}$ is without loss of generality in our search for optimal mechanisms. In our paper, the principal has no control over the ways in which new information in a subsequent stage is revealed to bidders. A shortlisted bidder will be fully informed about her true value v_i after incurring the entry cost. As such, we are not concerned about the discriminatory information disclosure issue studied in Li and Shi (2017).

As in Esö and Szentes, we can focus on equivalent direct mechanisms that require bidders to report s_i's, rather than v_i's. Note that reporting (α_i', v_i') is equivalent to reporting $(\alpha_i', s_i' = H_{\alpha_i'}(v_i'))$.[19]

[16] It is easily seen that s_i is uniformly distributed over $[0, 1]$, and is hence statistically independent of the initial information α_i.

[17] G_i could be assumed to be uniform on $[0, 1]$. More generally, all s_i's satisfying $u(\alpha_i, s_i) \equiv v_i$ are positive monotonic transformation of each other (Lemma 1 in Esö and Szentes).

[18] In Esö and Szentes, there is no such need for interim verification, as their allocation and payment rules are executed at the end of the mechanism.

[19] Alternatively we can derive optimal mechanisms based on buyers' second-stage reported values as in Pavan, Segal

Let $\mathbf{N} = \{1, 2, ..., N\}$ denote the set of all the potential buyers and $2^{\mathbf{N}}$ denote the collection of all the subsets (subgroups) of $\mathbf{N}$, including the empty set, ϕ.

3 Analysis with Single-Round Shortlisting

3.1 Two-Stage Mechanisms

The first-stage mechanism is characterized by the shortlisting rule $A^g(\boldsymbol{\alpha})$ and payment rule $x_i(\boldsymbol{\alpha}), i = 1, 2, ..., N$. Given the reported profile $\boldsymbol{\alpha}$, the shortlisting rule, $A^g : [\underline{\alpha}, \overline{\alpha}]^N \to [0, 1]$, assigns a probability to each subgroup $g \in 2^{\mathbf{N}}$, where $\sum_{g \in 2^{\mathbf{N}}} A^g(\boldsymbol{\alpha}) = 1$. The payment rule $x_i : [\underline{\alpha}, \overline{\alpha}]^N \to \mathbb{R}$, specifies bidder i's first-stage payment given the reported profile $\boldsymbol{\alpha}$.

Given the first-stage reported profile $\boldsymbol{\alpha}$, and that group g is shortlisted, the second-stage mechanism is characterized by $p_i^g(\boldsymbol{\alpha}, \mathbf{s}^g)$, the probability with which the asset is allocated to buyer $i \in g$, and $t_i^g(\boldsymbol{\alpha}, \mathbf{s}^g)$, the payment to the seller made by buyer $i \in g, \forall g \in 2^{\mathbf{N}}$.

We will identify the revenue-maximizing selling mechanism in two steps. First, we establish a revenue upper bound by considering a relaxed problem in which the second-stage signals $\mathbf{s}$ are public information among the shortlisted buyers. In this relaxed problem, we ignore the second-stage incentive compatibility condition (IC) and individual rationality condition (IR). Second, we will identify a feasible mechanism (satisfying IC and IR in both stages) in the original setting, which achieves the above revenue bound.

3.2 A Revenue Upper Bound with Public $\mathbf{s}$

In this subsection, we identify an upper bound for the expected revenue in a relaxed setting with public $\mathbf{s}$ for the shortlisted buyers and the seller. We drop the IC and IR constraints for the shortlisted bidders in the second stage so that all shortlisted bidders must incur entry costs to learn their second-stage signals as in our original setup, and regardless of their second-stage signals, they must participate in the second-stage selling mechanism and report their second-stage signals truthfully.

It is clear that the highest possible expected revenue achievable in this relaxed setting imposes an upper bound for the expected revenue that can be obtained in our original setup, where the bidders' second-stage IC and IR must both be satisfied. We will identify this upper bound first.

In this relaxed setting, the mechanisms are specified the same as in Section 3.1. All potential bidders report their types α_i, giving rise to a reported type profile $\boldsymbol{\alpha}$. The mechanism specifies the first-stage shortlisting rule $A^g(\boldsymbol{\alpha})$ and payment rule $x_i(\alpha_i, \boldsymbol{\alpha}_{-i})$. Every shortlisted bidder j incurs cost c to discover her second-stage signal s_j. The second-stage selling mechanism specifies the winning

and Toikka (2014). We choose to work with the orthogonalized signals instead as that allows us to further study revelation policies regarding first-stage reports, which can be done by establishing the revenue bound in a relaxed environment with public orthogonalized signal s.

probability $p_i^g(\boldsymbol{\alpha}, \mathbf{s}^g)$ and payment rule $t_i^g(\boldsymbol{\alpha}, \mathbf{s}^g)$, $\forall i \in g$, $\forall g \in 2^N$.

Given the announced $\boldsymbol{\alpha}$ and s_i, we define the interim winning probability and expected payment rule as $P_i^g(\boldsymbol{\alpha}, s_i) = E_{\mathbf{s}_{-i}^g} p_i^g(\boldsymbol{\alpha}, \mathbf{s}^g)$ and $T_i^g(\boldsymbol{\alpha}, s_i) = E_{\mathbf{s}_{-i}^g} t_i^g(\boldsymbol{\alpha}, \mathbf{s}^g)$, where $\mathbf{s}_{-i}^g = \mathbf{s}^g \backslash \{s_i\}$, $\forall i \in g$ and $\forall g \in 2^{\mathbf{N}}$. Let g_i denote a shortlisted subgroup that contains bidder i. For shortlisted bidder $i \in g_i$ with type α_i, her interim expected payoff when she reports $\hat{\alpha}_i$ and others report truthfully is given by

$$\pi_i(\alpha_i, \hat{\alpha}_i) = E_{\alpha_{-i}}\{\sum_{g_i} A^{g_i}(\hat{\alpha}_i, \boldsymbol{\alpha}_{-i})[E_{s_i}((u(\alpha_i, s_i)P_i^{g_i}(\hat{\alpha}_i, \boldsymbol{\alpha}_{-i}, s_i) - T_i^{g_i}(\hat{\alpha}_i, \boldsymbol{\alpha}_{-i}, s_i)) - c] - x_i(\hat{\alpha}_i, \boldsymbol{\alpha}_{-i})\}. \tag{1}$$

Standard arguments based on the envelope theorem (cf. Theorem 2 in Milgrom and Segal (2002)) lead to the following result.

Lemma 1. With public **s,** first stage IC leads to the following expression for the seller's expected revenue:

$$ER = E_{\boldsymbol{\alpha}} \sum_g \left\{ A^g(\boldsymbol{\alpha}) \left[\begin{array}{c} E_{\mathbf{s}} \left[\sum_{i \in g} p_i^g(\boldsymbol{\alpha}, \mathbf{s}^g) \left(u(\alpha_i, s_i) - \frac{1-F(\alpha_i)}{f(\alpha_i)} u_1(\alpha_i, s_i) \right) \right] \\ -|g|c \end{array} \right] \right\} - \sum_{i=1}^N \pi_i(\underline{\alpha}, \underline{\alpha}). \tag{2}$$

To maximize ER, the seller sets $\pi_i(\underline{\alpha}, \underline{\alpha}) = 0$ for all $i = 1, 2, ..., N$; i.e., no rent should be given to the buyer with the lowest possible (pre-entry) type.

Define the *virtual value adjusted by the second-stage signal* as follows:

$$w(\alpha_i, s_i) = u(\alpha_i, s_i) - \frac{1 - F(\alpha_i)}{f(\alpha_i)} u_1(\alpha_i, s_i). \tag{3}$$

Given the revealed $\boldsymbol{\alpha}$ and the shortlisted group g, $\forall \mathbf{s}^g$, we set[20]

$$p_i^{*g}(\boldsymbol{\alpha}, \mathbf{s}^g) = \begin{cases} 1 & \text{if } i = \arg\max_{j \in g}\{w(\alpha_j, s_j)\} \text{ and } w(\alpha_i, s_i) \geq 0, \\ 0 & \text{otherwise,} \end{cases} \quad \forall g, \forall i \in g. \tag{4}$$

Define the expected *virtual surplus* (the virtual value less the entry cost) as follows:

$$w^{*g}(\boldsymbol{\alpha}) = E_{\mathbf{s}} \left[\sum_{i \in g} p_i^{*g}(\boldsymbol{\alpha}, \mathbf{s}^g) w(\alpha_i, s_i) - |g|c \right]. \tag{5}$$

[20] Ties occur with probability zero and are hence ignored.

Given the revealed $\boldsymbol{\alpha}$, we set the first-stage shortlisting rule as follows:[21]

$$A^{*g}(\boldsymbol{\alpha}) = \begin{cases} 1 & \text{if } g = \arg\max_{\tilde{g}}\{w^{*\tilde{g}}(\boldsymbol{\alpha})\} \text{ and } w^{*g}(\boldsymbol{\alpha}) \geq 0, \\ 0 & \text{otherwise,} \end{cases} \forall g. \tag{6}$$

Let

$$ER^* = E_{\boldsymbol{\alpha}} \sum_g \left\{ A^{*g}(\boldsymbol{\alpha}) E_{\mathbf{s}} \left[\sum_{i \in g} p_i^{*g}(\boldsymbol{\alpha}, \mathbf{s}^g) w(\alpha_i, s_i) - |g|c \right] \right\}. \tag{7}$$

It is clear that ER^* provides an upper bound for the expected revenue in the original setting.

3.3 Revenue-Maximizing Mechanisms in The Original Setting

In this section, we will establish that ER^*can be achieved by a feasible mechanism (satisfying IC and IR in both stages) in the original setting.

To this end, we will first establish necessary conditions implied by IC conditions in both stages. We start with the second stage. Suppose group g is shortlisted, and the profile $\tilde{\boldsymbol{\alpha}}$ reported in the first stage is revealed as public information to the shortlisted bidders.

First, suppose $\boldsymbol{\alpha}$ is truthfully reported at the first stage and group g is shortlisted. Assume that they follow the recommendation and incur the information acquisition cost c to discover $\mathbf{s}^g$.[22]

The bidder i's second-stage interim expected payoff when she observes s_i but reports $\hat{s}_i$ is as follows:

$$\tilde{\pi}_i^g(\boldsymbol{\alpha}; s_i, \hat{s}_i) = E_{\mathbf{s}_{-i}^g}[u(\alpha_i, s_i) p_i^g(\boldsymbol{\alpha}, \hat{s}_i, \mathbf{s}_{-i}^g) - t_i^g(\boldsymbol{\alpha}, \hat{s}_i, \mathbf{s}_{-i}^g)] = u(\alpha_i, s_i) P_i^g(\boldsymbol{\alpha}, \hat{s}_i) - T_i^g(\boldsymbol{\alpha}, \hat{s}_i).$$

The second-stage incentive compatibility (IC) conditions require that

$$\tilde{\pi}_i^g(\boldsymbol{\alpha}; s_i, \hat{s}_i) \leq \tilde{\pi}_i^g(\boldsymbol{\alpha}; s_i, s_i), \forall g, \boldsymbol{\alpha}, s_i, \hat{s}_i. \tag{8}$$

It is standard in the traditional screening literature that when group g is shortlisted upon on truthful revelation of $\boldsymbol{\alpha}$, the second stage mechanism is IC if and only if $P_i^g(\boldsymbol{\alpha}, s_i), \forall i \in g$ is increasing.

According to Lemma 2 in Liu et al. (2020), in an incentive-compatible two-stage mechanism, a buyer with first stage type α_i who reported $\hat{\alpha}_i$ and observed signal s_i in the second stage will report

[21] Again ties occur with probability zero and are hence ignored.

[22] As will be shown, the equilibrium expected profit from going forward is positive for a buyer upon entry, so in equilibrium, a bidder does have an incentive to follow the recommendation to acquire (costly) information and participate in the final auction once admitted (as dropping out only results in zero profit).

$\hat{s}_i = \sigma_i(\alpha_i, \hat{\alpha}_i, s_i)$ such that: If $\hat{\alpha}_i < \alpha_i$, then

$$\sigma_i(\alpha_i, \hat{\alpha}_i, s_i) = \begin{cases} 1, \text{ if } u(\alpha_i, s_i) > u(\hat{\alpha}_i, 1), \\ \tilde{\sigma}_i(\alpha_i, \hat{\alpha}_i, s_i), \text{ otherwise.} \end{cases} \tag{9}$$

If $\hat{\alpha}_i > \alpha_i$, then

$$\sigma_i(\alpha_i, \hat{\alpha}_i, s_i) = \begin{cases} 0, \text{ if } \quad u(\alpha_i, s_i) < u(\hat{\alpha}_i, 0), \\ \tilde{\sigma}_i(\alpha_i, \hat{\alpha}_i, s_i), \text{ otherwise.} \end{cases} \tag{10}$$

Here, $\tilde{\sigma}_i(\alpha_i, \hat{\alpha}_i, s_i) \in [0, 1]$ is first defined by Esö and Szentes (2007), which is the unique signal such that $u(\alpha_i, s_i) = u(\hat{\alpha}_i, \tilde{\sigma}_i(\alpha_i, \hat{\alpha}_i, s_i))$ when $u(\alpha_i, s_i) \in [u(\hat{\alpha}_i, 0), u(\hat{\alpha}_i, 1)]$. Reporting $\tilde{\sigma}_i(\alpha_i, \hat{\alpha}_i, s_i)$ after a lie $\hat{\alpha}_i$ is equivalent to revealing v_i truthfully regardless of the first-stage report. The optimality of this strategy has been established in general for Markov environments by Pavan, Segal, and Toikka (2014). Our two-stage setting resembles the Markov environment defined in Pavan, Segal, and Toikka since the agents' payoffs depend only on their second-stage true types (v_i's) and the allocation outcome, but not on their first-stage true types. However, in our setting, due to shifting supports, it is not always possible to fully correct the first stage lies. Whenever it is impossible, our lie correction strategy specified in (9) and (10) requires the buyer to correct her first-stage lie to the possible greatest extent.

Note that $\hat{s}_i$ does not depend on $\boldsymbol{\alpha}_{-i}$, g, or $\mathbf{s}^g_{-i}$. Define

$$\begin{aligned} \tilde{\pi}^g_i(\boldsymbol{\alpha}, \hat{\alpha}_i; s_i, \hat{s}_i) &= E_{\mathbf{s}^g_{-i}}[u(\alpha_i, s_i) p^g_i(\hat{\alpha}_i, \boldsymbol{\alpha}_{-i}, \hat{s}_i, \mathbf{s}^g_{-i}) - t^g_i(\hat{\alpha}_i, \boldsymbol{\alpha}_{-i}, \hat{s}_i, \mathbf{s}^g_{-i})] \\ &= u(\alpha_i, s_i) P^g_i(\hat{\alpha}_i, \boldsymbol{\alpha}_{-i}, \hat{s}_i) - T^g_i(\hat{\alpha}_i, \boldsymbol{\alpha}_{-i}, \hat{s}_i); \\ \tilde{\pi}^g_i(\alpha_i, \hat{\alpha}_i; \boldsymbol{\alpha}_{-i}) &= E_{s_i} \tilde{\pi}^g_i(\boldsymbol{\alpha}, \hat{\alpha}_i; s_i, \hat{s}_i = \sigma_i(\alpha_i, \hat{\alpha}_i, s_i)). \end{aligned}$$

$\tilde{\pi}^g_i(\alpha_i, \hat{\alpha}_i; \boldsymbol{\alpha}_{-i})$ is the expected second-stage payoff for the type-α_i bidder if she reported $\hat{\alpha}_i$ in the first stage (and everyone else reported truthfully) given her opponents' types being $\boldsymbol{\alpha}_{-i}$.

We are now ready to consider the implication of first-stage IC, provided that second-stage IC holds upon the first-stage truthful revelation. Let $\pi_i(\alpha_i, \hat{\alpha}_i)$ be the expected payoff (net of the entry cost) for a type-α_i bidder who reports $\hat{\alpha}_i$ in the first stage. By the lie correction strategy, we have

$$\begin{aligned} \pi_i(\alpha_i, \hat{\alpha}_i) &= E_{\boldsymbol{\alpha}_{-i}} \left\{ \sum_{g_i} A^{g_i}(\hat{\alpha}_i, \boldsymbol{\alpha}_{-i}) [\tilde{\pi}^{g_i}_i(\alpha_i, \hat{\alpha}_i; \boldsymbol{\alpha}_{-i}) - c] - x_i(\hat{\alpha}_i, \boldsymbol{\alpha}_{-i}) \right\} \\ &= E_{\boldsymbol{\alpha}_{-i}} \left\{ \sum_{g_i} A^{g_i}(\hat{\alpha}_i, \boldsymbol{\alpha}_{-i}) \left[E_{s_i} \tilde{\pi}^{g_i}_i(\boldsymbol{\alpha}, \hat{\alpha}_i; s_i, \hat{s}_i = \sigma_i(\alpha_i, \hat{\alpha}_i, s_i)) - c \right] \right\} - x_i(\hat{\alpha}_i), \end{aligned} \tag{11}$$

where $\hat{s}_i = \sigma_i(\alpha_i, \hat{\alpha}_i, s_i)$ and $x_i(\hat{\alpha}_i) = E_{\boldsymbol{\alpha}_{-i}} x_i(\hat{\alpha}_i, \boldsymbol{\alpha}_{-i})$. The first-stage IC requires $\pi_i(\alpha_i, \alpha_i) \geq \pi_i(\alpha_i, \hat{\alpha}_i)$.

Lemma 2. If the two-stage mechanism is incentive compatible, then buyer i's expected payoff (as a function of her pre-entry type) can be expressed as

$$ER = E_{\boldsymbol{\alpha}} \sum_g \left\{ A^g(\boldsymbol{\alpha}) \left[\begin{array}{c} E_{\mathbf{s}} \left[\sum_{i \in g} p_i^g(\boldsymbol{\alpha}, \mathbf{s}^g) \left(u(\alpha_i, s_i) - \frac{1-F(\alpha_i)}{f(\alpha_i)} u_1(\alpha_i, s_i) \right) \right] \\ -|g|c \end{array} \right] \right\} - \sum_{i=1}^{N} \pi_i(\underline{\alpha}, \underline{\alpha}), \tag{12}$$

which coincides with the expected revenue with public $\mathbf{s}$ given by (2).

If allocation rule $(A^{*g}(\boldsymbol{\alpha}), p_i^{*g}(\boldsymbol{\alpha}, \mathbf{s}^g))$ defined in (6) and (4) can be truthfully implemented by some appropriately defined payment rule $(x_i^{*g}(\boldsymbol{\alpha}), t_i^{*g}(\boldsymbol{\alpha}, \mathbf{s}^g))$, under which $\pi_i(\underline{\alpha}, \underline{\alpha}) = 0$, then the revenue bound ER^* in (7) can be achieved.

We will proceed to show such payment rule $(x_i^{*g}(\boldsymbol{\alpha}), t_i^{*g}(\boldsymbol{\alpha}, \mathbf{s}^g))$ exists. Under Assumptions 1 and 2, we can establish the following properties of the optimal second-stage allocation rule:

Corollary 1. (i) $p_i^{*g_i}(\boldsymbol{\alpha}, \mathbf{s}^{g_i})$ increases in both α_i and s_i, $\forall i \in g_i$, $\forall g_i$, $\boldsymbol{\alpha}_{-i}$, and $\mathbf{s}_{-i}^{g_i}$, which implies that $P_i^{*g_i}(\alpha_i, \boldsymbol{\alpha}_{-i}, s_i)$ increases in both α_i and s_i, $\forall g_i$, $\boldsymbol{\alpha}_{-i}$; (ii) If $\alpha_i > \hat{\alpha}_i$, $s_i < \hat{s}_i$ and $u(\alpha_i, s_i) \geq u(\hat{\alpha}_i, \hat{s}_i)$, then $p_i^{*g_i}(\alpha_i, \boldsymbol{\alpha}_{-i}, s_i, \mathbf{s}_{-i}^{g_i}) \geq p_i^{*g_i}(\hat{\alpha}_i, \boldsymbol{\alpha}_{-i}, \hat{s}_i, \mathbf{s}_{-i}^{g_i})$, which implies $P_i^{*g_i}(\alpha_i, \boldsymbol{\alpha}_{-i}, s_i) \geq P_i^{*g_i}(\hat{\alpha}_i, \boldsymbol{\alpha}_{-i}, \hat{s}_i), \forall g_i$, $\boldsymbol{\alpha}_{-i}$.

Property (ii) above suggests that whenever $\alpha_i > \hat{\alpha}_i$, $s_i < \hat{s}_i$ and $u(\alpha_i, s_i) \geq u(\hat{\alpha}_i, \hat{s}_i)$, the allocation rule $p_i^{*g_i}(\boldsymbol{\alpha}, \mathbf{s}^{g_i})$ favors the "truth-telling" pair (α_i, s_i). Esö and Szentes (2007) first establish these properties for the case of $u(\alpha_i, s_i) = u(\hat{\alpha}_i, \hat{s}_i)$ in an environment with common support. Liu et. al. (2020) extend these properties to the environment with moving support.

Given α_i, let $s(\alpha_i)$ be defined such that $w(\alpha_i, s(\alpha_i)) = 0$. To identify properties of the shortlisting rule $A^{*g}(\boldsymbol{\alpha})$, we define a truncated random variable as follows:

$$w_i^+(\alpha_i, s_i) = \begin{cases} w(\alpha_i, s_i) & \text{if } w(\alpha_i, s_i) \geq 0 \text{ or equivalently } s_i \geq s(\alpha_i) \\ 0 & \text{otherwise} \end{cases} \forall i.$$

Note that conditional on $\boldsymbol{\alpha}$, w_i^+'s are independent across $i \in g$.

Let $\Delta \tilde{S}^g(\alpha_i; \boldsymbol{\alpha}_{-i})$ denote buyer i's marginal contribution to the expected virtual value, $i \in g$, then

$$\Delta \tilde{S}^g(\alpha_i; \boldsymbol{\alpha}_{-i}) = \tilde{S}(\boldsymbol{\alpha}^g) - \tilde{S}(\boldsymbol{\alpha}_{-i}^g), i \in g, \forall \boldsymbol{\alpha}^g,$$

where $\boldsymbol{\alpha}_{-i}^g = \boldsymbol{\alpha}^g \backslash \{\alpha_i\}$ and

$$\tilde{S}(\boldsymbol{\alpha}^g) = E_{\mathbf{s}^g} \max_{i \in g} \{w_i^+(\alpha_i, s_i)\}, \forall g, \forall \boldsymbol{\alpha}^g.$$

Given our regularity assumption, the following two properties are obvious:

(1) $\Delta\tilde{S}^g(\alpha_i;\boldsymbol{\alpha}_{-i})$ increases with α_i, and decreases with $\alpha_j, \forall j \neq i, \forall i \in g, \forall g$.

(2) $\Delta\tilde{S}^g(\alpha_i;\boldsymbol{\alpha}_{-i}) \geq \Delta\tilde{S}^{g'}(\alpha_i;\boldsymbol{\alpha}_{-i})$, $\forall \boldsymbol{\alpha}_{-i}, \forall i \in g, \forall g \subset g'$.

The optimal shortlisting rule can be alternatively described as follows. Given $\boldsymbol{\alpha}$, we can rank all α_is from the highest to the lowest. The seller admits bidders one by one in descending order of α_i's as long as the bidder's marginal contribution to the expected virtual value is greater than c, i.e.

$$\Delta\tilde{S}^g(\alpha_i;\boldsymbol{\alpha}_{-i}) = \tilde{S}(\boldsymbol{\alpha}^g) - \tilde{S}(\boldsymbol{\alpha}^g_{-i}) \geq c,$$

where g denotes the group of bidders with the highest $|g|$ types before entry.

The optimal shortlisting rule (6) described above resembles the greedy algorithm proposed by Chade and Smith (2006). While Chade and Smith consider simultaneous search with complete information, our setting is more complicated as the seller's payoffs depend on bidders' pre-entry types which are private information. Taking into account incentive compatibility, the payoff function in our setting should be defined in terms of the expected virtual surplus $\left(\Delta\tilde{S}^g(\alpha_i;\boldsymbol{\alpha}_{-i}) - c\right)$. It is easily verified that given the regularity assumption, the expected virtual surplus satisfies the *downward recursive* (DR) condition (with respect to the sets of shortlisted bidders) as defined in Chade and Smith. As such, it is not surprising that our optimal shortlisting coincides with the greedy algorithm.[23]

Let $g^*(\alpha)$ denote the set of bidders admitted under the optimal shortlisting rule. Two properties of shortlisting rule A^{*g} thus follow:

Corollary 2. (i) Given $\boldsymbol{\alpha}_{-i}$, if bidder i with α_i is shortlisted, then she would also be shortlisted with a higher type $\tilde{\alpha}_i(> \alpha_i)$; (ii) Given $\boldsymbol{\alpha}_{-i}$, bidder i will be shortlisted as long as α_i is higher than a threshold $\hat{\alpha}_i(\boldsymbol{\alpha}_{-i})$. As α_i increases, the shortlisted group weakly shrinks. As α_i increases from $\hat{\alpha}_i(\boldsymbol{\alpha}_{-i})$, the bidders in $g^*(\boldsymbol{\alpha})\backslash\{i\}$ would be excluded one by one (with the lowest type originally shortlisted being excluded first).

We are now ready to show that the optimal final good allocation and entry right allocation rules (4) and (6) are truthfully implementable by some well constructed payment rules in both stages.

Note that $u(\alpha_i, s_i)$ increases with s_i and by Assumption 1, $u_1(\alpha_i, s_i)$ (weakly) decreases with s_i. This implies that $w(\alpha_i, s_i)$ increases with s_i. By the final good allocation rule (4), the winning probability $P_i^{*g}(\boldsymbol{\alpha}, s_i)$ is weakly increasing in s_i. By Lemma 2 in Myerson (1981), the second-stage mechanism is incentive compatible (given $\boldsymbol{\alpha}$ and g). Thus, given the truthfully revealed $\boldsymbol{\alpha}$ and shortlisted group g, a second-stage payment rule, say, $t_i^{*g}(\boldsymbol{\alpha}, \mathbf{s}^g), \forall i \in g, \forall g$, can be constructed to truthfully implement the second-stage allocation rule $p_i^{*g}(\boldsymbol{\alpha}, \mathbf{s}^g), \forall i \in g, \forall g$ while maintaining the second-stage IR constraints (to participate in the second-stage mechanism), i.e. $\tilde{\pi}_i^g(\boldsymbol{\alpha}, \alpha_i; s_i, s_i) \geq 0$ on equilibrium path.

[23] The greedy algorithm has nice computational properties and is also employed by Milgrom and Segal (2020) in their analysis of incentive auctions.

We use $\tilde{\pi}_i^{*g_i}(\alpha_i, \hat{\alpha}_i; \boldsymbol{\alpha}_{-i})$ to denote the second-stage expected payoff to buyer i of type α_i if she announces $\hat{\alpha}_i$ and is shortlisted in group g_i, given that everyone else announces $\boldsymbol{\alpha}_{-i}$ truthfully at the first stage. Construct the first-stage payment rule as follows:

$$\begin{aligned} x_i^*(\boldsymbol{\alpha}) &= \sum_{g_i} A^{*g_i}(\alpha_i, \boldsymbol{\alpha}_{-i})[\tilde{\pi}_i^{*g_i}(\alpha_i, \alpha_i; \boldsymbol{\alpha}_{-i}) - c] \\ &\quad - \int_{\underline{\alpha}}^{\alpha_i} \int u_1(y, s_i) \cdot \sum_{g_i} \left[E_{\boldsymbol{\alpha}_{-i}} A^{*g_i}(y, \boldsymbol{\alpha}_{-i}) P_i^{*g_i}(y, \boldsymbol{\alpha}_{-i}, s_i)\right] dG_i(s_i) dy \end{aligned} \tag{13}$$

Note that by (11), we have

$$\pi_i^*(\alpha_i, \alpha_i) = E_{\boldsymbol{\alpha}_{-i}} \left\{ \sum_{g_i} A^{*g_i}(\alpha_i, \boldsymbol{\alpha}_{-i})[\tilde{\pi}_i^{*g_i}(\alpha_i, \alpha_i; \boldsymbol{\alpha}_{-i}) - c] - x_i^*(\alpha_i, \boldsymbol{\alpha}_{-i}) \right\}. \tag{14}$$

Substituting (13) into (14), we can verify that

$$\pi_i^*(\alpha_i, \alpha_i) = \int_{\underline{\alpha}}^{\alpha_i} \int u_1(y, s_i) \cdot \sum_{g_i} \left[E_{\boldsymbol{\alpha}_{-i}} A^{*g_i}(y, \boldsymbol{\alpha}_{-i}) P_i^{*g_i}(y, \boldsymbol{\alpha}_{-i}, s_i)\right] dG_i(s_i) dy. \tag{15}$$

Note that we have $\pi_i^*(\alpha_i, \alpha_i) \geq 0$. In particular, $\pi_i^*(\underline{\alpha}, \underline{\alpha}) = 0$.

Proposition 1. Under Assumptions 1 and 2, together with the payment rule $(x_i^*(\boldsymbol{\alpha}),\ t_i^{*g}(\boldsymbol{\alpha}, \mathbf{s}^g))$ defined above, the optimal final good allocation and entry right allocation rule $(A^{*g}(\boldsymbol{\alpha}), p_i^{*g}(\boldsymbol{\alpha}, \mathbf{s}^g))$ described in (4) and (6) are IR and IC implementable. Moreover, $\pi_i^*(\underline{\alpha}, \underline{\alpha}) = 0$.

Proof. See Appendix. □

In the discussions prior to the proposition, we have established the second-stage IC and IR given the first-stage truthful revelation and IR. We have also shown $\pi_i^*(\underline{\alpha}, \underline{\alpha}) = 0$. To fully establish Proposition 1, we only need to show the first-stage IC under payment rules x_i^* and t_i^{*g}. Our proof of the first-stage IC crucially relies on the properties of allocation rule $(A^{*g}(\boldsymbol{\alpha}), p_i^{*g}(\boldsymbol{\alpha}, \mathbf{s}^g))$, which are stated in Corollaries 1 and 2. In particular, Corollary 2 shows that if a buyer i over-reports her first-stage type, she will more likely be shortlisted, and more likely be shortlisted in a smaller group; Corollary 1 shows that buyer i's lie correction strategy of (9) and (10) would let her win with a higher chance in the second stage even if she is shortlisted in the same group. These properties together imply that over-reporting increases players' overall winning chances. Recall that in the single stage setting of Myerson (1981), the monotonicity of winning chances is sufficient for IC. Our first stage problem resembles that of Myerson (1981). The fact that over-reporting increases players' overall winning chances similarly leads to the first-stage IC in our setting.

It is worth noting that Assumptions 1 and 2 are sufficient but not necessary for the optimal shortlisting rule to be truthfully implementable: the necessary and sufficient condition is that Δ

defined in (26) is non-positive, which is also the *integral monotonicity* condition characterized by Pavan, Segal, and Toikka (2014).

Proposition 1 reveals that allocation rule $(A^{*g}(\boldsymbol{\alpha}), p_i^{*g}(\boldsymbol{\alpha}, \mathbf{s}^g))$ and payment rule $(x_i^{*g}(\boldsymbol{\alpha}), t_i^{*g}(\boldsymbol{\alpha}, \mathbf{s}^g))$ constitute a feasible (both IC and IR) two-stage mechanism which also ensures $\pi_i(\underline{\alpha}, \underline{\alpha}) = 0$. Clearly, by (12) this mechanism achieves the revenue bound ER^* in (7). Therefore, these rules constitute the revenue-maximizing two-stage mechanism in the original setting.

Proposition 2. Under Assumptions 1 and 2, allocation rule $(A^{*g}(\boldsymbol{\alpha}), p_i^{*g}(\boldsymbol{\alpha}, \mathbf{s}^g))$ and payment rule $(x_i^{*g}(\boldsymbol{\alpha}), t_i^{*g}(\boldsymbol{\alpha}, \mathbf{s}^g))$ constitute the revenue-maximizing two-stage selling mechanism in the original setting, which achieves revenue bound ER^* in (7).

So consistent with the results identified by Esö and Szentes, the asset should be awarded to the bidder with the highest non-negative virtual value adjusted by the second-stage signal, which is a generalization of the optimal allocation rule in Myerson (1981). Our analysis thus shows that the generalized Myerson allocation rule is robust to settings with costly entry, which affects the final allocation only through its effect on the entry right allocation rule.

The optimal shortlisting rule $A^{*g}(\boldsymbol{\alpha})$ admits the set of bidders that gives rise to the maximal expected virtual surplus. Alternatively, given our regularity assumption, the optimal shortlisting rule admits the bidders in descending order of their marginal contribution to the expected virtual surplus – the bidder with the highest contribution first, the bidder with the second-highest contribution second, etc. – provided that their marginal contribution is positive. As a result, the seller must shortlist a group of bidders with the highest first-stage types for information acquisition.

Quint and Hendricks (2018) study how information acquisition of bidders can be coordinated by indicative bidding in our two-stage setting. By restricting indicative bids to a finite discrete domain, Quint and Hendricks (2018) show that a symmetric equilibrium exists in weakly-monotone strategies. However, since bidder first-stage types "pool" over a finite number of bids, the highest-type bidders are not always selected for information acquisition in the first stage. Without safely selecting the most qualified bidders for the final sale, their mechanism is thus not optimal in maximizing expected revenue.

In our preceding analysis of the revenue-maximizing two-stage mechanism, we focus on the revelation policy whereby first-stage reports are fully revealed to the shortlisted bidders. Given this particular revelation policy, one concern is that there might be some loss of generality in identifying optimal mechanisms. However, Proposition 2 shows that our derived mechanism with this disclosure policy achieves ER^*, the revenue upper bound from the relaxed setting with public $\mathbf{s}$. In this relaxed setting, we drop the IC and IR constraints for the shortlisted bidders in the second stage (all shortlisted bidders must incur entry costs to learn their second-stage signals and regardless of their second-stage signals, they must participate in the second-stage mechanism and report truthfully

their second-stage signals). Therefore, regardless of how to disclose the first-stage reports, the highest possible expected revenue achievable in this relaxed setting (i.e. ER^*) imposes an upper bound for the expected revenue that can be obtained in our original setting, where the bidders' second-stage IC and IR constraints must both be satisfied. We thus have the following result:

Corollary 3. In maximizing expected revenue in our setting, there is no loss of generality to assume that first-stage reports are fully revealed to the shortlisted bidders.

Basically, we demonstrate that the virtual surplus in our original setting is the same as in the relaxed setting. This observation goes back to Eso and Szentes (2007), and it is used also by Pavan, Segal, and Toikka (2014) to derive the dynamic envelope formula more generally, which immediately yields their expression for dynamic virtual surplus.[24] So that observation per se is by now standard. The question is just whether the allocation rule that maximizes the virtual surplus in the relaxed setting is implementable in the original setting. Since this is verified, the bidders do not get any information rent on their new orthogonalized information, and the principal's payoff is the same as in the relaxed setting, which is (weakly) greater than that in any setting where the agents have to report their second-stage types. This is true irrespective of what the agents may learn about the first-stage reports of the other agents.

3.4 Applications

Our optimal mechanism analysis is general enough to encompass many existing models in the literature on auctions with costly entry. Below we demonstrate how we can apply our general optimal mechanism to special models previously studied.

1. Bidders do not have pre-entry types and only learn about their values after entry (e.g., McAfee and McMillan, 1987; Tan, 1992; and Levin and Smith, 1994). In this case, $u(\alpha_i, s_i) = s_i$. Hence $w(\alpha_i, s_i) = s_i$, which implies that the optimal auction is *ex post* efficient, and the optimal entry is to select a set of bidders that results in the maximal expected social surplus. Since bidders are identical before entry, optimal entry is entirely characterized by n^*, the optimal number of bidders to be selected. The implementation is somewhat simple: the second round is a standard auction (first-price, second-price, or English auction). The first round (entry stage) is to select exactly n^* bidders, and whomever selected is required to pay an upfront entry fee e^*, which is set so that no rent is left for the entrants *ex ante*.

2. Bidders know their values before entry, and entry is merely a bid preparation process (without value updating) (e.g. Samuelson,1985; Stegeman, 1996; Campbell, 1998; Menezes and Monteiro,

[24] Eso and Szentes (2017) offer a general irrelevance result on this. They show that the maximal expected revenue is the same as if the principal could observe the agent's orthogonalized types after the initial period. In this sense, the dynamic nature is irrelevant: the agent only receives information rents for her initial private information.

2000; Tan and Yilankaya, 2006; Cao and Tian, 2009; and Lu, 2009). In this setting, $u(\alpha_i, s_i) = \alpha_i$ and $s_i \equiv 0$,[25] and hence $w(\alpha_i, s_i) = \alpha_i - (1 - F(\alpha_i))/f(\alpha_i)$. It is easily verified that according to Proposition 1, the optimal allocation rules can be described as follows: the bidder with the highest "type" (α_i) is admitted as the sole entrant to win the item, as long as her contribution to the virtual surplus $w(\alpha_i, s_i) - c$ is positive.

3. Each bidder is endowed with pre-entry type α_i, and learns an additional private value component s_i (e.g., Ye, 2007; Quint and Hendricks, 2018). The total value is given by $u(\alpha_i, s_i) = \alpha_i + s_i$. We assume that α_i is distributed uniformly over $[0, 1]$ and s_i follows an arbitrary continuous distribution, and let $c \in (0, 1)$. Hence $w(\alpha_i, s_i) = \alpha_i + s_i - (1 - F(\alpha_i))/f(\alpha_i) = 2\alpha_i + s_i - 1$. The optimal second-stage allocation rule thus requires that the asset be allocated to the entrant bidder with the highest virtual value $w(\alpha_i, s_i)$ provided that it is nonnegative. The optimal entry rule requires that bidders be admitted in descending order of their pre-entry types, as long as their contribution to the expected virtual surplus is nonnegative. If only one buyer (the one with the highest type $\alpha_{(1)}$) is admitted, the expected virtual surplus is given by $w_1 = E_{s_{i_1}}\{(2\alpha_{(1)} + s_{i_1} - 1) \vee 0\} - c$.[26] So the optimal number of entrants $n^* \geq 1$ if $w_1 \geq 0$. If two top buyers are admitted, the expected virtual surplus is given by

$$w_2 = E_{(s_{i_1}, s_{i_2})}\left[\max\left\{2\alpha_{(1)} + s_{i_1} - 1, 2\alpha_{(2)} + s_{i_2} - 1, 0\right\}\right] - 2c.$$

So the optimal number of entrants $n^* \geq 2$ if the incremental expected virtual surplus

$$\begin{aligned} & w_2 - w_1 \\ = \; & E_{(s_{i_1}, s_{i_2})}\left[\max\left\{2\alpha_{(1)} + s_{i_1} - 1, 2\alpha_{(2)} + s_{i_2} - 1, 0\right\}\right] - E_{s(1)}\{(2\alpha_{(1)} + s_{i_1} - 1) \vee 0\} \geqslant c. \end{aligned}$$

Continuing this procedure of calculation, it can be verified that $n^* \geq n$ if $w_n - w_{n-1} \geqslant 0$, or

$$\left\{ \begin{array}{c} E_{(s_{i_1}, s_{i_2}, \ldots, s_{i_n})}\left[(\max_{k=1,\ldots,n}\{2\alpha_{(k)} + s_{i_k} - 1\}) \vee 0\right] \\ -E_{(s_{i_1}, s_{i_2}, \ldots, s_{i_{n-1}})}\left[(\max_{k=1,\ldots,n-1}\{2\alpha_{(k)} + s_{i_k} - 1\}) \vee 0\right] \end{array} \right\} \geq c. \tag{16}$$

4 Analysis with Sequential Shortlisting

Now we move to the setting where the seller may conduct sequential shortlisting, which consists of M (≥ 2) shortlisting rounds/stages and one final allocation stage $M + 1$. Specifically, the procedure is described as follows.[27]

[25] Therefore, there is no issue of reporting s_i in the second stage.

[26] Let i_k be the index of bidder who possesses α_k, the kth highest value of α.

[27] We again use the null message ϕ to denote a buyer's report if she is not required to report her s_i.

At stage 1, all bidders report their initial types of α_i. Denote the reports by $\mathbf{m}_1 = (m_{1,i})$ where $m_{1,i} \in [\underline{\alpha}, \overline{\alpha}]$, $\forall i \in \mathbf{N}$ is i's report. Let $g_0 = \varnothing$. $\forall g_1 \subset 2^{\mathbf{N}}$, the probability that g_1 is shortlisted is $A^{g_1}(\mathbf{m}_1|g_0)$. The payment of $i \in \mathbf{N}$ is $t_{1,i}(\mathbf{m}_1)$. All bidders in shortlisted group g_1 incur their information acquisition cost c to discover their ex post values, v_i, $i \in g_1$.

At stage 2, if group g_1 is shortlisted and discover their values v's at stage 1, they are asked to report their s_i's. The bidders' second-stage reports about s_i's are denoted by $\mathbf{m}_2 = (m_{2,i})$ where $m_{2,i} \in [0,1]$ if $i \in g_1$ and $m_{2,i} = \phi$ if $i \notin g_1$. $\forall g_2 \subset 2^{\mathbf{N}\setminus g_1}$, the probability for g_2 to be shortlisted is $A^{g_2}(\mathbf{m}_1, \mathbf{m}_2|g_0, g_1)$ and i's payment is $t_{2,i}(\mathbf{m}_1, \mathbf{m}_2)$ for any $i \in \mathbf{N}$. All bidders in shortlisted group g_2 incur their information acquisition cost c to discover their ex post values, v_i, $i \in g_2$.

At stage 3, if group $g_2 \in 2^{\Omega\setminus g_1}$ is shortlisted and discover their values v_i's at stage 2, they are asked to report their s_i's. The bidders' stage-3 reports about s_i's are denoted by $\mathbf{m}_3 = (m_{3,i})$ where $m_{3,i} \in [0,1]$ if $i \in g_2$ and $m_{3,i} = \phi$ if $i \notin g_2$. $\forall g_3 \subset 2^{\mathbf{N}\setminus(g_1 \cup g_2)}$, the probability for g_3 to be shortlisted is $A^{g_3}(\mathbf{m}_1, \mathbf{m}_2, \mathbf{m}_3|g_0, g_1, g_2)$, and i's payment is $t_{3,i}(\mathbf{m}_1, \mathbf{m}_2, \mathbf{m}_3)$ for any $i \in \mathbf{N}$. All bidders in shortlisted group g_3 incur their information acquisition cost c to discover their ex post values, v_i, $i \in g_3$.

The procedure proceeds analogously up to stage M. At stage M, if group g_{M-1} is shortlisted and the bidders in g_{M-1}discover their values v_i's at stage $M-1$, they are asked to report their s_i's. The bidders' stage-M reports about s_i's are denoted by $\mathbf{m}_M = (m_{M,i})$ where $m_{M,i} \in [0,1]$ if $i \in g_{M-1}$ and $m_{M,i} = \phi$ if $i \notin g_{M-1}$. $\forall g_M \subset 2^{\mathbf{N}\setminus(\cup_{j=1}^{M-1} g_j)}$, the probability for g_M to be shortlisted is $A^{g_M}(\mathbf{m}_1, \mathbf{m}_2, ..., \mathbf{m}_M|g_0, g_1, g_2, ..., g_{M-1})$, and i's payment is $t_{M,i}(\mathbf{m}_1, \mathbf{m}_2, ..., \mathbf{m}_M)$ for any $i \in \mathbf{N}$. All bidders in shortlisted group g_M incur their information acquisition cost c to discover their ex post values, v_i, $i \in g_M$.

At the final stage, i.e. stage $M+1$, if g_M is shortlisted and discover their values v_is at the end of stage M, their reports are denoted by $\mathbf{m}_{M+1} = (m_{M+1,i})$ where $m_{M+1,i} \in [0,1]$ if $i \in g_M$; $m_{M+1,i} = \phi$ if $i \notin g_M$. Denote the sequence of shortlisting outcome by vector $\mathbf{g} = (g_1, g_2, ..., g_M)$ with $|\mathbf{g}| = M$. Let $G_{\mathbf{g}}$ denote the set of all agents shortlisted in sequence $\mathbf{g}$. Given the final shortlisted group $G_{\mathbf{g}}$, let $p_i^{G_{\mathbf{g}}}(\mathbf{m}_1, \mathbf{m}_2, ..., \mathbf{m}_{M+1})$ be agent i's winning probability, for agent $i \in G_{\mathbf{g}}$,[28] and agent i's payment is $t_{M+1,i}(\mathbf{m}_1, \mathbf{m}_2, ..., \mathbf{m}_M, \mathbf{m}_{M+1})$ for any $i \in \mathbf{N}$.

We use $\{\mathbf{A}, \mathbf{p}, \mathbf{t}, M\}$ to denote the procedure specified above. Without loss of generality, we can focus on the cases where $M \geq N$. The mechanisms with $\tilde{M}(< N)$ can be trivially duplicated by a mechanism with $M = N$, in which case the shortlisting stops at stage $\tilde{M}$.

Our analysis proceeds as in Section 3. We first consider a relaxed environment where the agents are only endowed with private information $\boldsymbol{\alpha}$, where s_i's become public once they are discovered. The optimal solution for this relaxed environment provides an upper bound for the seller's expected

[28] We will show that in equilibrium, the agents who are shortlisted have incentives to incur their information acquisition costs to discover their values.

revenue in the original environment where the discovered s_i's are private information for the shortlisted bidders. We will establish that this upper bound is actually achievable in the original environment.

4.1 The Relaxed Environment

For a given mechanism $\{\mathbf{A}, \mathbf{p}, \mathbf{t}, M\}$, and message sequence $(\mathbf{m}_k, k = 1, 2, ..., M)$, the probability of a shortlisting outcome $\mathbf{g} = (g_1, g_2, ..., g_M)$ is given by

$$\Pr(\mathbf{g}|(\mathbf{m}_i)_{i=1}^M) = \Pi_{k=1}^M A^{g_k}(\mathbf{m}_1, \mathbf{m}_2, ..., \mathbf{m}_k | g_0, g_1, g_2, ..., g_{k-1}).$$

As s_i becomes public once discovered in the relaxed environment, we have for $k \geq 2$, $m_{k,i} = s_i$, $i \in g_{k-1}$,and $m_{k,i} = \phi$, $i \notin g_{k-1}$. We use $\mathbf{m}_k^s$, $k \geq 2$ to denote these true types from stages 2 to $M+1$. Define $\Pr(\mathbf{g}|\boldsymbol{\alpha}, \mathbf{s}) = \Pr(\mathbf{g}|\boldsymbol{\alpha}, \mathbf{m}_2^{\mathbf{s}}, ..., \mathbf{m}_{M+1}^{\mathbf{s}})$, and for any $G \in 2^N$, define

$$\Pr(G|\boldsymbol{\alpha}, \mathbf{s}) = \sum_{\forall \mathbf{g} \text{ s.t. } G_{\mathbf{g}} = G} \Pr(\mathbf{g}|\boldsymbol{\alpha}, \mathbf{s}),$$

where, as before, $G_{\mathbf{g}}$ denotes the set of all agents shortlisted in sequence $\mathbf{g}$. Following standard procedure, we can establish the following lemma:

Lemma 3. For any $\{\Pr(G), \forall G \in 2^{\Omega}\}$ derived from any shortlisting rule, to maximize the expected revenue ER, the seller sets $\pi_i(\underline{\alpha}, \underline{\alpha}) = 0$ and allocates the object to the shortlisted bidder whose virtual value is the highest, provided that it is positive. Ties are randomly broken. In this case, the expected revenue is given by

$$ER = E_{\boldsymbol{\alpha}} E_{\mathbf{s}} \left\{ \sum_{G \in 2^{\mathbf{N}}} \Pr(G|\boldsymbol{\alpha}, \mathbf{s}) [\max\{w_i^+(\alpha_i, s_i)\}_{i \in G} - \sum_{i \in G} c] \right\}.$$

Proof. See Appendix. □

4.1.1 Optimal Shortlisting

We now turn to the optimal sequential shortlisting rule. Before searching for the optimal shortlisting rule, we first establish the following lemma:

Lemma 4. There is no loss of generality to consider shortlisting rules, under which the only possible set of player(s) shortlisted at each stage is the single agent who has the highest α_i among the remaining bidders before the shortlisting process is completed.

Proof. See Appendix. □

In the proof, we first show that there is no loss of generality to consider shortlisting rules, under which any possible set of players shortlisted at each stage must be a singleton before the shortlisting process is completed. We then show that we can focus on the rules where at each stage the seller either shortlists an agent with probability 1 or stops shortlisting. Finally, we demonstrate that the agent shortlisted should be the one with the highest α_i among the remaining bidders. Based on Lemma 4, we have the following optimal shortlisting rule.

Proposition 3. Without loss of generality, we assume α_i decreases with i. $\forall \mathbf{s}$, for $k = 1, 2, ..., N$, bidder k is shortlisted if and only if $E_{s_k}[(w_k^+(\alpha_k, s_k) - \max\{w_l^+(\alpha_l, s_l)\}_{1\leq l<k}) \vee 0] \geq c$. If bidder k is not shortlisted in stage k, then no buyer is shortlisted in subsequent stages.

In stage 1, the seller considers whether he should shortlist agent 1. Recall the expression for expected revenue in Lemma 3. If he does not shortlist agent 1, then the shortlisting process ends and he gets 0 revenue. If he shortlists agent 1 and simply ends the process, then his expected revenue is $E_{s_1}[w_1^+(\alpha_1, s_1)] - c$. If he shortlists agent 1 and follows the optimal shortlisting strategy after that, his expected revenue can be higher than $E_{s_1}[w_1^+(\alpha_1, s_1)] - c$. This means that the seller should definitely shortlist agent 1 in stage 1 if $E_{s_1}[w_1^+(\alpha_1, s_1)] \geq c$. If $E_{s_1}[w_1^+(\alpha_1, s_1)] < c$, then we must have $E_{s_k}[(w_k^+(\alpha_k, s_k) - \max\{w_l^+(\alpha_l, s_l)\}_{1\leq l<k}) \vee 0] < c, \forall k = 2, ..., N$. This means that for $k = N, N-1, ..., 2$, even if the first $k-1$ bidders are shortlisted, the next bidder should not be shortlisted. This further implies that agent 1 should not be shortlisted in stage 1.

In stage 2, suppose agent 1 is shortlisted in stage 1, and s_1 is revealed. If the seller does not shortlist agent 2, he gets a revenue of $w_1^+(\alpha_1, s_1) - c$. If he shortlists agent 2 and simply ends the process, then his expected revenue is $E_{s_2} \max\{w_i^+(\alpha_i, s_i)\}_{i\in\{1,2\}} - \sum_{i\in\{1,2\}} c$. We have $E_{s_2} \max\{w_i^+(\alpha_i, s_i)\}_{i\in\{1,2\}} - \sum_{i\in\{1,2\}} c \geq w_1^+(\alpha_1, s_1) - c$ if and only if $E_{s_2}[(w_2^+(\alpha_2, s_2) - w_1^+(\alpha_1, s_1)) \vee 0] \geq c$. If he shortlists agent 2 and follows the optimal shortlisting strategy after that, his expected revenue can be higher than $E_{s_2} \max\{w_i^+(\alpha_i, s_i)\}_{i\in\{1,2\}} - \sum_{i\in\{1,2\}} c$. This means that the seller should definitely shortlist agent 2 in stage 2 if $E_{s_2}[(w_2^+(\alpha_2, s_2) - w_1^+(\alpha_1, s_1)) \vee 0] \geq c$. If $E_{s_2}[(w_2^+(\alpha_2, s_2) - w_1^+(\alpha_1, s_1)) \vee 0] < c$, then we must have $E_{s_k}[(w_k^+(\alpha_k, s_k) - \max\{w_l^+(\alpha_l, s_l)\}_{1\leq l<k}) \vee 0] < c, \forall k = 3, ..., N$. This means that for $k = N, N-1, ..., 3$, even if the first $k-1$ bidders are shortlisted, the next bidder should not be shortlisted. This further implies that agent 2 should not be shortlisted in stage 2. The reasoning for other stages is similar.

4.2 Incentive Compatibility in The Original Setting

Following the same procedure as in Section 3, we next verify that the allocation rules identified in the relaxed setting are implementable. We use $(\hat{\boldsymbol{\alpha}}, \mathbf{m}_2, ..., \mathbf{m}_{M+1})$ to denote the announcements of agents at different stages, $\mathbf{A}^* = \{A^{*g_k}(\hat{\boldsymbol{\alpha}}, \mathbf{m}_2, ..., \mathbf{m}_{k-1}; g_1, g_2, ..., g_{k-1}),\ k = 1, 2, ..., M,\ \forall \mathbf{g} = (g_1, g_2, ..., g_M)\}$ to denote the shortlisting rule described in Proposition 3, and $\mathbf{p}^* = \{p_i^{*G_\mathbf{g}}(\hat{\boldsymbol{\alpha}}, \mathbf{m}_2, ..., \mathbf{m}_{M+1}), i \in \mathbf{N},$

$\forall \mathbf{g} = (g_1, g_2, ..., g_M)\}$ to denote the allocation rule specified in Lemma (3). In addition,

$$\Pr{}^*(\mathbf{g}|(\mathbf{m}_i)_{i=1}^M) = \Pi_{k=1}^M A^{*g_k}(\mathbf{m}_1, \mathbf{m}_2, ..., \mathbf{m}_k | g_0, g_1, g_2, ..., g_{k-1}),$$

which is the probability that sequence $\mathbf{g}$ is shortlisted given messages reported $(\mathbf{m}_i)_{i=1}^M$.

Then $(\mathbf{A}^*, \mathbf{p}^*)$ imposes an upper bound for seller expected revenue in the relaxed environment where s_i's are public information. While we assume truthful reporting in stage 1 when deriving these rules, the stage-1 incentive compatibility has not yet been established in the relaxed environment. In this section, we will establish that these rules are indeed incentive compatible in the original environment where both (α_i, s_i) are private information of agent i. For this purpose, we will construct payment rules $\mathbf{t}^*$ that together with $(\mathbf{A}^*, \mathbf{p}^*)$ induce truthful information revelation on the equilibrium path. In addition, we will ensure that $\pi_i(\underline{\alpha}, \underline{\alpha}) = 0, \forall i$.

Proposition 4. Under Assumptions 1 and 2, the optimal shortlisting and final good allocation rules $(\mathbf{A}^*, \mathbf{p}^*)$ are IR and IC implementable.

Proof. See Appendix. □

Note that by reporting a higher first-stage signal, a bidder would be shortlisted with a higher probability, and would be shortlisted in a smaller group with a higher probability, given $(\boldsymbol{\alpha}_{-i}, \mathbf{s})$. Given any shortlisted group, reporting a higher first-stage signal and correcting the lie later would raise the bidder's virtual value, and thus the winning probability. We thus conclude that reporting a higher first-stage type leads to a higher winning probability at the end. This suggests that the monotonicity of the winning probability, which is typically required for incentive compatibility, is also satisfied.

Given Proposition 4, we establish that the proposed mechanism $(\mathbf{A}^*, \mathbf{p}^*, \mathbf{t}^*)$ achieves the revenue bound identified in the relaxed environment in Section 4.1

Proposition 5. Mechanism $(\mathbf{A}^*, \mathbf{p}^*, \mathbf{t}^*)$ generates the same expected revenue as in the relaxed environment of Section 4.1, in which the bidders' second-stage additional information is public.

This result is clear since at the optima of both settings, the shortlisting rule and the object allocation rule are exactly the same, and the expected payoffs for bidders with the lowest type are set to be zero. This implies that the total expected surplus in both environments is the same. In addition, given (28) and (38) in the Appendix, a bidder with the same type enjoys the same expected payoff across the two scenarios. Since the expected revenue is the difference between the total expected surplus and expected bidder payoff, the seller revenue must be the same in the two environments.

By Proposition 5, we conclude that mechanism $(\mathbf{A}^*, \mathbf{p}^*, \mathbf{t}^*)$ must be optimal when there is $M = N$ stages of shortlisting. Any $M(>= N)$ would induce the same expected revenue at the optimum, and

any $M(< N)$ is dominated by $M = N$. We can thus set the optimal number of shortlisting stages $M^* = N$.

5 Extensions and Numerical Examples

5.1 Allowing Direct Sale in The First Stage

In our previous analysis we model information acquisition as entry, in the sense that a bidder is not allowed to bid without going through the "due diligence" process. This assumption is due to the specific institutional setup we are trying to model.[29] Given the complexity and high-stakes nature of the sale, it is unlikely that a seller would be comfortable accepting a bid from someone who did not go through such an important information acquisition process. On the other hand, buyers may not be willing to bid in the dark either, in particular due to the potential shareholder lawsuits if the investment does not turn out well. As such, we believe that it is appropriate to model information acquisition as mandatory for making a bid in our environment. Nevertheless, from theoretical perspectives, it would be interesting to identify optimal mechanisms in environments where bidders are allowed to bid without having to go through information acquisition. In this subsection, we allow for sale in the first stage, before bidders go through due diligence to acquire more information about the object for sale. So some bidders may submit their final bids based on their prior estimates only (without incurring information acquisition costs), and the sale may be terminated as long as some first-stage bid is accepted.

5.1.1 Analysis with Single-round Shortlisting

The two-stage mechanism is the same as specified in Section 3 except that in the first stage a buyer i obtains the object with $p_i(\boldsymbol{\alpha})$, $i = 1, 2, ..., N$, before a group of buyers is shortlisted for information acquisition. If the object is unsold in the first stage, the shortlisting rule, $A^g : [\underline{\alpha}, \overline{\alpha}]^N \to [0, 1]$, assigns a probability to each subgroup $g \in 2^{\mathbf{N}}$ for information acquisition. The payment rule $x_i : [\underline{\alpha}, \overline{\alpha}]^N \to \mathbb{R}$, specifies bidder i's first-stage payment given the reported profile $\boldsymbol{\alpha}$.

Given the first-stage reported profile $\boldsymbol{\alpha}$, and that group g is shortlisted for information acquisition, the second-stage mechanism is characterized by $p_i^g(\boldsymbol{\alpha}, \mathbf{s}^g)$, the probability with which the asset is allocated to buyer $i \in g$, and $t_i^g(\boldsymbol{\alpha}, \mathbf{s}^g)$, the payment to the seller made by buyer $i \in g, \forall g \in 2^{\mathbf{N}}$.

In Section A of the online appendix, we fully characterize the following revenue-maximizing two-stage mechanism allowing sale in the first stage. In the second stage, given the revealed $\boldsymbol{\alpha}$ and the

[29] In a typical electrical generating asset sale in the US as described by Vallen and Bullinger (1999), before submitting a final bid, each bidder (more precisely, bidding team) usually needs to go through the due diligence process to meet with senior management and personnel, study equipment conditions and operating history, evaluate supply contracts and employment agreements, etc. This process is strictly controlled and closely monitored by the auctioneer (typically an investment banker serving as the financial advisor for the selling party).

shortlisted group g, $\forall \mathbf{s}^g$, $p_i^{*g}(\boldsymbol{\alpha}, \mathbf{s}^g)$ takes the same form as in (4):

$$p_i^{*g}(\boldsymbol{\alpha}, \mathbf{s}^g) = \begin{cases} 1 & \text{if } i = \arg\max_{j \in g}\{w(\alpha_j, s_j)\} \text{ and } w(\alpha_i, s_i) \geq 0 \\ 0 & \text{otherwise} \end{cases} \forall g, \forall i \in g.$$

Recall that the expected *virtual surplus* $w^{*g}(\boldsymbol{\alpha})$ (the virtual value less the entry cost) is defined in (5). At the first stage, given the revealed $\boldsymbol{\alpha}$, the optimal shortlisting rule is specified in (6):

$$A^{*g}(\boldsymbol{\alpha}) = \begin{cases} 1 & \text{if } g = \arg\max_{\tilde{g}}\{w^{*\tilde{g}}(\boldsymbol{\alpha})\} \text{ and } w^{*g}(\boldsymbol{\alpha}) \geq 0 \\ 0 & \text{otherwise} \end{cases} \forall g.$$

Recall that $g^*(\boldsymbol{\alpha})$ denotes the set of bidders admitted under the optimal shortlisting rule. The highest expected revenue generated from the second-stage sale is

$$R_2^{g^*(\boldsymbol{\alpha})}(\boldsymbol{\alpha}) = E_{\mathbf{s}}\left[\sum_{i \in g^*(\boldsymbol{\alpha})} p_i^{g^*(\boldsymbol{\alpha})}(\boldsymbol{\alpha}, \mathbf{s}^{g^*(\boldsymbol{\alpha})})\left(u(\alpha_i, s_i) - \frac{1 - F(\alpha_i)}{f(\alpha_i)} u_1(\alpha_i, s_i)\right) - |g^*(\boldsymbol{\alpha})|c\right],$$

and the highest expected revenue generated from the first-stage sale is

$$R_1^*(\boldsymbol{\alpha}) = E_s\left(u(\alpha_{(1)}, s) - \frac{1 - F(\alpha_{(1)})}{f(\alpha_{(1)})} u_1(\alpha_{(1)}, s)\right),$$

where $\alpha_{(1)}$ denotes the highest first-stage type among all buyers, and s is uniformly distributed over $[0, 1]$.

Clearly, the optimal first-stage selling probability is given by:

$$p_i^*(\boldsymbol{\alpha}) = \begin{cases} 1 & \text{if } \alpha_i \geq \alpha_j, \forall j \text{ and } R_1^*(\boldsymbol{\alpha}) \geq R_2^{g^*(\boldsymbol{\alpha})}(\boldsymbol{\alpha}), \\ 0 & \text{otherwise,} \end{cases} \forall i.$$

In other words, given first-stage type profile $\boldsymbol{\alpha}$, the object is sold in the first stage if and only if by doing so it generates higher expected revenue than that generated from the first-stage optimal shortlisting and second-stage optimal selling mechanism.

Allocation rule $(p_i^{*g}(\boldsymbol{\alpha}, \mathbf{s}^g),\ A^{*g}(\boldsymbol{\alpha}),\ p_i^*(\boldsymbol{\alpha}))$ together with properly specified payment rules give rise to the following expected revenue:

$$ER^{**} = E_{\boldsymbol{\alpha}}\left\{\begin{array}{c} \sum_i [p_i^*(\boldsymbol{\alpha}) E_{s_i} w(\alpha_i, s_i)] + \\ [1 - \sum_i p_i^*(\boldsymbol{\alpha})] \sum_g A^{*g}(\boldsymbol{\alpha}) E_{\mathbf{s}}\left[\sum_{i \in g} p_i^{*g}(\boldsymbol{\alpha}, \mathbf{s}^g) w(\alpha_i, s_i) - |g|c\right] \end{array}\right\}. \tag{17}$$

5.1.2 Analysis with Multi-round Shortlisting

Now we allow for sequential shortlisting. The mechanism is specified the same way as in Section 4 except that a first-stage selling probability $p_i(\mathbf{m}_1)$ is also specified, with $\sum_{i\in\mathbf{N}} p_i(\mathbf{m}_1) \leq 1$; and only if the object is unsold in the first stage, each subgroup $g_1 \in 2^{\mathbf{N}}$ would be shortlisted with probability $A^{g_1}(\mathbf{m}_1|g_0)$ for information acquisition, with $\sum_{g\in 2^{\mathbf{N}}} A^g(\mathbf{m}_1|g_0) = 1$. We will continue to use the same notation as in Section 4.

In Section B of the online appendix, we fully characterize the following revenue-maximizing multi-stage mechanism allowing first-stage sale. The sequential shortlisting rule and the final-stage selling rule remain the same as in Section 4.

We define

$$R_2^*(\boldsymbol{\alpha}) = E_{\mathbf{s}}\left[\sum_{G\in 2^{\mathbf{N}}} Pr^*(G|\boldsymbol{\alpha},\mathbf{s})\left(\max\{w_i^+(\alpha_i,s_i)\}_{i\in G} - \sum_{i\in G} c\right)\right], \tag{18}$$

where $\Pr^*(G|\boldsymbol{\alpha},\mathbf{s})$ denotes the probability that the set of bidders G is admitted under the optimal shortlisting rule described in Proposition 3.

The first stage selling rule is as follows:

$$\tilde{p}_i^*(\boldsymbol{\alpha}) = \begin{cases} 1 & \text{if } \alpha_i \geq \alpha_j, \forall j \text{ and } R_1^*(\boldsymbol{\alpha}) \geq R_2^*(\boldsymbol{\alpha}) \\ 0 & \text{otherwise} \end{cases} \forall i.$$

In other words, given first-stage type profile $\boldsymbol{\alpha}$, the object is sold in the first stage if and only if by doing so the expected revenue generated is higher than that from the optimal sequential shortlisting and final-stage optimal selling mechanism.

Clearly, the revenue-maximizing mechanism gives rise to the following expected revenue:

$$ER^{***} = E_{\boldsymbol{\alpha}}\left\{\begin{array}{c} \sum_i \tilde{p}_i^*(\boldsymbol{\alpha}) E_{s_i} w(\alpha_i, s_i) \\ +\left[1-\sum_i \tilde{p}_i^*(\boldsymbol{\alpha})\right] E_{\mathbf{s}}\left[\sum_{G\in 2^{\mathbf{N}}} Pr^*(G|\boldsymbol{\alpha},\mathbf{s})\left(\max\{w_i^+(\alpha_i,s_i)\}_{i\in G} - \sum_{i\in G} c\right)\right] \end{array}\right\}. \tag{19}$$

5.2 Revenue Comparisons: Numerical Examples

We present numerical examples in this section to compare expected revenues among different mechanisms. First, we follow Quint and Hendricks (2018) to consider the following example: α_i is uniformly distributed over $[0, 100]$, s_i follows exponential distribution with $\lambda = 0.12$, and entry cost $c = 5$. We numerically compute the expected revenue for the following cases: 1) unrestricted entry, where bidders make their own entry decisions simultaneously and independently;[30] 2) indicative bidding as cheap

[30] In equilibrium, bidders enter the auction if and only if their pre-entry types are above some entry threshold.

talk as analyzed by Quint and Hendricks (2018); 3) two-stage mechanisms analyzed by Ye (2007); 4) single-round shortlisting without allowing sale in the first stage; 5) sequential shortlisting without allowing sale in the first stage; 6) single-round shortlisting allowing sale in the first stage; 7) sequential shortlisting allowing sale in the first stage. Our simulations produce expected revenues for all these mechanisms for the following varying numbers of potential bidders: $N = 3, 4, 5, 7, 10, 20, 50, 200$. Details are reported in Table 1 below:

Table 1: $\alpha_i \sim U[0, 100]$, $s_i \sim exp(0.12)$, and $c = 5$

Potential bidders (N)	3	4	5	7	10	20	50	200
Unrestricted Entry	50.04	58.24	63.58	70.15	75.65	83.22	89.27	93.28
Quint and Hendricks (2018)	50.50	59.52	65.63	73.42	79.98	88.77	95.20	99.50
Ye (2007)	50.897	60.165	65.720	73.632	80.806	90.632	97.226	101.048
Two-Stage w/o 1st stage sale	56.114	64.227	70.724	78.559	85.309	93.812	99.400	102.341
Multi-Stage w/o 1st stage sale	56.123	64.242	70.747	78.590	85.351	93.900	99.592	102.838
Two-Stage with 1st stage sale	60.599	68.976	75.599	83.530	90.306	98.812	104.400	107.342
Multi-Stage with 1st stage sale	60.599	68.976	75.599	83.530	90.306	98.812	104.400	107.346

As is clear from the table, the comparison is quite intuitive, which is consistent with our expectation. First, compared to Ye (2007) and Quint and Hendricks (2018), our current optimal mechanism design approach results in higher expected revenue; Second, single-round shortlisting is revenue-dominated by sequential shortlisting. Note, however, the difference between these two mechanisms is insignificant in our example, which renders one justification for the widespread use of two-stage auctions; Third, allowing sale in the first stage further improves expected revenue.

Also note that when the first-stage sale is allowed, single-round shortlisting and sequential shortlisting generate basically the same expected revenue when N is small ($N \leq 20$ in our simulations). The reason is that when N is small, $\alpha_{(1)}$ differs much from $\alpha_{(2)}$, which invalidates the need for shortlisting. Since sale occurs in the first stage, sequential shortlisting does not improves the revenue. When N is large, say, $N \geq 50$, $\alpha_{(1)}$ and $\alpha_{(2)}$ are closer, hence shortlisting is desirable, which leads to the superiority of sequential shortlisting over single-round shortlisting, although in this example the difference continues to be quite small.

Intuitively, we might expect a shortlisting procedure to be more frequently used in real settings when information updating is substantial after entry, or when the information acquisition cost is not too big. To examine this intuition, we next focus on the revenue comparison between single-round shortlisting and sequential shortlisting, both when the first-stage sale is allowed and not allowed. Table 2 is obtained by fixing $\alpha_i \sim U[0, 100]$, $s_i \sim exp(\lambda)$, $c = 5$, and varying λ ($\lambda = 0.06, 0.12, 0.24$). Table 3 is obtained by fixing $\alpha_i \sim U[0, 100]$, $s_i \sim exp(0.12)$, and varying information acquisition cost c ($c = 1, 2.5, 5$).

Since the expected value of S is $1/\lambda$, a lower λ implies more substantial post-entry value updating. As is clear from Table 2, when first-stage sale is not allowed, sequential shortlisting revenue-dominates single-round shortlisting when $\lambda = .06$ and 0.12, but not when $\lambda = 0.24$: when post-entry value

Table 2: $\alpha_i \sim U[0, 100]$, $s_i \sim exp(\lambda)$, and $c = 5$

	N	3	4	5	7	10	20	50	200
Two-Stage w/o 1st stage sale	$\lambda = 0.06$	63.757	72.680	78.464	87.498	93.792	103.138	109.592	113.717
	$\lambda = 0.12$	56.114	64.227	70.724	78.559	85.309	93.812	99.400	102.342
	$\lambda = 0.24$	52.182	60.353	66.546	74.400	81.011	89.703	95.228	98.167
Multi-Stage w/o 1st stage sale	$\lambda = 0.06$	63.915	72.929	78.755	87.894	94.382	104.250	111.508	117.181
	$\lambda = 0.12$	56.123	64.242	70.747	78.590	85.351	93.900	99.592	102.838
	$\lambda = 0.24$	52.182	60.353	66.546	74.400	81.011	89.703	95.228	98.167
Two-Stage with 1st stage sale	$\lambda = 0.06$	68.139	77.229	83.064	92.085	98.200	107.109	112.740	115.673
	$\lambda = 0.12$	60.599	68.976	75.600	83.530	90.306	98.812	104.400	107.342
	$\lambda = 0.24$	56.597	65.061	71.398	79.365	86.007	94.703	100.228	103.167
Multi-Stage with 1st stage sale	$\lambda = 0.06$	68.186	77.319	83.160	92.221	98.449	107.723	114.253	119.318
	$\lambda = 0.12$	60.599	68.976	75.599	83.530	90.306	98.812	104.400	107.346
	$\lambda = 0.24$	56.597	65.061	71.398	79.365	86.007	94.703	100.228	103.167

updating is not substantial, allowing sequential shortlisting does not improve sale revenue. When first-stage sale is allowed, sequential shortlisting revenue-dominates single-round shortlisting only when $\lambda = .06$; when $\lambda = 0.12$ and $\lambda = 0.24$, sales occur in the first stage and shortlisting does not help.

Table 3: $\alpha_i \sim U[0, 100]$ and $s_i \sim exp(0.12)$

	N	3	4	5	7	10	20	50	200
Two-Stage w/o 1st stage sale	$c = 5$	56.114	64.227	70.724	78.559	85.309	93.811	99.400	102.342
	$c = 2.5$	58.547	66.697	73.620	80.797	87.723	96.677	102.553	106.108
	$c = 1$	60.408	68.691	74.809	82.658	89.822	99.045	105.949	111.330
Multi-Stage w/o 1st stage sale	$c = 5$	56.123	64.242	70.747	78.590	85.351	93.900	99.592	102.838
	$c = 2.5$	58.583	66.752	73.694	80.904	87.888	96.963	103.177	107.520
	$c = 1$	60.435	68.729	74.854	82.724	89.942	99.269	106.524	112.672
Two-Stage with 1st stage sale	$c = 5$	60.599	68.976	75.599	83.530	90.306	98.812	104.400	107.342
	$c = 2.5$	60.739	69.005	75.983	83.178	90.068	98.890	104.442	107.341
	$c = 1$	61.202	69.486	75.598	83.394	90.462	99.457	106.055	111.330
Multi-Stage with 1st stage sale	$c = 5$	60.599	68.976	75.599	83.530	90.306	98.812	104.400	107.346
	$c = 2.5$	60.751	69.023	76.009	83.213	90.128	99.014	104.827	108.681
	$c = 1$	61.241	69.544	75.667	83.499	90.637	99.789	106.817	112.857

As is clear from Table 3, when first-stage sale is not allowed, sequential shortlisting revenue-dominates single-round shortlisting for all the cases $c = 1, 2.5, 5$. When first-stage sale is allowed, sequential shortlisting revenue-dominates single-round shortlisting only when $c = 1, 2.5$; when $c = 5$, sales occur in the first stage and shortlisting does not help. Our numerical example thus suggests that shortlisting is more likely to occur when c decreases.

6 Concluding Remarks

Our paper contributes to the literature on two fronts. First, it characterizes optimal mechanisms, either when shortlisting is restricted or unrestricted to a single round, for an environment of multi-stage auctions, which are commonly employed in sales of complicated and high-valued business assets, procurements, privatization, takeover, and merger and acquisition contests. Our analysis is general enough to nest many existing studies in the literature of auctions with costly entry. Second, our paper

contributes to the literature on sequential screening by introducing costly and endogenous information acquisition into a dynamic auction framework. Information acquisition makes the optimal mechanism design more challenging, as now it must balance bidders' information acquisition incentives and information elicitation in the final good allocation stage, which are interdependent.

Since single-round shortlisting can be trivially replicated by sequential shortlisting, the optimal two-stage mechanism characterized in Section 3 must be revenue-dominated by the optimal mechanism allowing for sequential shortlisting characterized in Section 4.[31] This is true when there is no time discounting. When time discounting is taken into account, however, an obvious drawback of running a multi-stage mechanism is the potential of delay, which would be too costly and therefore favors a more time-efficient two-stage mechanism. We believe that this consideration, along with the practical difficulty in administering multiple rounds of the due diligence process,[32] leads to the "norm" of the two-stage auction format widely used in the real world.

Implementation of the optimal mechanism characterized in this paper may face some practical obstacles. First, the industry may not be comfortable with the idea of paying entry fees before knowing the auction outcome, and this is the major reason, we believe, that contributes to the common use of nonbinding indicative bidding. Second, the optimal mechanism is so complicated that the industry bidders might face great difficulties in developing bidding strategies for different rounds (although such a concern is alleviated to some extent if professional or sophisticated experts are hired to help). For these reasons the nature of our analysis is primarily normative, offering a "market design" approach to guide a potential refinement of an extremely important transaction procedure widely used in the industry. Despite this limitation, our analysis does conform to the "norm" of business in at least two aspects. First, a defining feature of our optimal mechanism is the shortlisting rule, which is also central in the two-stage auction practices. Second, we demonstrate that the optimal number shortlisted is endogenously determined, which is also consistent with the fact that in real sales, the number of finalists is often not pre-determined.[33]

Our analysis offers a theoretical benchmark for evaluating various two-stage or multi-stage auctions currently used in the real world. The information structure modeled in this research has recently received attention not only from theorists but also from econometricians and empiricists. For example, Marmer, Shneyerov, and Xu (2013) and Gentry and Li (2014) have successfully proposed nonparametric specification tests on a so-called affiliated-signal (AS) model with entry, and Roberts and Sweeting (2013) estimate a parametric variant of the AS model using data on California timber auctions. The

[31] However, our examples in Section 5.2 illustrate that the revenue sacrificed by conducting a single round shortlisting can often be negligible.

[32] Just imagine, for example, the hassle of arranging multiple meetings with senior management.

[33] For example, in the sale of PGW (Philadelphia Gas Works), a recent application of two-stage auctions, a "smaller number" of firms were invited to submit final bids after the first round – although this number was neither pre-announced nor disclosed (CBS Phily, November 19, 2013, "Sell-off of Philadelphia's Natural Gas Utility Goes To Binding Bidding," by Mike Dunn).

affiliated-signal models can be regarded as a special case in the framework studied in our paper, and the optimal mechanism characterized in this paper may potentially serve as a calibration benchmark for counter-factual simulations for related empirical work to come.

7 Appendix

Proof of Lemma 2: The first-stage IC requires $\pi_i(\alpha_i, \alpha_i) \geq \pi_i(\alpha_i, \hat{\alpha}_i)$. Note that if we replace $\hat{s}_i$ by truth type s_i in (11), we still have $\pi_i(\alpha_i, \alpha_i) \geq \pi_i(\alpha_i, \hat{\alpha}_i)$, which leads to

$$\frac{d\pi_i(\alpha_i, \alpha_i)}{d\alpha_i} = \frac{\partial \pi_i(\alpha_i, \hat{\alpha}_i)}{\partial \alpha_i}|_{\hat{\alpha}_i = \alpha_i} = \int u_1(\alpha_i, s_i) \cdot \sum_{g_i} \left[E_{\boldsymbol{\alpha}_{-i}} A^{g_i}(\alpha_i, \boldsymbol{\alpha}_{-i}) P_i^{g_i}(\alpha_i, \boldsymbol{\alpha}_{-i}, s_i) \right] dG_i(s_i).$$

which leads to the following result: if the two-stage mechanism is incentive compatible, then buyer i's expected payoff (as a function of her pre-entry type) can be expressed as

$$\pi_i(\alpha_i, \alpha_i) = \pi_i(\underline{\alpha}, \underline{\alpha}) + \int_{\underline{\alpha}}^{\alpha_i} \int u_1(y, s_i) \cdot \sum_{g_i} \left[E_{\boldsymbol{\alpha}_{-i}} A^{g_i}(y, \boldsymbol{\alpha}_{-i}) P_i^{g_i}(y, \boldsymbol{\alpha}_{-i}, s_i) \right] dG_i(s_i) dy. \tag{20}$$

Thus

$$\sum_{i=1}^{N} E\pi_i(\alpha_i, \alpha_i) = \sum_{i=1}^{N} \pi_i(\underline{\alpha}, \underline{\alpha}) + E_{\boldsymbol{\alpha}} \left\{ \sum_g A^g(\boldsymbol{\alpha}) E_{\mathbf{s}} \left[\sum_{i \in g} p_i^g(\boldsymbol{\alpha}, \mathbf{s}^g) \frac{1 - F(\alpha_i)}{f(\alpha_i)} u_1(\alpha_i, s_i) \right] \right\}. \tag{21}$$

The total expected surplus from the two-stage mechanism is

$$TS = E_{\boldsymbol{\alpha}} \sum_g \left\{ A^g(\boldsymbol{\alpha}) E_{\mathbf{s}} \left[\sum_{i \in g} p_i^g(\boldsymbol{\alpha}, \mathbf{s}^g) u(\alpha_i, s_i) - |g| c \right] \right\}. \tag{22}$$

The seller's expected revenue is thus given by

$$\begin{aligned} ER &= TS - \sum_{i=1}^{N} E\pi_i(\alpha_i, \alpha_i) \\ &= E_{\boldsymbol{\alpha}} \sum_g \left\{ A^g(\boldsymbol{\alpha}) \left[\begin{array}{c} E_{\mathbf{s}} \left[\sum_{i \in g} p_i^g(\boldsymbol{\alpha}, \mathbf{s}^g) \left(u(\alpha_i, s_i) - \frac{1 - F(\alpha_i)}{f(\alpha_i)} u_1(\alpha_i, s_i) \right) \right] \\ -|g|c \end{array} \right] \right\} - \sum_{i=1}^{N} \pi_i(\underline{\alpha}, \underline{\alpha}), \end{aligned}$$

which coincides with the expected revenue with public $\mathbf{s}$ given by (2). □

Proof of Proposition 1: The second-stage IC and IR given the first-stage truthful revelation and IR have been established in the discussions prior to the proposition. We will next show the first-stage

IC under payment rule x_i^*, together with the second-stage payment rule t_i^{*g}.

Suppose that all buyers except i report their types $\boldsymbol{\alpha}_{-i}$ truthfully. Consider buyer i with α_i contemplating to misreport $\hat{\alpha}_i < \alpha_i$. The deviation payoff is

$$\Delta = \pi_i^*(\alpha_i, \hat{\alpha}_i) - \pi_i^*(\alpha_i, \alpha_i) = [\pi_i^*(\alpha_i, \hat{\alpha}_i) - \pi_i^*(\hat{\alpha}_i, \hat{\alpha}_i)] + [\pi_i^*(\hat{\alpha}_i, \hat{\alpha}_i) - \pi_i^*(\alpha_i, \alpha_i)].$$

Since (15) is satisfied by the construction of $x_i^*(\boldsymbol{\alpha})$, we have

$$\pi_i^*(\hat{\alpha}_i, \hat{\alpha}_i) - \pi_i^*(\alpha_i, \alpha_i) = -\int_{\hat{\alpha}_i}^{\alpha_i} \int u_1(y, s_i) \cdot \sum_{g_i} \left[E_{\boldsymbol{\alpha}_{-i}} A^{*g_i}(y, \boldsymbol{\alpha}_{-i}) P_i^{*g_i}(y, \boldsymbol{\alpha}_{-i}, s_i)\right] dG_i(s_i) dy.$$

Recall the definition of $\pi_i^*(\alpha_i, \hat{\alpha}_i)$ in (11) and that g_i denotes a group including bidder i. Note that $\frac{\partial \tilde{\pi}_i^{g_i}(\boldsymbol{\alpha}, \hat{\alpha}_i; s_i, \hat{s}_i = \sigma_i(\alpha_i, \hat{\alpha}_i, s_i))}{\partial \hat{s}_i} \frac{\partial \hat{s}_i}{\partial \alpha_i} = 0$ since either $\frac{\partial \tilde{\pi}_i^{g_i}(\boldsymbol{\alpha}, \hat{\alpha}_i; s_i, \hat{s}_i = \sigma_i(\alpha_i, \hat{\alpha}_i, s_i))}{\partial \hat{s}_i} = 0$ (interior $\hat{s}_i$) or $\frac{\partial \hat{s}_i}{\partial \alpha_i} = 0$ (boundary $\hat{s}_i$). This observation implies

$$\frac{d\tilde{\pi}_i^{g_i}(\boldsymbol{\alpha}, \hat{\alpha}_i; s_i, \sigma_i(\alpha_i, \hat{\alpha}_i, s_i))}{d\alpha_i} = \frac{\partial \tilde{\pi}_i^{g_i}(\boldsymbol{\alpha}, \hat{\alpha}_i; s_i, \hat{s}_i)}{\partial \alpha_i}|_{\hat{s}_i = \sigma_i(\alpha_i, \hat{\alpha}_i, s_i)} = u_1(\alpha_i, s_i) P_i^{g_i}(\hat{\alpha}_i, \boldsymbol{\alpha}_{-i}, \sigma_i(\alpha_i, \hat{\alpha}_i, s_i)).$$

We thus have

$$\frac{\partial \tilde{\pi}_i^{g_i}(\alpha_i, \hat{\alpha}_i; \boldsymbol{\alpha}_{-i})}{\partial \alpha_i} = E_{s_i} u_1(\alpha_i, s_i) P_i^{g_i}(\hat{\alpha}_i, \boldsymbol{\alpha}_{-i}, \sigma_i(\alpha_i, \hat{\alpha}_i, s_i)), \tag{23}$$

which leads to the following result.[34]

Suppose $\boldsymbol{\alpha}_{-i}$ is truthfully revealed from the first stage and the second-stage mechanism is incentive compatible given truthfully revealed $\boldsymbol{\alpha}$. If buyer i of type α_i who reported $\hat{\alpha}_i$ in the first stage is shortlisted in group g_i, her expected payoff from the second stage is given by

$$\tilde{\pi}_i^{g_i}(\alpha_i, \hat{\alpha}_i; \boldsymbol{\alpha}_{-i}) = \tilde{\pi}_i^{g_i}(\hat{\alpha}_i, \hat{\alpha}_i; \boldsymbol{\alpha}_{-i}) + \int \int_{\hat{\alpha}_i}^{\alpha_i} u_1(y, s_i) P_i^{g_i}(\hat{\alpha}_i, \boldsymbol{\alpha}_{-i}, \sigma_i(y, \hat{\alpha}_i, s_i)) dy dG_i(s_i). \tag{24}$$

Therefore, we have

$$\begin{aligned} & \pi_i^*(\alpha_i, \hat{\alpha}_i) - \pi_i^*(\hat{\alpha}_i, \hat{\alpha}_i) \\ = \; & E_{\boldsymbol{\alpha}_{-i}} \left\{ \sum_{g_i} A^{g_i}(\hat{\alpha}_i, \boldsymbol{\alpha}_{-i}) [\tilde{\pi}_i^{g_i}(\alpha_i, \hat{\alpha}_i; \boldsymbol{\alpha}_{-i}) - \tilde{\pi}_i^{g_i}(\hat{\alpha}_i, \hat{\alpha}_i; \boldsymbol{\alpha}_{-i})] \right\} \\ = \; & \int_{\hat{\alpha}_i}^{\alpha_i} \int u_1(y, s_i) \cdot \sum_{g_i} \left[E_{\boldsymbol{\alpha}_{-i}} A^{*g_i}(\hat{\alpha}_i, \boldsymbol{\alpha}_{-i}) P_i^{*g_i}(\hat{\alpha}_i, \boldsymbol{\alpha}_{-i}, \sigma_i(y, \hat{\alpha}_i, s_i))\right] dG_i(s_i) dy. \end{aligned} \tag{25}$$

[34] This procedure is followed by Liu et. al (2020) when establishing their Lemma 3.

Therefore, we have

$$\begin{aligned}\Delta &= \int_{\hat{\alpha}_i}^{\alpha_i} E_{\boldsymbol{\alpha}_{-i}} \sum_{g_i} \left[\begin{array}{c} A^{*g_i}(y, \boldsymbol{\alpha}_{-i}) \\ \cdot \int u_1(y, s_i)[P_i^{*g_i}(\hat{\alpha}_i, \boldsymbol{\alpha}_{-i}, \sigma_i(y, \hat{\alpha}_i, s_i)) - P_i^{*g_i}(y, \boldsymbol{\alpha}_{-i}, s_i)] dG_i(s_i) \end{array} \right] dy \\ &+ \int_{\hat{\alpha}_i}^{\alpha_i} E_{\boldsymbol{\alpha}_{-i}} \sum_{g_i} \left[\begin{array}{c} [A^{*g_i}(\hat{\alpha}_i, \boldsymbol{\alpha}_{-i}) - A^{*g_i}(y, \boldsymbol{\alpha}_{-i})] \\ \cdot \int u_1(y, s_i) P_i^{*g_i}(\hat{\alpha}_i, \boldsymbol{\alpha}_{-i}, \sigma_i(y, \hat{\alpha}_i, s_i)) dG_i(s_i) \end{array} \right] dy. \end{aligned} \tag{26}$$

From Corollary 1 (ii), we have $P_i^{*g_i}(\hat{\alpha}_i, \boldsymbol{\alpha}_{-i}, \sigma_i(y, \hat{\alpha}_i, s_i)) - P_i^{*g_i}(y, \boldsymbol{\alpha}_{-i}, s_i) \leq 0$, which implies that the first term in Δ is negative.

We now consider the second term in Δ when $y > \hat{\alpha}_i$. By Corollary 2, the optimal shortlisting rule implies that given $\boldsymbol{\alpha}_{-i}$, when buyer i is admitted with a higher α_i, she must be admitted to a group with a weakly smaller size. If y and $\hat{\alpha}_i$ are admitted in the same group, then $A^{*g_i}(\hat{\alpha}_i, \boldsymbol{\alpha}_{-i}) = A^{*g_i}(y, \boldsymbol{\alpha}_{-i})$ and this term in Δ is zero.

We now turn to the case where $g^*(\hat{\alpha}_i, \boldsymbol{\alpha}_{-i}) \supset g^*(y, \boldsymbol{\alpha}_{-i}) \supset \{i\}$. Note that $A^{*g_i}(\cdot, \boldsymbol{\alpha}_{-i})$ is 1 for the shortlisted group, and 0 for all other groups. Therefore,

$$\begin{aligned} & \sum_{g_i} [A^{*g_i}(\hat{\alpha}_i, \boldsymbol{\alpha}_{-i}) - A^{*g_i}(y, \boldsymbol{\alpha}_{-i})] u_1(y, s_i) P_i^{*g_i}(\hat{\alpha}_i, \boldsymbol{\alpha}_{-i}, \sigma_i(y, \hat{\alpha}_i, s_i)) \\ = \; & u_1(y, s_i) \left[P_i^{*g^*(\hat{\alpha}_i, \boldsymbol{\alpha}_{-i})}(\hat{\alpha}_i, \boldsymbol{\alpha}_{-i}, \sigma_i(y, \hat{\alpha}_i, s_i)) - P_i^{*g^*(y, \boldsymbol{\alpha}_{-i})}(\hat{\alpha}_i, \boldsymbol{\alpha}_{-i}, \sigma_i(y, \hat{\alpha}_i, s_i)) \right] \\ \leq \; & 0, \end{aligned}$$

which implies that the second term in Δ is negative. Since $g^*(\hat{\alpha}_i, \boldsymbol{\alpha}_{-i}) \supset g^*(y, \boldsymbol{\alpha}_{-i}) \supset \{i\}$, we must have $P_i^{*g^*(\hat{\alpha}_i, \boldsymbol{\alpha}_{-i})}(\hat{\alpha}_i, \boldsymbol{\alpha}_{-i}, \sigma_i(y, \hat{\alpha}_i, s_i)) \leq P_i^{*g^*(y, \boldsymbol{\alpha}_{-i})}(\hat{\alpha}_i, \boldsymbol{\alpha}_{-i}, \sigma_i(y, \hat{\alpha}_i, s_i))$, i.e. entrant i wins with a smaller probability if a strictly bigger group is shortlisted.

A similar argument can be used to rule out deviating to $\hat{\alpha}_i > \alpha_i$. $\square$

Proof of Lemma 3: Agent i's expected payoff when i is endowed with α_i but announces $\hat{\alpha}_i$ is given by:

$$\begin{aligned} & \pi_i(\alpha_i, \hat{\alpha}_i) \\ = \; & E_{\boldsymbol{\alpha}_{-i}} E_{\mathbf{s}} \left\{ \begin{array}{c} -\sum_{\forall \mathbf{g}} \left[\Pr(\mathbf{g} | (\hat{\alpha}_i, \boldsymbol{\alpha}_{-i}), \mathbf{m}_2^{\mathbf{s}}, ..., \mathbf{m}_{M+1}^{\mathbf{s}}) \sum_{k=1}^{M} t_{k+1,i}((\hat{\alpha}_i, \boldsymbol{\alpha}_{-i}), \mathbf{m}_2^{\mathbf{s}}, ..., \mathbf{m}_{k+1}^{\mathbf{s}}) \right] \\ + \sum_{\forall \mathbf{g} \text{ s.t. } i \in G_{\mathbf{g}}} \Pr(\mathbf{g} | (\hat{\alpha}_i, \boldsymbol{\alpha}_{-i}), \mathbf{m}_2^{\mathbf{s}}, ..., \mathbf{m}_{M+1}^{\mathbf{s}}) \left[u(\alpha_i, s_i) p_i^{G_{\mathbf{g}}}((\hat{\alpha}_i, \boldsymbol{\alpha}_{-i}), \mathbf{m}_2^{\mathbf{s}}, ..., \mathbf{m}_{M+1}^{\mathbf{s}}) - c \right] \end{array} \right\} \\ & - E_{\boldsymbol{\alpha}_{-i}} [t_{1,i}((\hat{\alpha}_i, \boldsymbol{\alpha}_{-\mathbf{i}}))]. \end{aligned}$$

Incentive compatibility together with the envelop theorem gives:

$$
\begin{aligned}
& \frac{d\pi_i(\alpha_i,\alpha_i)}{d\alpha_i} \\
= \; & E_{\boldsymbol{\alpha}_{-i}} E_{\mathbf{s}} \left\{ \sum_{\forall \mathbf{g} \text{ s.t. } i \in G_{\mathbf{g}}} \frac{\partial u(\alpha_i, s_i)}{\partial \alpha_i} \left[\begin{array}{c} \Pr\left(g|(\alpha_i, \boldsymbol{\alpha}_{-\mathbf{i}}), \mathbf{m}_2^{\mathbf{s}}, ..., \mathbf{m}_{M+1}^{\mathbf{s}}\right) \\ \cdot p_i^{G_{\mathbf{g}}}((\alpha_i, \boldsymbol{\alpha}_{-\mathbf{i}}), \mathbf{m}_2^{\mathbf{s}}, ..., \mathbf{m}_{M+1}^{\mathbf{s}}) \end{array} \right] \right\}. \qquad (27)
\end{aligned}
$$

Thus, we have

$$
\begin{aligned}
& \pi_i(\alpha_i,\alpha_i) = \pi_i(\underline{\alpha},\underline{\alpha}) \qquad (28) \\
& + E_{\boldsymbol{\alpha}_{-i}} \int_{\underline{\alpha}}^{\alpha_i} E_{\mathbf{s}} \left\{ \sum_{\forall \mathbf{g} \text{ s.t. } i \in G_{\mathbf{g}}} u_1(y, s_i) \left[\begin{array}{c} \Pr\left(g|(y, \boldsymbol{\alpha}_{-\mathbf{i}}), \mathbf{m}_2^{\mathbf{s}}, ..., \mathbf{m}_{M+1}^{\mathbf{s}}\right) \\ \cdot p_i^{G_{\mathbf{g}}}((y, \boldsymbol{\alpha}_{-\mathbf{i}}), \mathbf{m}_2^{\mathbf{s}}, ..., \mathbf{m}_{M+1}^{\mathbf{s}}) \end{array} \right] \right\} dy.
\end{aligned}
$$

The expected social surplus given $\boldsymbol{\alpha}$ is as follows:

$$
TS(\boldsymbol{\alpha}) = E_{\mathbf{s}} \left\{ \sum_{\forall \mathbf{g}} \left[\Pr(\mathbf{g}|\boldsymbol{\alpha}, \mathbf{m}_2^{\mathbf{s}}, ..., \mathbf{m}_{M+1}^{\mathbf{s}}) \sum_{i \in G_{\mathbf{g}}} \left(p_i^{G_{\mathbf{g}}}(\boldsymbol{\alpha}, \mathbf{m}_2^{\mathbf{s}}, ..., \mathbf{m}_{M+1}^{\mathbf{s}}) u(\alpha_i, s_i) - c \right) \right] \right\}.
$$

The seller seeks to maximize the expected revenue:

$$
ER = E_{\boldsymbol{\alpha}} \left[TS(\boldsymbol{\alpha}) - \sum_{i \in \mathbf{N}} \pi_i(\alpha_i, \alpha_i) \right].
$$

Note we have

$$
\sum_{\forall \mathbf{g} \text{ s.t. } i \in G_{\mathbf{g}}} \Pr(\mathbf{g}|\boldsymbol{\alpha}, \mathbf{s}) = \sum_{\forall G \text{ s.t. } i \in G} \Pr(G|\boldsymbol{\alpha}, \mathbf{s}).
$$

By the standard procedure, we can rewrite the seller's objective as follows:

$$
ER = E_{\boldsymbol{\alpha}} E_{\mathbf{s}} \sum_{\forall \mathbf{g}} \left[\begin{array}{c} \Pr\left(\mathbf{g}|\boldsymbol{\alpha}, \mathbf{m}_2^{\mathbf{s}}, ..., \mathbf{m}_{M+1}^{\mathbf{s}}\right) \\ \cdot \sum_{i \in G_{\mathbf{g}}} \left[p_i^{G_{\mathbf{g}}}(\boldsymbol{\alpha}, \mathbf{m}_2^{\mathbf{s}}, ..., \mathbf{m}_{M+1}^{\mathbf{s}}) w(\alpha_i, s_i) - c \right] \end{array} \right] - \sum_i \pi_i(\underline{\alpha},\underline{\alpha}),
$$

where $w(\alpha_i, s_i) = u(\alpha_i, s_i) - u_1(\alpha_i, s_i) \frac{1-F(\alpha_i)}{f(\alpha_i)}$.

Note that

$$
\begin{aligned}
& E_{\alpha_i}[\pi_i(\alpha_i, \alpha_i)] \\
= \; & \pi_i(\underline{\alpha}, \underline{\alpha}) + \\
& E_{\boldsymbol{\alpha}_{-i}} E_{\alpha_i} \int_{\underline{\alpha}}^{\alpha_i} E_{\mathbf{s}} \left\{ \sum_{\forall \mathbf{g} \text{ s.t. } i \in G_{\mathbf{g}}} u_1(y, s_i) \left[\begin{array}{c} \Pr(g|(y, \boldsymbol{\alpha}_{-\mathbf{i}}), \mathbf{m}_2^{\mathbf{s}}, ..., \mathbf{m}_{M+1}^{\mathbf{s}}) \\ \cdot p_i^{G_{\mathbf{g}}}((y, \boldsymbol{\alpha}_{-\mathbf{i}}), \mathbf{m}_2^{\mathbf{s}}, ..., \mathbf{m}_{M+1}^{\mathbf{s}}) \end{array} \right] \right\} dy
\end{aligned}
$$

$$
\begin{aligned}
&= \pi_i(\underline{\alpha},\underline{\alpha}) + \\
&\quad E_{\boldsymbol{\alpha}_{-i}} \int_{\underline{\alpha}}^{\overline{\alpha}} \int_{\underline{\alpha}}^{\alpha_i} E_{\mathbf{s}} \left\{ \sum_{\forall \mathbf{g} \text{ s.t. } i \in G_{\mathbf{g}}} u_1(y,s_i) \left[\begin{array}{c} \Pr(g|(y,\boldsymbol{\alpha}_{-\mathbf{i}}),\mathbf{m}_2^{\mathbf{s}},...,\mathbf{m}_{M+1}^{\mathbf{s}}) \\ \cdot p_i^{G_{\mathbf{g}}}((y,\boldsymbol{\alpha}_{-\mathbf{i}}),\mathbf{m}_2^{\mathbf{s}},...,\mathbf{m}_{M+1}^{\mathbf{s}}) \end{array} \right] \right\} dy f(\alpha_i) d\alpha_i \\
&= \pi_i(\underline{\alpha},\underline{\alpha}) + \\
&\quad E_{\boldsymbol{\alpha}_{-i}} \int_{\underline{\alpha}}^{\overline{\alpha}} \int_{y}^{\overline{\alpha}} E_{\mathbf{s}} \left\{ \sum_{\forall \mathbf{g} \text{ s.t. } i \in G_{\mathbf{g}}} u_1(y,s_i) \left[\begin{array}{c} \Pr(g|(y,\boldsymbol{\alpha}_{-\mathbf{i}}),\mathbf{m}_2^{\mathbf{s}},...,\mathbf{m}_{M+1}^{\mathbf{s}}) \\ \cdot p_i^{G_{\mathbf{g}}}((y,\boldsymbol{\alpha}_{-\mathbf{i}}),\mathbf{m}_2^{\mathbf{s}},...,\mathbf{m}_{M+1}^{\mathbf{s}}) \end{array} \right] \right\} f(\alpha_i) d\alpha_i dy \\
&= \pi_i(\underline{\alpha},\underline{\alpha}) + \\
&\quad E_{\boldsymbol{\alpha}_{-i}} \int_{\underline{\alpha}}^{\overline{\alpha}} E_{\mathbf{s}} \left\{ \sum_{\forall \mathbf{g} \text{ s.t. } i \in G_{\mathbf{g}}} u_1(y,s_i) \left[\begin{array}{c} \Pr(g|(y,\boldsymbol{\alpha}_{-\mathbf{i}}),\mathbf{m}_2^{\mathbf{s}},...,\mathbf{m}_{M+1}^{\mathbf{s}}) \\ \cdot p_i^{G_{\mathbf{g}}}((y,\boldsymbol{\alpha}_{-\mathbf{i}}),\mathbf{m}_2^{\mathbf{s}},...,\mathbf{m}_{M+1}^{\mathbf{s}}) \end{array} \right] \right\} [\int_{y}^{\overline{\alpha}} f(\alpha_i) d\alpha_i] dy \\
&= \pi_i(\underline{\alpha},\underline{\alpha}) + \\
&\quad E_{\boldsymbol{\alpha}_{-i}} \int_{\underline{\alpha}}^{\overline{\alpha}} E_{\mathbf{s}} \left\{ \sum_{\forall \mathbf{g} \text{ s.t. } i \in G_{\mathbf{g}}} u_1(y,s_i) \left[\begin{array}{c} \Pr(g|(y,\boldsymbol{\alpha}_{-\mathbf{i}}),\mathbf{m}_2^{\mathbf{s}},...,\mathbf{m}_{M+1}^{\mathbf{s}}) \\ \cdot p_i^{G_{\mathbf{g}}}((y,\boldsymbol{\alpha}_{-\mathbf{i}}),\mathbf{m}_2^{\mathbf{s}},...,\mathbf{m}_{M+1}^{\mathbf{s}}) \end{array} \right] \right\} \frac{1-F(y)}{f(y)} f(y) dy \\
&= \pi_i(\underline{\alpha},\underline{\alpha}) + \\
&\quad E_{\boldsymbol{\alpha}_{-i}} E_{\alpha_i} E_{\mathbf{s}} \left\{ \sum_{\forall \mathbf{g} \text{ s.t. } i \in G_{\mathbf{g}}} u_1(\alpha_i,s_i) \left[\begin{array}{c} \Pr(g|(\alpha_i,\boldsymbol{\alpha}_{-\mathbf{i}}),\mathbf{m}_2^{\mathbf{s}},...,\mathbf{m}_{M+1}^{\mathbf{s}}) \\ \cdot p_i^{G_{\mathbf{g}}}((\alpha_i,\boldsymbol{\alpha}_{-\mathbf{i}}),\mathbf{m}_2^{\mathbf{s}},...,\mathbf{m}_{M+1}^{\mathbf{s}}) \end{array} \right] \right\} \frac{1-F(\alpha_i)}{f(\alpha_i)} \\
&= \pi_i(\underline{\alpha},\underline{\alpha}) + E_{\boldsymbol{\alpha}} E_{\mathbf{s}} \left\{ \sum_{\forall \mathbf{g} \text{ s.t. } i \in G_{\mathbf{g}}} u_1(\alpha_i,s_i) \left[\begin{array}{c} \Pr(g|(\boldsymbol{\alpha},\mathbf{m}_2^{\mathbf{s}},...,\mathbf{m}_{M+1}^{\mathbf{s}}) \\ \cdot p_i^{G_{\mathbf{g}}}(\boldsymbol{\alpha},\mathbf{m}_2^{\mathbf{s}},...,\mathbf{m}_{M+1}^{\mathbf{s}}) \end{array} \right] \right\} \frac{1-F(\alpha_i)}{f(\alpha_i)}.
\end{aligned}
$$

Therefore, the seller's expected revenue is given by

$$
\begin{aligned}
ER &= E_{\boldsymbol{\alpha}} \left[TS(\boldsymbol{\alpha}) - \sum_{i \in \mathbf{N}} \pi_i(\alpha_i,\alpha_i) \right] \\
&= E_{\boldsymbol{\alpha}} E_{\mathbf{s}} \left\{ \sum_{\forall \mathbf{g}} \left[\Pr(\mathbf{g}|\boldsymbol{\alpha},\mathbf{m}_2^{\mathbf{s}},...,\mathbf{m}_{M+1}^{\mathbf{s}}) \sum_{i \in G_{\mathbf{g}}} \left(p_i^{G_{\mathbf{g}}}(\boldsymbol{\alpha},\mathbf{m}_2^{\mathbf{s}},...,\mathbf{m}_{M+1}^{\mathbf{s}}) u(\alpha_i,s_i) - c \right) \right] \right\} \\
&\quad - \sum_i \pi_i(\underline{\alpha},\underline{\alpha}) - E_{\boldsymbol{\alpha}} E_{\mathbf{s}} \left\{ \sum_i \sum_{\forall \mathbf{g} \text{ s.t. } i \in G_{\mathbf{g}}} u_1(\alpha_i,s_i) \left[\begin{array}{c} \Pr(g|(\boldsymbol{\alpha},m_2^{\mathbf{s}},...,m_{M+1}^{\mathbf{s}}) \\ \cdot p_i^{G_{\mathbf{g}}}(\boldsymbol{\alpha},m_2^{\mathbf{s}},...,m_{M+1}^{\mathbf{s}}) \end{array} \right] \frac{1-F(\alpha_i)}{f(\alpha_i)} \right\} \\
&= E_{\boldsymbol{\alpha}} E_{\mathbf{s}} \left\{ \sum_{\forall \mathbf{g}} \left[\Pr(\mathbf{g}|\boldsymbol{\alpha},\mathbf{m}_2^{\mathbf{s}},...,\mathbf{m}_{M+1}^{\mathbf{s}}) \sum_{i \in G_{\mathbf{g}}} \left(p_i^{G_{\mathbf{g}}}(\boldsymbol{\alpha},\mathbf{m}_2^{\mathbf{s}},...,\mathbf{m}_{M+1}^{\mathbf{s}}) u(\alpha_i,s_i) - c \right) \right] \right\} \\
&\quad - E_{\boldsymbol{\alpha}} E_{\mathbf{s}} \left\{ \sum_{\forall \mathbf{g}} \left[\begin{array}{c} \Pr(\mathbf{g}|\mathbf{c},\mathbf{m}_2^{\mathbf{s}},...,\mathbf{m}_{M+1}^{\mathbf{s}}) \\ \cdot \sum_{i \in G_{\mathbf{g}}} \left[p_i^{G_{\mathbf{g}}}(\boldsymbol{\alpha},m_2^{\mathbf{s}},...,m_{M+1}^{\mathbf{s}}) u_1(\alpha_i,s_i) \frac{1-F(\alpha_i)}{f(\alpha_i)} \right] \end{array} \right] \right\} - \sum_i \pi_i(\underline{\alpha},\underline{\alpha}) \\
&= E_{\boldsymbol{\alpha}} E_{\mathbf{s}} \sum_{\forall \mathbf{g}} \Pr(\mathbf{g}|\boldsymbol{\alpha},\mathbf{m}_2^{\mathbf{s}},...,\mathbf{m}_{M+1}^{\mathbf{s}}) \sum_{i \in G_{\mathbf{g}}} \left\{ \left[\begin{array}{c} p_i^{G_{\mathbf{g}}}(\boldsymbol{\alpha},\mathbf{m}_2^{\mathbf{s}},...,\mathbf{m}_{M+1}^{\mathbf{s}}) \\ \cdot \left[u(\alpha_i,s_i) - u_1(\alpha_i,s_i) \frac{1-F(\alpha_i)}{f(\alpha_i)} \right] \end{array} \right] - c \right\} - \sum_i \pi_i(\underline{\alpha},\underline{\alpha}) \\
&= E_{\boldsymbol{\alpha}} E_{\mathbf{s}} \sum_{\forall \mathbf{g}} \Pr(\mathbf{g}|\boldsymbol{\alpha},\mathbf{m}_2^{\mathbf{s}},...,\mathbf{m}_{M+1}^{\mathbf{s}}) \sum_{i \in G_{\mathbf{g}}} \left[p_i^{G_{\mathbf{g}}}(\boldsymbol{\alpha},\mathbf{m}_2^{\mathbf{s}},...,\mathbf{m}_{M+1}^{\mathbf{s}}) w(\alpha_i,s_i) - c \right] - \sum_i \pi_i(\underline{\alpha},\underline{\alpha}).
\end{aligned}
$$

Lemma 3 thus follows.

□

Proof of Lemma 4: First we show that there is no loss of generality to consider shortlisting rules, under which any possible set of players shortlisted at each stage must be a singleton before the shortlisting process is completed.

Consider any given sequential shortlisting rule. By Lemma 3, the expected revenue is determined by the derived $\{\Pr(G|\boldsymbol{\alpha},\mathbf{s}), \forall G \in 2^{\mathbf{N}}, \boldsymbol{\alpha}, \mathbf{s}\}$. We next show that any $\{\Pr(G|\boldsymbol{\alpha},\mathbf{s}), \forall G \in 2^{\mathbf{N}}, \boldsymbol{\alpha}, \mathbf{s}\}$ can be generated by an one-agent-per-stage-until-the-last shortlisting rule as described in the first paragraph of this proof.

Let $\mathbf{G}^+(\boldsymbol{\alpha},\mathbf{s}) = \{G|\Pr(G|\boldsymbol{\alpha},\mathbf{s}) > 0, G \in 2^{\mathbf{N}}\}$. $\forall G^+ \in \mathbf{G}^+(\boldsymbol{\alpha},\mathbf{s})$ and $G^+ \neq \varnothing$, there are at most N agents in it. We can rank these agents in G^+ in ascending order by their indexes.

If $\varnothing \in \mathbf{G}^+(\boldsymbol{\alpha},\mathbf{s})$, we let $A^{g_1=\varnothing}(\boldsymbol{\alpha}|g_0) = \Pr(\varnothing|\boldsymbol{\alpha},\mathbf{s})$ and further let $A^{g_k=\varnothing}(\boldsymbol{\alpha},\mathbf{m}_2^{\mathbf{s}},...,\mathbf{m}_k^{\mathbf{s}}|g_0,g_1,g_2,...,g_{k-1}) = 1$ for $g_{k-1} = \varnothing$, $k = 2,...,M$, which says that if no one is shortlisted in stage $k \geq 1$, then no one will be shortlisted in subsequent stages.

$\forall i \in \mathbf{N}$, let $A^{g_1=\{i\}}(\boldsymbol{\alpha}|g_0) = \sum_{\forall G \text{ s.t. } j\geq i, \forall j\in G} \Pr(G|\boldsymbol{\alpha},\mathbf{s})$. In other words, the probability that agent i is shortlisted at stage 1 is the sum of the probabilities of all the final shortlisted groups that contain i as the smallest indexed within the group. If agent i does not belong to any final shortlisted group with him as the smallest indexed, then we let $A^{g_1=\{i\}}(\boldsymbol{\alpha}|g_0) = 0$. Note by the above constructions, we must have $A^{g_1=\varnothing}(\boldsymbol{\alpha}|g_0) + \sum_{i\in\mathbf{N}} A^{g_1=\{i\}}(\boldsymbol{\alpha}|g_0) = 1$. Therefore, we let $A^{g_1=\{i\}}(\boldsymbol{\alpha}|g_0) = 0$ for any group g_1 that contains more than two agents.

We now move to constructing the stage-2 shortlisting rule. We only need to focus on the nonempty single-element g_1's that have positive chances of being shortlisted at stage 1, since any group g_1 with more than two agents is never shortlisted in stage 1, and when no agent is shortlisted in stage 1, then the future shortlist rule has been fully pinned down. Take any of these g_1's and focus on all G^+'s that contain the single agent in g_1 as the smallest indexed. Let $\mathbf{G}^+(g_1)$ denote this set of G^+'s. We are now ready to define $A^{g_2}(\boldsymbol{\alpha},\mathbf{m}_1^s|g_0,g_1), \forall g_2 \in 2^{\mathbf{N}\backslash g_1}$. For $g_2 = \varnothing$, we define $A^{g_2=\varnothing}(\boldsymbol{\alpha},\mathbf{m}_1^s|g_0,g_1) = 0$ if $g_1 \notin \mathbf{G}^+(g_1)$, and $A^{g_2=\varnothing}(\boldsymbol{\alpha},\mathbf{m}_1^s|g_0,g_1) = \frac{\Pr(G^+=g_1|\boldsymbol{\alpha},\mathbf{s})}{A^{g_1}(\boldsymbol{\alpha}|g_0)}$ if $g_1 \in \mathbf{G}^+(g_1)$. We let $A^{g_2}(\boldsymbol{\alpha},\mathbf{m}_1^s|g_0,g_1) = 0$ for any group $g_2 \in 2^{\mathbf{N}\backslash g_1}$that contains more than two agents. We use $\mathbf{N}(g_1)$ to denote the pool of the second smallest indexed agents in the G^+'s in $\mathbf{G}^+(g_1)$. If $\mathbf{N}(g_1)$ is empty, then we let $A^{g_2=\{j\}}(\boldsymbol{\alpha},\mathbf{m}_1^s|g_0,g_1) = 0$, $\forall j \in \mathbf{N}\backslash g_1$. If $\mathbf{N}(g_1)$ is not empty, we let

$$A^{g_2=\{j\}}(\boldsymbol{\alpha},\mathbf{m}_1^s|g_0,g_1) = \begin{cases} \frac{\sum_{G^+\in\mathbf{G}^+(g_1) \text{ s.t. } k\geq j, \forall k\in G^+\backslash g_1} \Pr(G^+|\boldsymbol{\alpha},\mathbf{s})}{A^{g_1}(\boldsymbol{\alpha}|g_0)} & \forall j \in \mathbf{N}(g_1), \\ 0 & \forall j \notin \mathbf{N}(g_1). \end{cases}$$

In words, for any agent $j \in \mathbf{N}(g_1)$, the probability that j is shortlisted conditional on that g_1 has been shortlisted is simply the ratio between the sum of probabilities of G^+'s that contain the single agent in g_1 as the smallest indexed and j as the second smallest indexed and the sum of probabil-

ities of G^+'s that contain the single agent in g_1. By construction, we have $A^{g_2=\varnothing}(\boldsymbol{\alpha}, \mathbf{m}_1^s|g_0, g_1) + \sum_{j\in\mathbf{N}(g_1)} A^{g_2=\{j\}}(\boldsymbol{\alpha}, \mathbf{m}_1^s|g_0, g_1) = 1$.

Similarly, for stage 3, we only need to consider these shortlisting histories (g_1, g_2) where both g_1 and g_2 are nonempty single-element groups. We define $\mathbf{G}^+(g_1, g_2)$ as the set of all G^+'s that contain the single element in g_1 as the smallest indexed, and the single element in $g_2 \in \mathbf{G}^+(g_1)\backslash\varnothing$ as the second smallest indexed. We are now ready to define $A^{g_3}(\boldsymbol{\alpha}, \mathbf{m}_1^s, \mathbf{m}_2^s|g_0, g_1, g_2), \forall g_3 \in 2^{\mathbf{N}\backslash\{g_1\cup g_2\}}$. For $g_3 = \varnothing$, we define $A^{g_3=\varnothing}(\boldsymbol{\alpha}, \mathbf{m}_1^s, \mathbf{m}_2^s|g_0, g_1, g_2) = 0$ if $g_1 \cup g_2 \notin \mathbf{G}^+(g_1, g_2)$ and $A^{g_3=\varnothing}(\boldsymbol{\alpha}, \mathbf{m}_1^s, \mathbf{m}_2^s|g_0, g_1, g_2) = \frac{\Pr(G^+=g_1\cup g_2|\boldsymbol{\alpha},\mathbf{s})}{A^{g_1}(\boldsymbol{\alpha}|g_0)\cdot A^{g_2}(\boldsymbol{\alpha},\mathbf{m}_1^s|g_0,g_1)}$ if $g_1 \cup g_2 \in \mathbf{G}^+(g_1, g_2)$. We let $A^{g_3}(\boldsymbol{\alpha}, \mathbf{m}_1^s, \mathbf{m}_2^s|g_0, g_1, g_2) = 0$ for any group $g_3 \in 2^{\mathbf{N}\backslash\{g_1\cup g_2\}}$ that contains more than two agents. We use $\mathbf{N}(g_1, g_2)$ to denote the pool of the third smallest indexed agents in the G^+s in $\mathbf{G}^+(g_1, g_2)$. If $\mathbf{N}(g_1, g_2)$ is empty, then we let $A^{g_3=\{j\}}(\boldsymbol{\alpha}, \mathbf{m}_1^s, \mathbf{m}_2^s|g_0, g_1, g_2) = 0$, $\forall j \in \mathbf{N}\backslash\{g_1 \cup g_2\}$. If $\mathbf{N}(g_1, g_2)$ is not empty, then $\forall j \in \mathbf{N}(g_1, g_2)$, we let $A^{g_3=\{j\}}(\boldsymbol{\alpha}, \mathbf{m}_1^s, \mathbf{m}_2^s|g_0, g_1, g_2) = \frac{\sum_{G^+\in\mathbf{G}^+(g_1,g_2)\text{ s.t. }k\geq j,\forall k\in G^+\backslash(g_1\cup g_2)}\Pr(G^+|\boldsymbol{\alpha},\mathbf{s})}{A^{g_1}(\boldsymbol{\alpha}|g_0)\cdot A^{g_2}(\boldsymbol{\alpha},\mathbf{m}_1^s|g_0,g_1)}$; and $\forall j \notin \mathbf{N}(g_1, g_2)$, we let $A^{g_3=\{j\}}(\boldsymbol{\alpha}, \mathbf{m}_1^s, \mathbf{m}_2^s|g_0, g_1, g_2) = 0$. By construction, we have $A^{g_3=\varnothing}(\boldsymbol{\alpha}, \mathbf{m}_1^s, \mathbf{m}_2^s|g_0, g_1, g_2) + \sum_{j\in\mathbf{N}(g_1,g_2)} A^{g_3=\{j\}}(\boldsymbol{\alpha}, \mathbf{m}_1^s, \mathbf{m}_2^s|g_0, g_1, g_2) = 1$.

This process continues analogously until we exhaust all agents in every group $G^+ \in \mathbf{G}^+(\boldsymbol{\alpha}, \mathbf{s})$. By construction, it is clear that this process of shortlisting one agent at each stage generates the same probabilities of $\{\Pr(G|\boldsymbol{\alpha}, \mathbf{s}), \forall G \in 2^{\mathbf{N}}, \boldsymbol{\alpha}, \mathbf{s}\}$. As a result, the above constructed shortlisting rule would generate the same expected revenue for the seller.

Note that the shortlisting rule established above allows for the possibility that each of the remaining bidders is shortlisted with a different probability at a particular stage. The point is that we can focus on shortlisting rules under which the seller shortlists one agent at each stage before stopping the shortlisting. It further implies that to search for the optimal shortlisting rule, without loss of generality, we can focus on the rules where at each stage the seller either shortlists an agent with probability 1 or stops shortlisting. The reason is as follows. At stage 1, the principal has $N + 1$ choices: shortlisting no one or shortlisting some i. Shortlisting no one implies that the shortlisting process stops at stage 1. According to Lemma 3, we have $ER = 0$. If the seller opts to shortlist agent i, he would continue to adopt the best shortlisting decisions in all subsequent stages. Suppose this leads to an optimal expected revenue of ER_i, $i = 1, 2, ..., N$. Clearly, there is no loss of generality for the seller to adopt an option which delivers the maximum revenue with probability 1. If this option is $\varnothing$, then the shortlisting process stops. If it is to shortlist agent i, then we move to the next stage with the new information revealed by i. At stage 2, similarly, the seller has N options: shortlisting no one or shortlisting any $j \in \mathbf{N}\backslash\{i\}$. Similarly, the seller should go for an option that delivers the maximum revenue with probability 1. The process continues until the seller runs out of agents or he decides to stop shortlisting. We are now ready to prove the lemma.

Consider any given $\boldsymbol{\alpha}$ and any shortlisting rule as described above

$$A^{g_k}(\boldsymbol{\alpha}, \mathbf{m}_1^s, \mathbf{m}_2^s, ..., \mathbf{m}_{k-1}^s; g_0, g_1, g_2, ...g_{k-1}).$$

$\Pr(G|\boldsymbol{\alpha}, \mathbf{s})$ denotes the conditional shortlisting probability of group G. Without loss of generality, we assume α_i decreases with i. In stage $k \geq 1$, which agent i_k is shortlisted with probability 1 depends on $(\boldsymbol{\alpha}, \mathbf{m}_1^s, \mathbf{m}_2^s, ..., \mathbf{m}_{k-1}^s; g_0, g_1, g_2, ...g_{k-1})$. Note signals $s_i, i \in \mathbf{N}$ are i.i.d. and they are independent of $\alpha_i, i \in \mathbf{N}$. We construct a new shortlisting rule under which the seller shortlists one agent at each stage before stopping the shortlisting, as follows. In stage 1, when i_1 is shortlisted, we replace her by agent 1. In stage 2, we use the same shortlisting rule $A^{g_2}(\boldsymbol{\alpha}, \mathbf{m}_1^s | g_0, g_1)$ treating as if $g_1 = \{i_1\}$, however, $\mathbf{m}_1^s$ now stands for agent 1's type instead. Because s_{i_1} and s_1 follow the same distribution, the rule $A^{g_2}(\boldsymbol{\alpha}, \mathbf{m}_1^s | g_0, g_1)$ would generate the same agent $i_2 \in \mathbf{N} \backslash \{i_1\}$ to be shortlisted in stage 2 while which i_2 is shortlisted might depend on the realization of $\mathbf{m}_1^s$. Whenever an i_2 is shortlisted in stage 2, we replace her by agent 2. In stage $k \geq 3$, we use the same shortlisting rule $A^{g_k}(\boldsymbol{\alpha}, \mathbf{m}_1^s, \mathbf{m}_2^s ..., \mathbf{m}_{k-1}^s | g_0, g_1, ..., g_{k-1})$ treating as if $g_h = \{i_h\}, 1 \leq h < k$, however, $\mathbf{m}_i^s$ now stands for agent i's type instead. Because s_{i_h} and s_h follow the same distribution, the rule $A^{g_k}(\boldsymbol{\alpha}, \mathbf{m}_1^s, \mathbf{m}_2^s ..., \mathbf{m}_{k-1}^s | g_0, g_1, ..., g_{k-1})$ would generate the same agent $i_k \in \mathbf{N} \backslash \cup_{h=1}^{k-1} \{i_h\}$ to be shortlisted in stage k while which i_k is shortlisted might depend on the realization of $(\mathbf{m_1^s}, \mathbf{m_2^s}, ..., \mathbf{m_{k-1}^s})$. Whenever an i_k is shortlisted in stage k, we replace her by agent k. We use $\hat{\Pr}(G|\boldsymbol{\alpha}, \mathbf{s})$ to denote the conditional shortlisting probability of group G under the new rule. Let $G_k = \{1, 2, ..., k\}, k = 1, 2, ..., N$. Note that $\hat{\Pr}(G|\boldsymbol{\alpha}, \mathbf{s}) = 0$ for any $G \in 2^{\mathbf{N}} \backslash \{\varnothing, G_k, k = 1, 2, ..., N\}$.

Based on the above construction, $\forall \boldsymbol{\alpha}$, for any $k = 1, 2, ..., N$ such that $E_{\mathbf{s}} \sum_{\forall G \text{ s.t. } |G|=k} \Pr(G|\boldsymbol{\alpha}, \mathbf{s}) > 0$, we must have $E_{\mathbf{s}} \hat{\Pr}(G_k|\boldsymbol{\alpha}, \mathbf{s}) = E_{\mathbf{s}} \sum_{\forall G \text{ s.t. } |G|=k} \Pr(G|\boldsymbol{\alpha}, \mathbf{s})$. Therefore,

$$\begin{aligned}
& E_{\mathbf{s}} \left\{ \sum_{G \in 2^{\mathbf{N}}} \Pr(G|\boldsymbol{\alpha}, \mathbf{s}) \left[\max\{w_i^+(\alpha_i, s_i)\}_{i \in G} - \sum_{i \in G} c \right] \right\} \\
= \; & E_{\mathbf{s}} \left\{ \sum_{k=1}^{N} \sum_{\forall G \text{ s.t. } |G|=k} \Pr(G|\boldsymbol{\alpha}, \mathbf{s}) \left[\max\{w_i^+(\alpha_i, s_i)\}_{i \in G} - kc \right] \right\} \\
\geq \; & E_{\mathbf{s}} \left\{ \sum_{k=1}^{N} \sum_{G_k} \hat{\Pr}(G_k|\boldsymbol{\alpha}, \mathbf{s}) [\max\{w_i^+(\alpha_i, s_i)\}_{i \in G_k} - kc] \right\} \\
= \; & E_{\mathbf{s}} \left\{ \sum_{G \in 2^{\mathbf{N}}} \hat{\Pr}(G|\boldsymbol{\alpha}, \mathbf{s}) [\max\{w_i^+(\alpha_i, s_i)\}_{i \in G} - \sum_{i \in G} c] \right\}.
\end{aligned}$$

Therefore, by Lemma 3, the constructed shortlisting rule generates higher expected revenue.

□

Proof of Proposition 4: Given stage-1 report $\hat{\boldsymbol{\alpha}}$, we use i_k to denote the agent whose report is ranked the kth highest. We first look at the reporting incentive at stages $k \geq 2$. Let $\mathbf{g}_{k,h} =$

$(g_1 = \{i_1\}, g_2 = \{i_2\}, ..., g_{k-1} = \{i_{k-1}\}, ..., g_h = \{i_h\}, g_{h+1} = \varnothing, ..., g_M = \varnothing)$, $h \geq k \geq 2$ be a sequence of shortlisted where h agents (including i_k) are shortlisted in total. Let $\hat{\mathbf{m}}_k$ denote the stage-k announcement (so its i_{k-1}−th element is $\widehat{s}_{i_{k-1}}$ and all the other elements are $\emptyset$). Given the announcement in the history $(\hat{\boldsymbol{\alpha}}, \mathbf{m}_2, ..., \mathbf{m}_{k-1})$, assuming agents $i_l, l \geq k$ truthfully reveal their information at stage $l+1$, agent i_{k-1}'s conditional expected payoff is

$$\begin{aligned}
& \pi_{i_{k-1}}(s_{i_{k-1}}, \widehat{s}_{i_{k-1}} | \hat{\boldsymbol{\alpha}}, \mathbf{m}_2, ..., \mathbf{m}_{k-1}) \\
= \; & E_{(s_{i_k}, ..., s_{i_N})} \left\{ \begin{array}{c} -\sum_{\forall h=k}^{N} \left[\begin{array}{c} \Pr{}^*(\mathbf{g}_{k-1,h} | \hat{\boldsymbol{\alpha}}, \mathbf{m}_2, ..., \mathbf{m}_{k-1}, \hat{\mathbf{m}}_k, \mathbf{m}^{\mathbf{s}}_{k+1} ..., \mathbf{m}^{\mathbf{s}}_{M+1}) \\ \cdot \sum_{l=k+1}^{M} t_{l,i_{k-1}}(\hat{\boldsymbol{\alpha}}, \mathbf{m}_2, ..., \mathbf{m}_{k-1}, \hat{\mathbf{m}}_k, \mathbf{m}^{\mathbf{s}}_{k+1}, ..., \mathbf{m}^{\mathbf{s}}_l) \end{array} \right] \\ +u(\alpha_{i_{k-1}}, s_{i_{k-1}}) \sum_{\forall h=k-1}^{N} \left[\begin{array}{c} \Pr{}^*(\mathbf{g}_{k-1,h} | \hat{\boldsymbol{\alpha}}, \mathbf{m}_2, ..., \mathbf{m}_{k-1}, \hat{\mathbf{m}}_k, \mathbf{m}^{\mathbf{s}}_{k+1} ..., \mathbf{m}^{\mathbf{s}}_{M+1}) \\ \cdot p^{*G_{\mathbf{g}_{k-1,h}}}_{i_{k-1}}(\hat{\boldsymbol{\alpha}}, \mathbf{m}_2, ..., \mathbf{m}_{k-1}, \hat{\mathbf{m}}_k, \mathbf{m}^{\mathbf{s}}_{k+1} ..., \mathbf{m}^{\mathbf{s}}_{M+1}) \end{array} \right] \end{array} \right\} \\
& -t_{k,i_{k-1}}(\hat{\boldsymbol{\alpha}}, \mathbf{m}_2, ..., \mathbf{m}_{k-1}, \hat{\mathbf{m}}_k).
\end{aligned}$$

We next identify the $t_{k,i}(\hat{\boldsymbol{\alpha}}, \mathbf{m}_2, ..., \mathbf{m}_{k-1}, \hat{\mathbf{m}}_k), i \in \mathbf{N}$ that induces truthful revelation at stage k whenever $\hat{\alpha}_{i_l} = \alpha_{i_l}, l \geq k-1$, i.e. whenever these first stage announcements are truthful. In particular, $t_{k,i_{k-1}}$ can be constructed following the standard Myersonian procedure with $\pi_{i_{k-1}}(0, 0 | \hat{\boldsymbol{\alpha}}, \mathbf{m}_2, ..., \mathbf{m}_{k-1})$ $= 0$, and $\forall k \geq 3$, $t_{k-1,i_{k-1}}$ can be set at $-c$ to induce information discovery of the shortlisted agent i_{k-1} at stage $k-1$. All other $t_{k,i}$ are set to zero for $i \neq i_{k-1}, i_k, \forall k \geq 2$. This means at each stage $k \geq 2$, transfers are nonzero only for the agents shortlisted in stages $k-1$ and k.

We first consider stage $N+1$. Suppose i_N is shortlisted in stage N. At stage N, we have

$$\begin{aligned}
& \pi_{i_N}(s_{i_N}, \widehat{s}_{i_N} | \hat{\boldsymbol{\alpha}}, \mathbf{m}_2, ..., \mathbf{m}_N) \\
= \; & u(\alpha_{i_N}, s_{i_N}) p^{*\mathbf{N}}_{i_N}(\hat{\boldsymbol{\alpha}}, \mathbf{m}_2, ..., \mathbf{m}_N, \hat{\mathbf{m}}_{N+1}, \mathbf{m}^{\mathbf{s}}_{N+2} ..., \mathbf{m}^{\mathbf{s}}_{M+1}) - t_{N+1,i_N}(\hat{\boldsymbol{\alpha}}, \mathbf{m}_2, ..., \mathbf{m}_N, \hat{\mathbf{m}}_{N+1}). \qquad (29)
\end{aligned}$$

By the envelop theorem, optimality of truthful revelation requires

$$\frac{d\pi_{i_N}(s_{i_N}, s_{i_N} | \hat{\boldsymbol{\alpha}}, \mathbf{m}_2, ..., \mathbf{m}_N)}{ds_{i_N}} = u_2(\alpha_{i_N}, s_{i_N}) p^{*\mathbf{N}}_{i_N}(\hat{\boldsymbol{\alpha}}, \mathbf{m}_2, ..., \mathbf{m}_N, \mathbf{m}^{\mathbf{s}}_{N+1}, \mathbf{m}^{\mathbf{s}}_{N+2} ..., \mathbf{m}^{\mathbf{s}}_{M+1}).$$

Recall that we set $\pi_{i_N}(0, 0 | \hat{\boldsymbol{\alpha}}, \mathbf{m}_2, ..., \mathbf{m}_N) = 0$. We thus have

$$\pi_{i_N}(s_{i_N}, s_{i_N} | \hat{\boldsymbol{\alpha}}, \mathbf{m}_2, ..., \mathbf{m}_N) = \int_0^{s_{i_N}} u_2(\alpha_{i_N}, y) p^{*\mathbf{N}}_{i_N}(\hat{\boldsymbol{\alpha}}, \mathbf{m}_2, ..., \mathbf{m}_N, \mathbf{m}^{\mathbf{s}}_{N+1}(y), \mathbf{m}^{\mathbf{s}}_{N+2} ..., \mathbf{m}^{\mathbf{s}}_{M+1}) dy, \qquad (30)$$

where in $\mathbf{m}^{\mathbf{s}}_{N+1}(y)$, s_{i_N} is replaced by y.

Given (29) and (30), we define

$$\begin{aligned}
& t^*_{N+1,i_N}(\hat{\boldsymbol{\alpha}}, \mathbf{m}_2, ..., \mathbf{m}_N, \mathbf{m}^{\mathbf{s}}_{N+1}) \\
= \; & u(\alpha_{i_N}, s_{i_N}) p^{*\mathbf{N}}_{i_N}(\hat{\boldsymbol{\alpha}}, \mathbf{m}_2, ..., \mathbf{m}_N, \mathbf{m}^{\mathbf{s}}_{N+1}, \mathbf{m}^{\mathbf{s}}_{N+2} ..., \mathbf{m}^{\mathbf{s}}_{M+1})
\end{aligned}$$

$$-\int_0^{s_{i_N}} u_2(\alpha_{i_N}, y) p_{i_N}^{*\mathbf{N}}(\hat{\boldsymbol{\alpha}}, \mathbf{m}_2, ..., \mathbf{m}_N, \mathbf{m}_{N+1}^{\mathbf{s}}(y), \mathbf{m}_{N+2}^{\mathbf{s}} ..., \mathbf{m}_{M+1}^{\mathbf{s}}) dy. \tag{31}$$

Therefore,

$t^*_{N+1,i_N}(\hat{\alpha}, m_2, ..., m_N, m^{\mathbf{s}}_{N+1}) = 0$ if agent i_N loses; and $t^*_{N+1,i_N}(\hat{\alpha}, m_2, ..., m_N, m^{\mathbf{s}}_{N+1}) = u(\alpha_{i_N}, \tilde{s}_{i_N})$ if she wins,

where $\tilde{s}_{i_N}(> 0)$ is agent i_N's minimum winning type in stage N based on winning rule p^*. Note that under Assumptions 1 and 2, we have that $w(\alpha_{i_k}, s_{i_k})$ increases with both α_{i_k} and s_{i_k}, $\forall k$. Thus $\tilde{s}_{i_N}$ is well defined, which is determined by

$$u(\alpha_{i_N}, \tilde{s}_{i_N}) - u_1(\alpha_{i_N}, \tilde{s}_{i_N}) \frac{1 - F(\alpha_{i_N})}{f(\alpha_{i_N})} = \left[0 \vee \max_{i_k, k<N} \left\{ u(\hat{\alpha}_{i_k}, s_{ik}) - u_1(\hat{\alpha}_{i_k}, s_{ik}) \frac{1 - F(\hat{\alpha}_{i_k})}{f(\hat{\alpha}_{i_k})} \right\} \right].$$

It is clear that (1) $\pi_{i_N}(s_{i_N}, \widehat{s}_{i_N} | \hat{\boldsymbol{\alpha}}, \mathbf{m}_2, ..., \mathbf{m}_N)$ satisfies the strict and smooth single crossing differences property in $(s_{i_N}, \widehat{s}_{i_N})$; (2) $p_{i_N}^{*\mathbf{N}}(\hat{\boldsymbol{\alpha}}, \mathbf{m}_2, ..., \mathbf{m}_N, \hat{\mathbf{m}}_{N+1}, \mathbf{m}^{\mathbf{s}}_{N+2} ..., \mathbf{m}^{\mathbf{s}}_{M+1})$ increases in $\widehat{s}_{i_N}$; (3) t^*_{N+1,i_N} defined above verifies the envelope theorem. By the constraint simplification theorem,[35] truthful revelation at stage $N+1$ is incentive compatible for $(\mathbf{p}^*, t^*_{N+1,i_N})$.

We now turn to stage N. First note that, given t^*_{N+1,i_N} constructed above, i_N would reveal truthfully at stage $N+1$ if shortlisted, regardless of i_{N-1}'s announcement at stage N. Let $\hat{\mathbf{m}}_N$ denote the stage-N announcement with the $i_{N-1}-$th element being $\widehat{s}_{i_{N-1}}$ and all other elements being $\emptyset$. Given announcement in the history $(\hat{\boldsymbol{\alpha}}, \mathbf{m}_2, ..., \mathbf{m}_{N-1})$, agent i_{N-1}'s conditional expected payoff is

$$\begin{aligned} & \pi_{i_{N-1}}(s_{i_{N-1}}, \widehat{s}_{i_{N-1}} | \hat{\boldsymbol{\alpha}}, \mathbf{m}_2, ..., \mathbf{m}_{N-1}) \\ = \; & E_{s_{i_N}} \left\{ u(\alpha_{i_{N-1}}, s_{i_{N-1}}) \sum_{\forall h = N-1}^{N} \left[\begin{array}{l} \Pr{}^*(\mathbf{g}_{N-1,h} | \hat{\boldsymbol{\alpha}}, \mathbf{m}_2, ..., \mathbf{m}_{N-1}, \hat{\mathbf{m}}_N, \mathbf{m}^{\mathbf{s}}_{N+1} ..., \mathbf{m}^{\mathbf{s}}_{M+1}) \\ \cdot p_{i_{N-1}}^{*G_{\mathbf{g}_{N-1,h}}}(\hat{\boldsymbol{\alpha}}, \mathbf{m}_2, ..., \mathbf{m}_{N-1}, \hat{\mathbf{m}}_N, \mathbf{m}^{\mathbf{s}}_{N+1} ..., \mathbf{m}^{\mathbf{s}}_{M+1}) \end{array} \right] \right\} \\ & - t_{N,i_{N-1}}(\hat{\boldsymbol{\alpha}}, \mathbf{m}_2, ..., \mathbf{m}_{N-1}, \hat{\mathbf{m}}_N). \end{aligned} \tag{32}$$

By the envelop theorem, optimality of truthful revelation requires

$$\begin{aligned} & \frac{d\pi_{i_{N-1}}(s_{i_{N-1}}, s_{i_{N-1}} | \hat{\boldsymbol{\alpha}}, \mathbf{m}_2, ..., \mathbf{m}_{N-1})}{ds_{i_{N-1}}} \\ = \; & u_2(\alpha_{i_{N-1}}, s_{i_{N-1}}) E_{s_{i_N}} \left\{ \sum_{\forall h = N-1}^{N} \left[\begin{array}{l} \Pr{}^*(\mathbf{g}_{N-1,h} | \hat{\boldsymbol{\alpha}}, \mathbf{m}_2, ..., \mathbf{m}_{N-1}, \mathbf{m}^{\mathbf{s}}_N, \mathbf{m}^{\mathbf{s}}_{N+1} ..., \mathbf{m}^{\mathbf{s}}_{M+1}) \\ \cdot \; p_{i_{N-1}}^{*G_{\mathbf{g}_{N-1,h}}}(\hat{\boldsymbol{\alpha}}, \mathbf{m}_2, ..., \mathbf{m}_{N-1}, \mathbf{m}^{\mathbf{s}}_N, \mathbf{m}^{\mathbf{s}}_{N+1} ..., \mathbf{m}^{\mathbf{s}}_{M+1}) \end{array} \right] \right\}. \end{aligned}$$

[35] A version of the constraint simplification theorem can be seen from Theorem 4.3 in Milgrom (2004, page 13).

Recall that we set $\pi_{i_{N-1}}(0,0|\hat{\boldsymbol{\alpha}},\mathbf{m}_2,...,\mathbf{m}_{N-1})=0$. We thus have

$$\begin{aligned}
&\pi_{i_{N-1}}(s_{i_{N-1}},s_{i_{N-1}}|\hat{\boldsymbol{\alpha}},\mathbf{m}_2,...,\mathbf{m}_{N-1}) \qquad (33)\\
=\ &\int_0^{s_{i_{N-1}}} u_2(\alpha_{i_{N-1}},y)E_{s_{i_N}}\left\{\sum_{\forall h=N-1}^{N}\left[\begin{array}{c}\Pr{}^*(\mathbf{g}_{N-1,h}|\hat{\boldsymbol{\alpha}},\mathbf{m}_2,...,\mathbf{m}_{N-1},\mathbf{m}_N^s(y),\mathbf{m}_{N+1}^{\mathbf{s}}...,\mathbf{m}_{M+1}^{\mathbf{s}})\\ \cdot\, p_{i_{N-1}}^{*G_{\mathbf{g}_{N-1,h}}}(\hat{\boldsymbol{\alpha}},\mathbf{m}_2,...,\mathbf{m}_{N-1},\mathbf{m}_N^s(y),\mathbf{m}_{N+1}^{\mathbf{s}}...,\mathbf{m}_{M+1}^{\mathbf{s}})\end{array}\right]\right\}dy,
\end{aligned}$$

where in $\mathbf{m}_N^{\mathbf{s}}(y)$, $s_{i_{N-1}}$ is replaced by y.

Based on (32) and (33), we define

$$\begin{aligned}
&t^*_{N,i_{N-1}}(\hat{\boldsymbol{\alpha}},\mathbf{m}_2,...,\mathbf{m}_{N-1},\mathbf{m}_N^{\mathbf{s}}) \qquad (34)\\
=\ &E_{s_{i_N}}\left\{u(\alpha_{i_{N-1}},s_{i_{N-1}})\sum_{\forall h=N-1}^{N}\left[\begin{array}{c}\Pr{}^*(\mathbf{g}_{N-1,h}|\hat{\boldsymbol{\alpha}},\mathbf{m}_2,...,\mathbf{m}_{N-1},\mathbf{m}_N^{\mathbf{s}},\mathbf{m}_{N+1}^{\mathbf{s}}...,\mathbf{m}_{M+1}^{\mathbf{s}})\\ \cdot p_{i_{N-1}}^{*G_{\mathbf{g}_{N-1,h}}}(\hat{\boldsymbol{\alpha}},\mathbf{m}_2,...,\mathbf{m}_{N-1},\mathbf{m}_N^{\mathbf{s}},\mathbf{m}_{N+1}^{\mathbf{s}}...,\mathbf{m}_{M+1}^{\mathbf{s}})\end{array}\right]\right\}\\
&-\int_0^{s_{i_{N-1}}} u_2(\alpha_{i_{N-1}},y)E_{s_{i_N}}\left\{\sum_{\forall h=N-1}^{N}\left[\begin{array}{c}\Pr{}^*(\mathbf{g}_{N-1,h}|\hat{\boldsymbol{\alpha}},\mathbf{m}_2,...,\mathbf{m}_{N-1},\mathbf{m}_N^s(y),\mathbf{m}_{N+1}^{\mathbf{s}}...,\mathbf{m}_{M+1}^{\mathbf{s}})\\ \cdot\, p_{i_{N-1}}^{*G_{\mathbf{g}_{N-1,h}}}(\hat{\boldsymbol{\alpha}},\mathbf{m}_2,...,\mathbf{m}_{N-1},\mathbf{m}_N^s(y),\mathbf{m}_{N+1}^{\mathbf{s}}...,\mathbf{m}_{M+1}^{\mathbf{s}})\end{array}\right]\right\}dy.
\end{aligned}$$

Let

$$\Phi(s_{i_{N-1}})\equiv\sum_{\forall h=N-1}^{N}\left[\begin{array}{c}\Pr{}^*(\mathbf{g}_{N-1,h}|\hat{\boldsymbol{\alpha}},\mathbf{m}_2,...,\mathbf{m}_{N-1},\mathbf{m}_N^s,\mathbf{m}_{N+1}^{\mathbf{s}}...,\mathbf{m}_{M+1}^{\mathbf{s}})\\ \cdot p_{i_{N-1}}^{*G_{\mathbf{g}_{N-1,h}}}(\hat{\boldsymbol{\alpha}},\mathbf{m}_2,...,\mathbf{m}_{N-1},\mathbf{m}_N^s,\mathbf{m}_{N+1}^{\mathbf{s}}...,\mathbf{m}_{M+1}^{\mathbf{s}})\end{array}\right]$$

be the probability that i_{N-1} is shortlisted and wins the asset at the end. We first show in Lemma 5 that this probability is increasing in $s_{i_{N-1}}$.

Lemma 5. $\Phi(s_{i_{N-1}})$ increases in $s_{i_{N-1}}$.

Proof of Lemma 5: To simplify notation, given $(\hat{\boldsymbol{\alpha}},\mathbf{m}_2,...,\mathbf{m}_{N-1},\mathbf{m}_N^s,\mathbf{m}_{N+1}^{\mathbf{s}}...,\mathbf{m}_{M+1}^{\mathbf{s}})$, let

$$\begin{aligned}
\alpha_1(s_{i_{N-1}}) &\equiv \Pr{}^*(\mathbf{g}_{N-1,N-1}|\hat{\boldsymbol{\alpha}},\mathbf{m}_2,...,\mathbf{m}_{N-1},\mathbf{m}_N^s,\mathbf{m}_{N+1}^{\mathbf{s}}...,\mathbf{m}_{M+1}^{\mathbf{s}}),\\
\alpha_2(s_{i_{N-1}}) &\equiv \Pr{}^*(\mathbf{g}_{N-1,N}|\hat{\boldsymbol{\alpha}},\mathbf{m}_2,...,\mathbf{m}_{N-1},\mathbf{m}_N^s,\mathbf{m}_{N+1}^{\mathbf{s}}...,\mathbf{m}_{M+1}^{\mathbf{s}}),\\
P_1(s_{i_{N-1}}) &\equiv p_{i_{N-1}}^{*G_{\mathbf{g}_{N-1,N-1}}}(\hat{\boldsymbol{\alpha}},\mathbf{m}_2,...,\mathbf{m}_{N-1},\mathbf{m}_N^s,\mathbf{m}_{N+1}^{\mathbf{s}}...,\mathbf{m}_{M+1}^{\mathbf{s}}),\\
P_2(s_{i_{N-1}}) &\equiv p_{i_{N-1}}^{*G_{\mathbf{g}_{N-1,N}}}(\hat{\boldsymbol{\alpha}},\mathbf{m}_2,...,\mathbf{m}_{N-1},\mathbf{m}_N^s,\mathbf{m}_{N+1}^{\mathbf{s}}...,\mathbf{m}_{M+1}^{\mathbf{s}}).
\end{aligned}$$

In addition, we let $\alpha_0(s_{i_{N-1}})$ be the probability that i_{N-1} is not shortlisted given $(\hat{\boldsymbol{\alpha}},\mathbf{m}_2,...,\mathbf{m}_{N-1},\mathbf{m}_N^s,\mathbf{m}_{N+1}^{\mathbf{s}}...,\mathbf{m}_{M+1}^{\mathbf{s}})$. So $\alpha_0(s_{i_{N-1}})=1-\alpha_1(s_{i_{N-1}})-\alpha_2(s_{i_{N-1}})$. We have

$$\begin{aligned}
\Phi(s_{i_{N-1}})&\equiv\sum_{\forall h=N-1}^{N}\left[\begin{array}{c}\Pr{}^*(\mathbf{g}_{N-1,h}|\hat{\boldsymbol{\alpha}},\mathbf{m}_2,...,\mathbf{m}_{N-1},\mathbf{m}_N^s,\mathbf{m}_{N+1}^{\mathbf{s}}...,\mathbf{m}_{M+1}^{\mathbf{s}})\\ \cdot p_{i_{N-1}}^{*G_{\mathbf{g}_{N-1,h}}}(\hat{\boldsymbol{\alpha}},\mathbf{m}_2,...,\mathbf{m}_{N-1},\mathbf{m}_N^s,\mathbf{m}_{N+1}^{\mathbf{s}}...,\mathbf{m}_{M+1}^{\mathbf{s}})\end{array}\right]\\
&=\left[\begin{array}{c}\Pr{}^*(\mathbf{g}_{N-1,N-1}|\hat{\boldsymbol{\alpha}},\mathbf{m}_2,...,\mathbf{m}_{N-1},\mathbf{m}_N^s,\mathbf{m}_{N+1}^{\mathbf{s}}...,\mathbf{m}_{M+1}^{\mathbf{s}})\\ \cdot\, p_{i_{N-1}}^{*G_{\mathbf{g}_{N-1,N-1}}}(\hat{\boldsymbol{\alpha}},\mathbf{m}_2,...,\mathbf{m}_{N-1},\mathbf{m}_N^s,\mathbf{m}_{N+1}^{\mathbf{s}}...,\mathbf{m}_{M+1}^{\mathbf{s}})\end{array}\right]
\end{aligned}$$

$$
\begin{aligned}
&+\left[\begin{array}{c}\Pr{}^*(\mathbf{g}_{N-1,N}|\hat{\boldsymbol{\alpha}},\mathbf{m}_2,...,\mathbf{m}_{N-1},\mathbf{m}_N^s,\mathbf{m}_{N+1}^{\mathbf{s}}...,\mathbf{m}_{M+1}^{\mathbf{s}})\\ \cdot p_{i_{N-1}}^{*G_{\mathbf{g}_{N-1,N}}}(\hat{\boldsymbol{\alpha}},\mathbf{m}_2,...,\mathbf{m}_{N-1},\mathbf{m}_N^s,\mathbf{m}_{N+1}^{\mathbf{s}}...,\mathbf{m}_{M+1}^{\mathbf{s}})\end{array}\right]\\
=\;& \alpha_1(s_{i_{N-1}})P_1(s_{i_{N-1}})+\alpha_2(s_{i_{N-1}})P_2(s_{i_{N-1}}).
\end{aligned}
$$

When $s_{i_{N-1}}$ increases, $\alpha_1(s_{i_{N-1}})$ increases, $\alpha_2(s_{i_{N-1}})$ decreases, and $\alpha_0(s_{i_{N-1}})$ also decreases. That is, the probability that i_{N-1} will be shortlisted as the last one shortlisted increases, while the probability that i_{N-1} and i_N will be both shortlisted decreases and the probability that i_{N-1} will not be shortlisted also decreases. It is also clear that $P_1(s_{i_{N-1}})$ and $P_2(s_{i_{N-1}})$ are both increasing in $s_{i_{N-1}}$. Besides, we have $P_1(s_{i_{N-1}}) \geq P_2(s_{i_{N-1}})$ (i_{N-1} has a better chance of winning with a smaller set of entrants). Thus, given $s'_{i_{N-1}}, s''_{i_{N-1}},\ \ s''_{i_{N-1}} > s'_{i_{N-1}}$, we have

$$
\begin{aligned}
&\Phi(s''_{i_{N-1}})-\Phi(s'_{i_{N-1}})\\
=\;& \alpha_1(s''_{i_{N-1}})P_1(s''_{i_{N-1}})-\alpha_1(s'_{i_{N-1}})P_1(s'_{i_{N-1}})+\alpha_2(s''_{i_{N-1}})P_2(s''_{i_{N-1}})-\alpha_2(s'_{i_{N-1}})P_2(s'_{i_{N-1}})\\
\geq\;& \left[\alpha_1(s''_{i_{N-1}})-\alpha_1(s'_{i_{N-1}})\right]\cdot P_1(s'_{i_{N-1}})+\left[\alpha_2(s''_{i_{N-1}})-\alpha_2(s'_{i_{N-1}})\right]\cdot P_2(s'_{i_{N-1}})\\
\geq\;& \left[\alpha_1(s''_{i_{N-1}})-\alpha_1(s'_{i_{N-1}})+\alpha_2(s''_{i_{N-1}})-\alpha_2(s'_{i_{N-1}})\right]\cdot P_2(s'_{i_{N-1}})\\
=\;& \left[\alpha_0(s'_{i_{N-1}})-\alpha_0(s''_{i_{N-1}})\right]\cdot P_2(s'_{i_{N-1}})\\
\geq\;& 0.
\end{aligned}
$$

□

Thus we have (1) $\pi_{i_{N-1}}(s_{i_{N-1}},\widehat{s}_{i_{N-1}}|\hat{\boldsymbol{\alpha}},\mathbf{m}_2,...,\mathbf{m}_N)$ satisfies the strict and smooth single crossing differences property in $(s_{i_{N-1}},\widehat{s}_{i_{N-1}})$; (2) $\Phi(s_{i_{N-1}})$ increases in $\widehat{s}_{i_{N-1}}$; (3) $t^*_{N,i_{N-1}}$ defined above verifies the envelope theorem. By the constraint simplification theorem again, it is incentive compatible for agent i_{N-1} to reveal $s_{i_{N-1}}$ truthfully at stage N.

Following the same procedure and going backward stage by stage, for $k = N-2, N-3, ..., 3, 2$, we can construct payment rules for the corresponding stages to establish that agent i_{k-1} would reveal her type $s_{i_{k-1}}$ truthfully given allocation rules $(\mathbf{A}^*, \mathbf{p}^*)$ and the constructed payments rule $\mathbf{t}^*$ as long as $\hat{\alpha}_{i_l} = \alpha_{i_l}$, for $l \geq k+1$. In particular, the constructed transfers are given by

$$
\begin{aligned}
&t^*_{k,i_{k-1}}(\hat{\boldsymbol{\alpha}},\mathbf{m}_2,...,\mathbf{m}_{k-1},\mathbf{m}_k^{\mathbf{s}}) \qquad (35)\\
=\;& E_{(s_{i_{k+1}},...,s_{i_N})}\left\{u(\alpha_{i_{k-1}},s_{i_{k-1}})\sum_{\forall h=k-1}^{N}\left[\begin{array}{c}\Pr{}^*(\mathbf{g}_{N-1,h}|\hat{\boldsymbol{\alpha}},\mathbf{m}_2,...,\mathbf{m}_{k-1},\mathbf{m}_k^{\mathbf{s}},\mathbf{m}_{k+1}^{\mathbf{s}}...,\mathbf{m}_{M+1}^{\mathbf{s}})\\ \cdot p_{i_{k-1}}^{*G_{\mathbf{g}_{k-1,h}}}(\hat{\boldsymbol{\alpha}},\mathbf{m}_2,...,\mathbf{m}_{k-1},\mathbf{m}_k^{\mathbf{s}},\mathbf{m}_{k+1}^{\mathbf{s}}...,\mathbf{m}_{M+1}^{\mathbf{s}})\end{array}\right]\right\}\\
&-\int_0^{s_{i_{k-1}}} u_2(\alpha_{i_{N-1}},y)E_{s_{i_N}}\left\{\sum_{\forall h=N-1}^{N}\left[\begin{array}{c}\Pr{}^*(\mathbf{g}_{N-1,h}|\hat{\boldsymbol{\alpha}},\mathbf{m}_2,...,\mathbf{m}_{k-1},\mathbf{m}_k^{\mathbf{s}}(y),\mathbf{m}_{k+1}^{\mathbf{s}}...,\mathbf{m}_{M+1}^{\mathbf{s}})\\ \cdot p_{i_{k-1}}^{*G_{\mathbf{g}_{k-1,h}}}(\hat{\boldsymbol{\alpha}},\mathbf{m}_2,...,\mathbf{m}_{k-1},\mathbf{m}_k^{\mathbf{s}}(y),\mathbf{m}_{k+1}^{\mathbf{s}}...,\mathbf{m}_{M+1}^{\mathbf{s}})\end{array}\right]\right\}dy,
\end{aligned}
$$

where in $m_k^{\mathbf{s}}(y)$, $s_{i_{N-1}}$ is replaced by y.

In fact, in the preceding arguments we show that so long as $\hat{\alpha}_{i_l} = \alpha_{i_l}, l \geq k+1$, truthful revelation for stages $l \geq k+1$ also follows. We now turn to stage 1. We will show that given $\hat{\boldsymbol{\alpha}}_{-i} = \boldsymbol{\alpha}_{-i}$, under shortlisting and allocation rule $(\mathbf{A}^*, \mathbf{p}^*)$, we can construct stage-1 payments such that agent i reveals α_i truthfully and $\pi_i(\underline{\alpha},\underline{\alpha}) = 0, \forall i$. In addition, the shortlisted agent has the incentive to incur cost c.

By the same logic as in Lemma 2 of Liu et. al. (2020), when agent i reports $\hat{\alpha}_i$ at stage 1, she will report $\sigma_i(\alpha_i, \hat{\alpha}_i, s_i)$ when she is shortlisted later and asked to report her second type, where $\sigma_i(\alpha_i, \hat{\alpha}_i, s_i)$ is defined in (10). By the argument in the proof of Corollary 1 in Esö and Szentes (2007), we have that if $\sigma_i(\alpha_i, \hat{\alpha}_i, s_i)$ is interior, then $w_i(\alpha_i, s_i) \lesseqgtr w_i(\hat{\alpha}_i, \sigma_i(\alpha_i, \hat{\alpha}_i, s_i))$ if and only if $\alpha_i \lesseqgtr \hat{\alpha}_i$. The result also holds when $\sigma_i(\alpha_i, \hat{\alpha}_i, s_i)$ is at boundaries of interval $[0, 1]$ since $w_i(\alpha_i, s_i)$ increases with s_i.

Let $r(\hat{\alpha}_i, \boldsymbol{\alpha}_{-i})$ denote the rank of $\hat{\alpha}_i$ in $(\hat{\alpha}_i, \boldsymbol{\alpha}_{-i})$, and $\mathbf{m}^{\boldsymbol{\sigma}_i}_{r(\hat{\alpha}_i,\boldsymbol{\alpha}_{-i})+1}$ denote the stage $r(\hat{\alpha}_i, \boldsymbol{\alpha}_{-i})$ reports in which agent i's report is $\sigma_i(\alpha_i, \hat{\alpha}_i, s_i)$. Further assume that all shortlisted agents get a subsidy of c from the seller besides the stage-1 transfer $t_{1,i}(\cdot)$ to make sure that they have the incentive to conduct the due diligence. At stage 1, agent i's expected payoff when i is of type α_i but announces $\hat{\alpha}_i$ is:

$$\begin{aligned}
&\pi_i(\alpha_i, \hat{\alpha}_i) \\
= \; & E_{\boldsymbol{\alpha}_{-i}} E_{\mathbf{s}} \left\{ \begin{array}{c} -\sum_{\forall h = r(\hat{\alpha}_i,\boldsymbol{\alpha}_{-i})}^{N} \left[\begin{array}{c} \Pr{}^*(\mathbf{g}_{1,h} | (\hat{\alpha}_i, \boldsymbol{\alpha}_{-\mathbf{i}}), \mathbf{m}^{\mathbf{s}}_2, ..., \mathbf{m}^{\sigma_i(\alpha_i,\hat{\alpha}_i,s_i)}_{r(\hat{\alpha}_i,\boldsymbol{\alpha}_{-i})+1}, ..., \mathbf{m}^{\mathbf{s}}_{M+1}) \\ \cdot \left(t^*_{r(\hat{\alpha}_i,\boldsymbol{\alpha}_{-i})+1,i}((\hat{\alpha}_i, \boldsymbol{\alpha}_{-\mathbf{i}}), \mathbf{m}^{\mathbf{s}}_2, ..., \mathbf{m}^{\sigma_i(\alpha_i,\hat{\alpha}_i,s_i)}_{r(\hat{\alpha}_i,\boldsymbol{\alpha}_{-i})+1}) - c \right) \end{array} \right] \\ +\sum_{\forall h = r(\hat{\alpha}_i,\boldsymbol{\alpha}_{-i})}^{N} \left[\begin{array}{c} \Pr{}^*(\mathbf{g}_{1,h} | (\hat{\alpha}_i, \boldsymbol{\alpha}_{-\mathbf{i}}), \mathbf{m}^{\mathbf{s}}_2, ..., \mathbf{m}^{\sigma_i(\alpha_i,\hat{\alpha}_i,s_i)}_{r(\hat{\alpha}_i,\boldsymbol{\alpha}_{-i})+1}, ..., \mathbf{m}^{\mathbf{s}}_{M+1}) \\ \cdot \left[u(\alpha_i, s_i) p_i^{*G_{\mathbf{g}_{1,h}}}((\hat{\alpha}_i, \boldsymbol{\alpha}_{-\mathbf{i}}), \mathbf{m}^{\mathbf{s}}_2, ..., \mathbf{m}^{\sigma_i(\alpha_i,\hat{\alpha}_i,s_i)}_{r(\hat{\alpha}_i,\boldsymbol{\alpha}_{-i})+1}, ..., \mathbf{m}^{\mathbf{s}}_{M+1}) - c \right] \end{array} \right] \end{array} \right\} \\
& - E_{\boldsymbol{\alpha}_{-i}}[t_{1,i}((\hat{\alpha}_i, \boldsymbol{\alpha}_{-\mathbf{i}}))]. \qquad (36)
\end{aligned}$$

We are now ready to pin down the transfer $t^*_{1,i}(\cdot)$ (net of the entry subsidy c) that induces truthful revelation in stage 1.

By the envelop theorem, optimality of truthful revelation requires

$$\begin{aligned}
& \frac{d\pi_i(\alpha_i, \alpha_i)}{d\alpha_i} \\
= \; & E_{\boldsymbol{\alpha}_{-i}} E_{\mathbf{s}} u_1(\alpha_i, s_i) \sum_{\forall h = r(\alpha_i,\boldsymbol{\alpha}_{-i})}^{N} \left[\begin{array}{c} \Pr{}^*(\mathbf{g}_{1,h} | (\alpha_i, \boldsymbol{\alpha}_{-\mathbf{i}}), \mathbf{m}^{\mathbf{s}}_2, ..., \mathbf{m}^{\mathbf{s}}_{r(\alpha_i,\boldsymbol{\alpha}_{-i})+1}, ..., \mathbf{m}^{\mathbf{s}}_{M+1}) \\ \cdot\, p_i^{*G_{\mathbf{g}_{1,h}}}((\alpha_i, \boldsymbol{\alpha}_{-\mathbf{i}}), \mathbf{m}^{\mathbf{s}}_2, ..., \mathbf{m}^{\mathbf{s}}_{r(\alpha_i,\boldsymbol{\alpha}_{-i})+1}, ..., \mathbf{m}^{\mathbf{s}}_{M+1}) \end{array} \right]. \qquad (37)
\end{aligned}$$

Recall that we set $\pi_i(\underline{\alpha}, \underline{\alpha}) = 0$. We thus have

$$\begin{aligned}
& \pi_i(\alpha_i, \alpha_i) \qquad (38) \\
= \; & \int_{\underline{\alpha}}^{\alpha_i} E_{\boldsymbol{\alpha}_{-i}} E_{\mathbf{s}} u_1(y, s_i) \sum_{\forall h = r(y,\boldsymbol{\alpha}_{-i})}^{N} \left[\begin{array}{c} \Pr{}^*(\mathbf{g}_{1,h} | (y, \boldsymbol{\alpha}_{-\mathbf{i}}), \mathbf{m}^{\mathbf{s}}_2, ..., \mathbf{m}^{\mathbf{s}}_{r(y,\boldsymbol{\alpha}_{-i})+1}, ..., \mathbf{m}^{\mathbf{s}}_{M+1}) \\ \cdot p_i^{*G_{\mathbf{g}_{1,h}}}((y, \boldsymbol{\alpha}_{-\mathbf{i}}), \mathbf{m}^{\mathbf{s}}_2, ..., \mathbf{m}^{\mathbf{s}}_{r(y,\boldsymbol{\alpha}_{-i})+1}, ..., \mathbf{m}^{\mathbf{s}}_{M+1}) \end{array} \right] dy.
\end{aligned}$$

By (36) and (38), we define

$$
\begin{aligned}
& t^*_{1,i}(\hat{\boldsymbol{\alpha}}) \qquad (39)\\
= \; & E_{\mathbf{s}} \left\{ \begin{array}{c} -\sum_{\forall h=r(\hat{\alpha}_i,\hat{\boldsymbol{\alpha}}_{-i})} \left[\begin{array}{c} \Pr{}^*(\mathbf{g}_{1,h}|(\hat{\alpha}_i,\hat{\boldsymbol{\alpha}}_{-\mathbf{i}}),\mathbf{m}^{\mathbf{s}}_2,...,\mathbf{m}^{\mathbf{s}}_{r(\hat{\alpha}_i,\hat{\boldsymbol{\alpha}}_{-i})+1},...,\mathbf{m}^{\mathbf{s}}_{M+1}) \\ \cdot t^*_{r(\hat{\alpha}_i,\hat{\boldsymbol{\alpha}}_{-i})+1,i}((\hat{\alpha}_i,\hat{\boldsymbol{\alpha}}_{-\mathbf{i}}),\mathbf{m}^{\mathbf{s}}_2,...,\mathbf{m}^{\mathbf{s}}_{r(\hat{\alpha}_i,\hat{\boldsymbol{\alpha}}_{-i})+1}) \end{array} \right] \\ +\sum_{\forall h=r(\hat{\alpha}_i,\hat{\boldsymbol{\alpha}}_{-i})} \left[\begin{array}{c} \Pr{}^*(\mathbf{g}_{1,h}|(\hat{\alpha}_i,\hat{\boldsymbol{\alpha}}_{-\mathbf{i}}),\mathbf{m}^{\mathbf{s}}_2,...,\mathbf{m}^{\mathbf{s}}_{r(\hat{\alpha}_i,\hat{\boldsymbol{\alpha}}_{-i})+1},...,\mathbf{m}^{\mathbf{s}}_{M+1}) \\ \cdot \left[u(\hat{\alpha}_i,s_i) p_i^{*G_{\mathbf{g}_{1,h}}}((\hat{\alpha}_i,\hat{\boldsymbol{\alpha}}_{-\mathbf{i}}),\mathbf{m}^{\mathbf{s}}_2,...,\mathbf{m}^{\mathbf{s}}_{r(\hat{\alpha}_i,\hat{\boldsymbol{\alpha}}_{-i})+1},...,\mathbf{m}^{\mathbf{s}}_{M+1}) \right] \end{array} \right] \end{array} \right\} \\
& - \int_{\underline{\alpha}}^{\hat{\alpha}_i} E_{\mathbf{s}} u_1(y,s_i) \sum_{\forall h=r(y,\boldsymbol{\alpha}_{-i})}^{N} \left[\begin{array}{c} \Pr{}^*(\mathbf{g}_{1,h}|(y,\boldsymbol{\alpha}_{-\mathbf{i}}),\mathbf{m}^{\mathbf{s}}_2,...,\mathbf{m}^{\mathbf{s}}_{r(y,\boldsymbol{\alpha}_{-i})+1},...,\mathbf{m}^{\mathbf{s}}_{M+1}) \\ \cdot p_i^{*G_{\mathbf{g}_{1,h}}}((y,\boldsymbol{\alpha}_{-\mathbf{i}}),\mathbf{m}^{\mathbf{s}}_2,...,\mathbf{m}^{\mathbf{s}}_{r(y,\boldsymbol{\alpha}_{-i})+1},...,\mathbf{m}^{\mathbf{s}}_{M+1}) \end{array} \right] dy.
\end{aligned}
$$

To show that it is incentive compatible for agent i to reveal truthfully at stage 1 under $(\mathbf{A}^*, \mathbf{p}^*, \mathbf{t}^*)$, we need to show

$$\pi_i(\alpha_i,\alpha_i) \geq \pi_i(\alpha_i,\hat{\alpha}_i), \forall \alpha_i, \hat{\alpha}_i.$$

It is not readily clear that $\pi_i(\alpha_i,\hat{\alpha}_i)$ satisfies the single crossing property. As such, to establish IC we will turn to an alternative argument other than the constraint simplification theorem.

Provided that the mechanism is truthful after stage 1 if stage-1 reports are truthful, similar to (25), we have

$$
\begin{aligned}
& \pi_i(\alpha_i,\hat{\alpha}_i) - \pi_i(\hat{\alpha}_i,\hat{\alpha}_i) \\
= \; & \int_{\hat{\alpha}_i}^{\alpha_i} E_{\boldsymbol{\alpha}_{-i}} E_{\mathbf{s}} u_1(y,s_i) \sum_{\forall h=r(\hat{\alpha}_i,\boldsymbol{\alpha}_{-i})}^{N} \left[\begin{array}{c} \Pr{}^*(\mathbf{g}_{1,h}|(\hat{\alpha}_i,\boldsymbol{\alpha}_{-\mathbf{i}}),\mathbf{m}^{\mathbf{s}}_2,...,\mathbf{m}^{\sigma_i(y,\hat{\alpha}_i,s_i)}_{r(\hat{\alpha}_i,\boldsymbol{\alpha}_{-i})+1},...,\mathbf{m}^{\mathbf{s}}_{M+1}) \\ \cdot p_i^{*G_{\mathbf{g}_{1,h}}}((\hat{\alpha}_i,\boldsymbol{\alpha}_{-\mathbf{i}}),\mathbf{m}^{\mathbf{s}}_2,...,\mathbf{m}^{\sigma_i(y,\hat{\alpha}_i,s_i)}_{r(\hat{\alpha}_i,\boldsymbol{\alpha}_{-i})+1},...,\mathbf{m}^{\mathbf{s}}_{M+1}) \end{array} \right] dy.
\end{aligned}
$$

Consider

$$
\begin{aligned}
& \Delta \\
= \; & \pi_i(\alpha_i,\alpha_i) - \pi_i(\alpha_i,\hat{\alpha}_i) \\
= \; & [\pi_i(\alpha_i,\alpha_i) - \pi_i(\hat{\alpha}_i,\hat{\alpha}_i)] + [\pi_i(\hat{\alpha}_i,\hat{\alpha}_i) - \pi_i(\alpha_i,\hat{\alpha}_i)] \\
= \; & \int_{\hat{\alpha}_i}^{\alpha_i} E_{\boldsymbol{\alpha}_{-i}} E_{\mathbf{s}} u_1(y,s_i) \sum_{\forall h=r(y,\boldsymbol{\alpha}_{-i})}^{N} \left[\begin{array}{c} \Pr{}^*(\mathbf{g}_{1,h}|(y,\boldsymbol{\alpha}_{-\mathbf{i}}),\mathbf{m}^{\mathbf{s}}_2,...,\mathbf{m}^{\mathbf{s}}_{r(y,\boldsymbol{\alpha}_{-i})+1},...,\mathbf{m}^{\mathbf{s}}_{M+1}) \\ \cdot p_i^{*G_{\mathbf{g}_{1,h}}}((y,\boldsymbol{\alpha}_{-\mathbf{i}}),\mathbf{m}^{\mathbf{s}}_2,...,\mathbf{m}^{\mathbf{s}}_{r(y,\boldsymbol{\alpha}_{-i})+1},...,\mathbf{m}^{\mathbf{s}}_{M+1}) \end{array} \right] dy \\
& - \int_{\hat{\alpha}_i}^{\alpha_i} E_{\boldsymbol{\alpha}_{-i}} E_{\mathbf{s}} u_1(y,s_i) \sum_{\forall h=r(\hat{\alpha}_i,\boldsymbol{\alpha}_{-i})}^{N} \left[\begin{array}{c} \Pr{}^*(\mathbf{g}_{1,h}|(\hat{\alpha}_i,\boldsymbol{\alpha}_{-\mathbf{i}}),\mathbf{m}^{\mathbf{s}}_2,...,\mathbf{m}^{\sigma_i(y,\hat{\alpha}_i,s_i)}_{r(\hat{\alpha}_i,\boldsymbol{\alpha}_{-i})+1},...,\mathbf{m}^{\mathbf{s}}_{M+1}) \\ \cdot p_i^{*G_{\mathbf{g}_{1,h}}}((\hat{\alpha}_i,\boldsymbol{\alpha}_{-\mathbf{i}}),\mathbf{m}^{\mathbf{s}}_2,...,\mathbf{m}^{\sigma_i(y,\hat{\alpha}_i,s_i)}_{r(\hat{\alpha}_i,\boldsymbol{\alpha}_{-i})+1},...,\mathbf{m}^{\mathbf{s}}_{M+1}) \end{array} \right] dy
\end{aligned}
$$

$$= \int_{\hat{\alpha}_i}^{\alpha_i} E_{\boldsymbol{\alpha}_{-i}} E_{\mathbf{s}} u_1(y, s_i) \left\{ \begin{array}{c} \sum_{\forall h = r(y, \boldsymbol{\alpha}_{-i})}^{N} \left[\begin{array}{c} \Pr{}^*(\mathbf{g}_{1,h} | (y, \boldsymbol{\alpha}_{-\mathbf{i}}), \mathbf{m}_2^{\mathbf{s}}, ..., \mathbf{m}_{r(y, \boldsymbol{\alpha}_{-i})+1}^{\mathbf{s}}, ..., \mathbf{m}_{M+1}^{\mathbf{s}}) \\ \cdot\, p_i^{*G_{\mathbf{g}_{1,h}}}((y, \boldsymbol{\alpha}_{-\mathbf{i}}), \mathbf{m}_2^{\mathbf{s}}, ..., \mathbf{m}_{r(y, \boldsymbol{\alpha}_{-i})+1}^{\mathbf{s}}, ..., \mathbf{m}_{M+1}^{\mathbf{s}}) \end{array} \right] \\ - \sum_{\forall h = r(\hat{\alpha}_i, \boldsymbol{\alpha}_{-i})}^{N} \left[\begin{array}{c} \Pr{}^*(\mathbf{g}_{1,h} | (\hat{\alpha}_i, \boldsymbol{\alpha}_{-\mathbf{i}}), \mathbf{m}_2^{\mathbf{s}}, ..., \mathbf{m}_{r(\hat{\alpha}_i, \boldsymbol{\alpha}_{-i})+1}^{\sigma_i(y, \hat{\alpha}_i, s_i)}, ..., \mathbf{m}_{M+1}^{\mathbf{s}}) \\ \cdot\, p_i^{*G_{\mathbf{g}_{1,h}}}((\hat{\alpha}_i, \boldsymbol{\alpha}_{-\mathbf{i}}), \mathbf{m}_2^{\mathbf{s}}, ..., \mathbf{m}_{r(\hat{\alpha}_i, \boldsymbol{\alpha}_{-i})+1}^{\sigma_i(y, \hat{\alpha}_i, s_i)}, ..., \mathbf{m}_{M+1}^{\mathbf{s}}) \end{array} \right] \end{array} \right\} dy.$$

Note that

$$\tilde{P}(y, \boldsymbol{\alpha}_{-i}, \mathbf{s}) = \sum_{\forall h = r(y, \boldsymbol{\alpha}_{-i})}^{N} \left[\begin{array}{c} \Pr{}^*(\mathbf{g}_{1,h} | (y, \boldsymbol{\alpha}_{-\mathbf{i}}), \mathbf{m}_2^{\mathbf{s}}, ..., \mathbf{m}_{r(y, \boldsymbol{\alpha}_{-i})+1}^{\mathbf{s}}, ..., \mathbf{m}_{M+1}^{\mathbf{s}}) \\ \cdot p_i^{*G_{\mathbf{g}_{1,h}}}((y, \boldsymbol{\alpha}_{-\mathbf{i}}), \mathbf{m}_2^{\mathbf{s}}, ..., \mathbf{m}_{r(y, \boldsymbol{\alpha}_{-i})+1}^{\mathbf{s}}, ..., \mathbf{m}_{M+1}^{\mathbf{s}}) \end{array} \right]$$

is agent i's winning probability given her type y and that she reports truthfully; and

$$\tilde{P}(\hat{\alpha}_i, \boldsymbol{\alpha}_{-i}, \mathbf{s}) = \sum_{\forall h = r(\hat{\alpha}_i, \boldsymbol{\alpha}_{-i})}^{N} \left[\begin{array}{c} \Pr{}^*(\mathbf{g}_{1,h} | (\hat{\alpha}_i, \boldsymbol{\alpha}_{-\mathbf{i}}), \mathbf{m}_2^{\mathbf{s}}, ..., \mathbf{m}_{r(\hat{\alpha}_i, \boldsymbol{\alpha}_{-i})+1}^{\sigma_i(y, \hat{\alpha}_i, s_i)}, ..., \mathbf{m}_{M+1}^{\mathbf{s}}) \\ \cdot p_i^{*G_{\mathbf{g}_{1,h}}}((\hat{\alpha}_i, \boldsymbol{\alpha}_{-\mathbf{i}}), \mathbf{m}_2^{\mathbf{s}}, ..., \mathbf{m}_{r(\hat{\alpha}_i, \boldsymbol{\alpha}_{-i})+1}^{\sigma_i(y, \hat{\alpha}_i, s_i)}, ..., \mathbf{m}_{M+1}^{\mathbf{s}}) \end{array} \right]$$

is agent i's winning probability given her type y and that she reports $\hat{\alpha}_i$ in stage 1 and corrects her lie when shortlisted. Recall that $w_i(y, s_i) \lesseqgtr w_i(\hat{\alpha}_i, \sigma_i(y, \hat{\alpha}_i, s_i))$ if and only if $y \lesseqgtr \hat{\alpha}_i$. $\forall \boldsymbol{\alpha}_{-i}, \mathbf{s}$, by definition of $(\mathbf{A}^*, \mathbf{p}^*)$, $\forall \boldsymbol{\alpha}_{-i}, \mathbf{s}$, we conclude that the $\tilde{P}(y, \boldsymbol{\alpha}_{-i}, \mathbf{s}) \geq \tilde{P}(\hat{\alpha}_i, \boldsymbol{\alpha}_{-i}, \mathbf{s})$ if and only if $y \geq \hat{\alpha}_i$ because in the latter case a (weakly) bigger group is shortlisted, and i's winning probability is smaller even when the group remains the same. Thus we must have $\Delta \geq 0$. $\square$

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Online Appendix: "Orchestrating Information Acquisition" by Lu, Ye and Feng

Details for Section IV.A: Allowing Direct Sale in The First Stage

This online appendix contains the details for deriving the revenue-maximizing mechanisms allowing sale in the first stage. As in the main text, we will proceed with two cases, the first with single-round shortlisting and the second with sequential shortlisting.

A Single-round Shortlisting

A.1 Mechanisms

The first-stage mechanism is characterized by the selling rule $p_i(\boldsymbol{\alpha})$, the shortlisting rule $A^g(\boldsymbol{\alpha})$, and payment rule $x_i(\boldsymbol{\alpha}), i = 1, 2, ..., N$. Given the reported profile $\boldsymbol{\alpha}$, the selling rule $p_i(\boldsymbol{\alpha}) : [\underline{\alpha}, \overline{\alpha}]^N \to [0, 1]$, assigns a selling probability to each buyer i, where $\sum_{i\in\mathbf{N}} p_i(\boldsymbol{\alpha}) \leq 1$; if the object is unsold in the first stage, the shortlisting rule, $A^g : [\underline{\alpha}, \overline{\alpha}]^N \to [0, 1]$, assigns a probability to each subgroup $g \in 2^{\mathbf{N}}$ for information acquisition, where $\sum_{g\in 2^{\mathbf{N}}} A^g(\boldsymbol{\alpha}) = 1$. The payment rule $x_i : [\underline{\alpha}, \overline{\alpha}]^N \to \mathbb{R}$, specifies bidder i's first-stage payment given the reported profile $\boldsymbol{\alpha}$.

Given the first-stage reported profile $\boldsymbol{\alpha}$, and that group g is shortlisted for information acquisition, the second-stage mechanism is characterized by $p_i^g(\boldsymbol{\alpha}, \mathbf{s}^g)$, the probability with which the asset is allocated to buyer $i \in g$, and $t_i^g(\boldsymbol{\alpha}, \mathbf{s}^g)$, the payment to the seller made by buyer $i \in g, \forall g \in 2^{\mathbf{N}}$.

We will identify the revenue-maximizing mechanism in two steps. First, we establish a revenue bound by considering a relaxed problem in which the second-stage signal s_i is known to the shortlisted buyer i. In this relaxed problem, we ignore the second-stage incentive compatibility condition (IC) and individual rationality condition (IR). Second, we will identify a feasible mechanism (satisfying IC and IR in both stages) in the original setting, which achieves the above revenue bound.

A.2 A Revenue Upper Bound with Public $\mathbf{s}$

We will first identify an upper bound for the expected revenue in a relaxed setting with public $\mathbf{s}$ for the shortlisted buyers. In this relaxed setting, the mechanisms are specified exactly the same as in Section 3.1. We drop the IC and IR constraints for the shortlisted bidders in the second stage so that all shortlisted bidders must incur entry costs to learn their second-stage signals as in our original setup, and regardless of their second-stage signals, they must participate in the second-stage selling mechanism and report their second-stage signals truthfully.

As a result, the highest possible expected revenue achievable in this relaxed setting imposes an upper bound for the expected revenue that can be obtained in our original setup, where the bidders' second-stage IC and IR constraints must both be satisfied. We next proceed to identify this bound.

Given the announced $\boldsymbol{\alpha}$ and s_i, let the interim winning probability and expected payment be, respectively, $P_i^g(\boldsymbol{\alpha}, s_i) = E_{\mathbf{s}_{-i}^g} p_i^g(\boldsymbol{\alpha}, \mathbf{s}^g)$ and $T_i^g(\boldsymbol{\alpha}, s_i) = E_{\mathbf{s}_{-i}^g} t_i^g(\boldsymbol{\alpha}, \mathbf{s}^g)$, where $\mathbf{s}_{-i}^g = \mathbf{s}^g \backslash \{s_i\}$, $\forall i \in g$ and $\forall g \in 2^{\mathbf{N}}$. Let g_i denote a shortlisted subgroup that contains bidder i. For shortlisted bidder $i \in g_i$ with type α_i, her interim expected payoff when she reports $\hat{\alpha}_i$ and others report truthfully is given by

$$\begin{aligned} \pi_i(\alpha_i, \hat{\alpha}_i) &= -E_{\boldsymbol{\alpha}_{-i}} x_i(\hat{\alpha}_i, \boldsymbol{\alpha}_{-i}) + \\ & E_{\boldsymbol{\alpha}_{-i}} \left\{ \begin{array}{c} p_i(\hat{\alpha}_i; \boldsymbol{\alpha}_{-i}) E_{s_i} u(\alpha_i, s_i) + \\ \left[\begin{array}{c} \left[1 - \sum_j p_j(\hat{\alpha}_i; \boldsymbol{\alpha}_{-i})\right] \\ \cdot \sum_{g_i} A^{g_i}(\hat{\alpha}_i, \boldsymbol{\alpha}_{-i}) \left[E_{s_i}\left(u(\alpha_i, s_i) P_i^{g_i}(\hat{\alpha}_i, \boldsymbol{\alpha}_{-i}, s_i) - T_i^{g_i}(\hat{\alpha}_i, \boldsymbol{\alpha}_{-i}, s_i)\right) - c\right] \end{array} \right] \end{array} \right\}. \end{aligned} \tag{40}$$

The IC condition requires $\pi_i(\alpha_i, \alpha_i) \geq \pi_i(\alpha_i, \hat{\alpha}_i)$. Standard arguments such as envelope theorem (cf. Theorem 2 in Milgrom and Segal (2002)) lead to the following result:

$$\begin{aligned} \frac{d\pi_i(\alpha_i, \alpha_i)}{d\alpha_i} &= \frac{\partial \pi_i(\alpha_i, \hat{\alpha}_i)}{\partial \alpha_i}\Big|_{\hat{\alpha}_i = \alpha_i} \\ &= E_{\boldsymbol{\alpha}_{-i}} \left\{ \begin{array}{c} p_i(\alpha_i; \boldsymbol{\alpha}_{-i}) E_{s_i} u_1(\alpha_i, s_i) + \\ \left[\begin{array}{c} \left[1 - \sum_j p_j(\alpha_i; \boldsymbol{\alpha}_{-i})\right] \\ \cdot \sum_{g_i} A^{g_i}(\alpha_i, \boldsymbol{\alpha}_{-i}) \left[E_{s_i} u_1(\alpha_i, s_i) P_i^{g_i}(\alpha_i, \boldsymbol{\alpha}_{-i}, s_i)\right] \end{array} \right] \end{array} \right\}. \end{aligned} \tag{41}$$

Therefore, we have

$$\begin{aligned} & \pi_i(\alpha_i, \alpha_i) \\ = \; & \pi_i(\underline{\alpha}, \underline{\alpha}) + E_{\boldsymbol{\alpha}_{-i}} \int_{\underline{\alpha}}^{\alpha_i} \left\{ \begin{array}{c} p_i(y; \boldsymbol{\alpha}_{-i}) E_{s_i} u_1(y, s_i) + \\ \left[\begin{array}{c} \left[1 - \sum_j p_j(y; \boldsymbol{\alpha}_{-i})\right] \\ \cdot \sum_{g_i} A^{g_i}(y, \boldsymbol{\alpha}_{-i}) \left[E_{s_i} u_1(y, s_i) P_i^{g_i}(y, \boldsymbol{\alpha}_{-i}, s_i)\right] \end{array} \right] \end{array} \right\} dy \\ = \; & \pi_i(\underline{\alpha}, \underline{\alpha}) + E_{\alpha_{-i}} \int_{\underline{\alpha}}^{\alpha_i} \int u_1(y, s_i) \cdot \left\{ \begin{array}{c} p_i(y; \boldsymbol{\alpha}_{-i}) + \\ \left[\begin{array}{c} \left[1 - \sum_j p_j(y; \boldsymbol{\alpha}_{-i})\right] \\ \cdot \sum_{g_i} A^{g_i}(y, \boldsymbol{\alpha}_{-i}) \left[P_i^{g_i}(y; \boldsymbol{\alpha}_{-i}, s_i)\right] \end{array} \right] \end{array} \right\} dG_i(s_i) dy. \end{aligned} \tag{42}$$

Taking expectation, we have

$$\begin{aligned} & E\pi_i(\alpha_i, \alpha_i) \\ = \; & \pi_i(\underline{\alpha}, \underline{\alpha}) + \int_{\underline{\alpha}}^{\overline{\alpha}} \int_{\underline{\alpha}}^{\alpha_i} E_{s_i} \left[E_{\alpha_{-i}} \left\{ \begin{array}{c} p_i(y; \boldsymbol{\alpha}_{-i}) u_1(y, s_i) + \\ \left[\begin{array}{c} \left[1 - \sum_j p_j(y; \boldsymbol{\alpha}_{-i})\right] \\ \cdot \sum_{g_i} \left[A^{g_i}(y, \boldsymbol{\alpha}_{-i}) P_i^{g_i}(y, \boldsymbol{\alpha}_{-i}, s_i) u_1(y, s_i)\right] \end{array} \right] \end{array} \right\} \right] dy dF(\alpha_i) \end{aligned}$$

$$
\begin{aligned}
&= \pi_i(\underline{\alpha},\underline{\alpha}) + E_{\alpha_i}\left[\frac{1-F(\alpha_i)}{f(\alpha_i)}E_{s_i}\left[E_{\alpha_{-i}}\left[\left[\begin{array}{c} p_i(\alpha_i;\boldsymbol{\alpha}_{-i})u_1(\alpha_i,s_i)+ \\ \left[1-\sum_j p_j(\alpha_i;\boldsymbol{\alpha}_{-i})\right] \\ \cdot\sum_{g_i}\left[A^{g_i}(\alpha_i,\boldsymbol{\alpha}_{-i})P_i^{g_i}(\alpha_i,\boldsymbol{\alpha}_{-i},s_i)u_1(\alpha_i,s_i)\right]\end{array}\right]\right]\right]\right] \\
&= \pi_i(\underline{\alpha},\underline{\alpha}) + E_{\boldsymbol{\alpha}}\left\{E_{s_i}\left[\frac{1-F(\alpha_i)}{f(\alpha_i)}\left\{\left[\begin{array}{c} p_i(\boldsymbol{\alpha})u_1(\alpha_i,s_i)+ \\ \left[1-\sum_j p_j(\boldsymbol{\alpha})\right] \\ \cdot\sum_{g_i}\left[A^{g_i}(\boldsymbol{\alpha})P_i^{g_i}(\boldsymbol{\alpha},s_i)u_1(\alpha_i,s_i)\right]\end{array}\right]\right\}\right]\right\}. \qquad (43)
\end{aligned}
$$

Thus

$$
\sum_{i=1}^{N} E\pi_i(\alpha_i,\alpha_i) = \sum_{i=1}^{N}\pi_i(\underline{\alpha},\underline{\alpha}) + E_{\boldsymbol{\alpha}}\left\{\left[\begin{array}{c} \sum_i\left[p_i(\boldsymbol{\alpha})\frac{1-F(\alpha_i)}{f(\alpha_i)}E_{s_i}u_1(\alpha_i,s_i)\right]+ \\ \left[1-\sum_j p_j(\boldsymbol{\alpha})\right] \\ \cdot\sum_g A^g(\boldsymbol{\alpha})E_{\mathbf{s}}\sum_{i\in g}\left[p_i^g(\boldsymbol{\alpha},\mathbf{s}^g)\frac{1-F(\alpha_i)}{f(\alpha_i)}u_1(\alpha_i,s_i)\right]\end{array}\right]\right\}. \qquad (44)
$$

The total expected surplus from the two-stage mechanism is

$$
TS = E_{\boldsymbol{\alpha}}\left\{\begin{array}{c} \sum_i p_i(\boldsymbol{\alpha})E_{s_i}u(\alpha_i,s_i) \\ +\left[1-\sum_i p_i(\boldsymbol{\alpha})\right]\sum_g\left\{A^g(\boldsymbol{\alpha})E_{\mathbf{s}}\left[\sum_{i\in g}p_i^g(\boldsymbol{\alpha},\mathbf{s}^g)u(\alpha_i,s_i)-|g|c\right]\right\}\end{array}\right\}. \qquad (45)
$$

The seller's expected revenue is thus given by

$$
\begin{aligned}
& ER \\
&= TS - \sum_{i=1}^{N} E\pi_i(\alpha_i,\alpha_i) \\
&= E_{\boldsymbol{\alpha}}\left\{\begin{array}{c} \sum_i\left[p_i(\boldsymbol{\alpha})E_{s_i}\left(u(\alpha_i,s_i)-\frac{1-F(\alpha_i)}{f(\alpha_i)}u_1(\alpha_i,s_i)\right)\right]+ \\ \left[1-\sum_i p_i(\boldsymbol{\alpha})\right]\sum_g A^g(\boldsymbol{\alpha})E_{\mathbf{s}}\left[\sum_{i\in g}p_i^g(\boldsymbol{\alpha},\mathbf{s}^g)\left(u(\alpha_i,s_i)-\frac{1-F(\alpha_i)}{f(\alpha_i)}u_1(\alpha_i,s_i)\right)-|g|c\right]\end{array}\right\} \\
&\quad -\sum_{i=1}^{N}\pi_i(\underline{\alpha},\underline{\alpha}). \qquad (46)
\end{aligned}
$$

Clearly, to maximize ER, the seller should set $\pi_i(\underline{\alpha},\underline{\alpha}) = 0$ for all $i = 1, 2, ..., N$.

Recall that the *virtual value adjusted by the second-stage signal is defined* in (3):

$$
w(\alpha_i,s_i) = u(\alpha_i,s_i) - \frac{1-F(\alpha_i)}{f(\alpha_i)}u_1(\alpha_i,s_i).
$$

From the expression of the expected revenue, we can derive the optimal allocation rules in both stages as follows. At the second stage, given the revealed $\boldsymbol{\alpha}$ and the shortlisted group g, $\forall\mathbf{s}^g$, $p_i^{*g}(\boldsymbol{\alpha},\mathbf{s}^g)$

takes the same form as in (4)[36]

$$p_i^{*g}(\boldsymbol{\alpha}, \mathbf{s}^g) = \begin{cases} 1 & \text{if } i = \arg\max_{j \in g}\{w(\alpha_j, s_j)\} \text{ and } w(\alpha_i, s_i) \geq 0 \\ 0 & \text{otherwise} \end{cases} \forall g, \forall i \in g.$$

Recall that the expected *virtual surplus* (the virtual value less the entry cost) is defined in (5):

$$w^{*g}(\boldsymbol{\alpha}) = E_{\mathbf{s}}\left[\sum_{i \in g} p_i^{*g}(\boldsymbol{\alpha}, \mathbf{s}^g) w(\alpha_i, s_i) - |g|c\right].$$

At the first stage, contingent on the revealed $\boldsymbol{\alpha}$, the optimal shortlisting rule is given in (6):[37]

$$A^{*g}(\boldsymbol{\alpha}) = \begin{cases} 1 & \text{if } g = \arg\max_{\tilde{g}}\{w^{*\tilde{g}}(\boldsymbol{\alpha})\} \text{ and } w^{*g}(\boldsymbol{\alpha}) \geq 0 \\ 0 & \text{otherwise} \end{cases} \forall g.$$

Recall that $g^*(\boldsymbol{\alpha})$ denotes the set of bidders admitted under the optimal shortlisting rule. The highest revenue generated from the second-stage sale is

$$R_2^{g^*(\boldsymbol{\alpha})}(\boldsymbol{\alpha}) = E_{\mathbf{s}}\left[\sum_{i \in g^*(\boldsymbol{\alpha})} p_i^{g^*(\boldsymbol{\alpha})}(\boldsymbol{\alpha}, \mathbf{s}^{g^*(\boldsymbol{\alpha})}) \left(u(\alpha_i, s_i) - \frac{1 - F(\alpha_i)}{f(\alpha_i)} u_1(\alpha_i, s_i)\right) - |g^*(\boldsymbol{\alpha})|c\right], \tag{47}$$

and the highest revenue generated from the first-stage sale is

$$\begin{aligned} R_1^*(\boldsymbol{\alpha}) &= \max_{\{i=1,2,..,N\}} E_{s_i}\left(u(\alpha_i, s_i) - \frac{1 - F(\alpha_i)}{f(\alpha_i)} u_1(\alpha_i, s_i)\right) \\ &= E_s\left(u(\alpha_{(1)}, s) - \frac{1 - F(\alpha_{(1)})}{f(\alpha_{(1)})} u_1(\alpha_{(1)}, s)\right), \end{aligned} \tag{48}$$

where $\alpha_{(1)}$ denotes the highest first-stage type among all buyers, and s is distributed uniformly over $[0, 1]$.[38]

Clearly, the optimal first-stage selling probabilities are:

$$p_i^*(\boldsymbol{\alpha}) = \begin{cases} 1 & \text{if } \alpha_i \geq \alpha_j, \forall j \text{ and } R_1^*(\boldsymbol{\alpha}) \geq R_2^{g^*(\boldsymbol{\alpha})}(\boldsymbol{\alpha}), \\ 0 & \text{otherwise,} \end{cases} \forall i. \tag{49}$$

In other words, given first-stage type profile $\boldsymbol{\alpha}$, the object is sold in the first stage if and only if

[36] Ties occur with probability zero and are hence ignored.

[37] Again ties occur with probability zero and are hence ignored.

[38] Assumptions 1 and 2 imply $u_{11} \leq 0$, and we have $u_1 > 0$ and $(\frac{1-F(\cdot)}{f(\cdot)})' \leq 0$. These imply that the buyer with $\alpha_{(1)}$ possesses the highest expected virtual value.

by doing so it generates higher expected revenue than that from first-stage optimal shortlisting and second-stage optimal selling mechanism.

Allocation rule $(p_i^{*g}(\boldsymbol{\alpha}, \mathbf{s}^g),\ A^{*g}(\boldsymbol{\alpha}),\ p_i^*(\boldsymbol{\alpha}))$ gives rise to the following bound for the seller's expected revenue:

$$ER^{**} = E_{\boldsymbol{\alpha}} \left\{ \begin{array}{c} \sum_i \left[p_i^*(\boldsymbol{\alpha}) E_{s_i} w(\alpha_i, s_i)\right] + \\ \left[1 - \sum_i p_i^*(\boldsymbol{\alpha})\right] \sum_g A^{*g}(\boldsymbol{\alpha}) E_{\mathbf{s}} \left[\sum_{i \in g} p_i^{*g}(\boldsymbol{\alpha}, \mathbf{s}^g) w(\alpha_i, s_i) - |g| c\right] \end{array} \right\}. \tag{50}$$

We have the following property for the first stage selling probabilities.

Lemma 6. Given $\boldsymbol{\alpha}_{-i}$, if $p_i^*(\alpha_i, \boldsymbol{\alpha}_{-i}) = 1$, then $p_i^*(\tilde{\alpha}_i, \boldsymbol{\alpha}_{-i}) = 1$ for $\tilde{\alpha}_i > \alpha_i$.

Proof of Lemma 6: To establish this result, it suffices to show that for any g_i, if

$$E_{s_i}\left(u(\alpha_i, s_i) - \frac{1 - F(\alpha_i)}{f(\alpha_i)} u_1(\alpha_i, s_i)\right) \geq R_2^{g_i}(\alpha_i, \boldsymbol{\alpha}_{-i}),$$

then

$$E_{s_i}\left(u(\tilde{\alpha}_i, s_i) - \frac{1 - F(\tilde{\alpha}_i)}{f(\tilde{\alpha}_i)} u_1(\alpha_i, s_i)\right) \geq R_2^{g_i}(\tilde{\alpha}_i, \boldsymbol{\alpha}_{-i}), \text{ for any } \tilde{\alpha}_i > \alpha_i.$$

We first consider the case $g_i = \{i\}$. In this case,

$$\begin{aligned} & E_{s_i}\left(u(\alpha_i, s_i) - \frac{1 - F(\alpha_i)}{f(\alpha_i)} u_1(\alpha_i, s_i)\right) \geq R_2^{g_i}(\alpha_i, \boldsymbol{\alpha}_{-i}) \\ \iff\ & E_{s_i}\left(u(\alpha_i, s_i) - \frac{1 - F(\alpha_i)}{f(\alpha_i)} u_1(\alpha_i, s_i)\right) \geq E_{s_i} \max\left\{\left(u(\alpha_i, s_i) - \frac{1 - F(\alpha_i)}{f(\alpha_i)} u_1(\alpha_i, s_i)\right), 0\right\} - c \\ \iff\ & c \geq -E_{s_i} \min\left\{\left(u(\alpha_i, s_i) - \frac{1 - F(\alpha_i)}{f(\alpha_i)} u_1(\alpha_i, s_i)\right), 0\right\}. \end{aligned}$$

Since $u(\alpha_i, s_i) - \frac{1-F(\alpha_i)}{f(\alpha_i)} u_1(\alpha_i, s_i)$ increases in α_i, we have

$$c \geq -E_{s_i} \min\left\{\left(u(\tilde{\alpha}_i, s_i) - \frac{1 - F(\tilde{\alpha}_i)}{f(\tilde{\alpha}_i)} u_1(\tilde{\alpha}_i, s_i)\right), 0\right\},$$

which further leads to

$$E_{s_i}\left(u(\tilde{\alpha}_i, s_i) - \frac{1 - F(\tilde{\alpha}_i)}{f(\tilde{\alpha}_i)} u_1(\tilde{\alpha}_i, s_i)\right) \geq R_2^{g_i}(\tilde{\alpha}_i, \boldsymbol{\alpha}_{-i}).$$

Hence we have $p_i^*(\tilde{\alpha}_i, \boldsymbol{\alpha}_{-i}) = 1$ for $\tilde{\alpha}_i > \alpha_i$.

We now turn to the case $g_i \supset \{i\}$. In this case, define

$$\xi = \max_{j \in g_i \setminus \{i\}} \left\{u(\alpha_j, s_j) - \frac{1 - F(\alpha_j)}{f(\alpha_j)} u_1(\alpha_j, s_j)\right\} \vee 0.$$

We have

$$
\begin{aligned}
& E_{s_i}\left(u(\alpha_i, s_i) - \frac{1-F(\alpha_i)}{f(\alpha_i)} u_1(\alpha_i, s_i)\right) \geq R_2^{g_i}(\alpha_i, \boldsymbol{\alpha}_{-i}) \\
\iff \quad & E_{s_i}\left(u(\alpha_i, s_i) - \frac{1-F(\alpha_i)}{f(\alpha_i)} u_1(\alpha_i, s_i)\right) \geq E_{s_i,\xi} \max\left\{u(\alpha_i, s_i) - \frac{1-F(\alpha_i)}{f(\alpha_i)} u_1(\alpha_i, s_i), \xi\right\} - |g_i|c \\
\iff \quad & |g_i|c + E_{s_i,\xi} \min\left\{\left(u(\alpha_i, s_i) - \frac{1-F(\alpha_i)}{f(\alpha_i)} u_1(\alpha_i, s_i)\right), \xi\right\} \geq 0.
\end{aligned}
$$

Since $u(\alpha_i, s_i) - \frac{1-F(\alpha_i)}{f(\alpha_i)} u_1(\alpha_i, s_i)$ increases in α_i, we have

$$
|g_i|c + E_{s_i,\xi} \min\left\{\left(u(\tilde{\alpha}, s_i) - \frac{1-F(\tilde{\alpha})}{f(\tilde{\alpha})} u_1(\tilde{\alpha}, s_i)\right), \xi\right\} \geq 0,
$$

which further leads to

$$
E_{s_i}\left(u(\tilde{\alpha}_i, s_i) - \frac{1-F(\tilde{\alpha}_i)}{f(\tilde{\alpha}_i)} u_1(\tilde{\alpha}_i, s_i)\right) \geq R_2^{g_i}(\tilde{\alpha}_i, \boldsymbol{\alpha}_{-i}),
$$

hence $p_i^*(\tilde{\alpha}_i, \boldsymbol{\alpha}_{-i}) = 1$ for $\tilde{\alpha}_i > \alpha_i$. □

It is clear that ER^{**} provides an upper bound for the seller expected revenue in the original setting with private $\mathbf{s}$.

A.3 Revenue-Maximizing Selling Mechanism in The Original Setting

We will establish that ER^{**} can be achieved by a feasible mechanism (satisfying IC and IR in both stages) in the original setting. To this end, we will first establish necessary conditions implied by IC conditions in both stages.

We start with the second stage. Suppose group g is shortlisted, and the profile $\tilde{\boldsymbol{\alpha}}$ reported in the first stage is revealed as public information to the shortlisted bidders. First, suppose $\boldsymbol{\alpha}$ is truthfully reported at the first stage and group g is shortlisted. Assume that they follow the recommendation and incur the information acquisition cost c to discover $\mathbf{s}^g$.

We are now ready to consider the implication of the first-stage IC. The lie correction strategy of (9) and (10) still hold. Let $\pi_i(\alpha_i, \hat{\alpha}_i)$ be the expected payoff (net of the entry cost) for a type-α_i bidder who reports $\hat{\alpha}_i$ in the first stage. In particular, we have

$$
\begin{aligned}
& \pi_i(\alpha_i, \hat{\alpha}_i) \\
= \quad & E_{\boldsymbol{\alpha}_{-i}}\left\{ p_i(\hat{\alpha}_i; \boldsymbol{\alpha}_{-i}) E_{s_i} u(\alpha_i, s_i) + \left[\begin{array}{c} [1 - \sum_j p_j(\hat{\alpha}_i; \boldsymbol{\alpha}_{-i})] \\ \cdot \sum_{g_i} A^{g_i}(\hat{\alpha}_i, \boldsymbol{\alpha}_{-i})[\tilde{\pi}_i^{g_i}(\alpha_i, \hat{\alpha}_i; \boldsymbol{\alpha}_{-i}) - c] \end{array} \right] \right. \\
& \qquad \left. -x_i(\hat{\alpha}_i, \boldsymbol{\alpha}_{-i}) \right\}
\end{aligned}
$$

$$
\begin{aligned}
&= E_{\boldsymbol{\alpha}_{-i}} \left\{ \begin{array}{c} p_i(\hat{\alpha}_i;\boldsymbol{\alpha}_{-i}) E_{s_i} u(\alpha_i, s_i) + \\ \left[\begin{array}{c} [1-\sum_j p_j(\hat{\alpha}_i;\boldsymbol{\alpha}_{-i})] \\ \cdot \sum_{g_i} A^{g_i}(\hat{\alpha}_i,\boldsymbol{\alpha}_{-i}) \left[E_{s_i} \left(u(\alpha_i,s_i) P_i^{g_i}(\hat{\alpha}_i,\boldsymbol{\alpha}_{-i},\hat{s}_i) - T_i^{g_i}(\hat{\alpha}_i,\boldsymbol{\alpha}_{-i},\hat{s}_i) \right) - c \right] \end{array} \right] \end{array} \right\} \\
&\quad -x_i(\hat{\alpha}_i), \qquad (51)
\end{aligned}
$$

where $\hat{s}_i = \sigma_i(\alpha_i, \hat{\alpha}_i, s_i)$ and $x_i(\hat{\alpha}_i) = E_{\boldsymbol{\alpha}_{-i}} x_i(\hat{\alpha}_i, \boldsymbol{\alpha}_{-i})$.

By similar arguments leading to (25), we have

$$
\begin{aligned}
&\frac{\partial \pi_i(\alpha_i, \hat{\alpha}_i)}{\partial \alpha_i} \qquad (52) \\
&= \int E_{\boldsymbol{\alpha}_{-i}} \left\{ \begin{array}{c} p_i(\hat{\alpha}_i;\boldsymbol{\alpha}_{-i}) u_1(\alpha_i, s_i) + \\ [1-\sum_j p_j(\hat{\alpha}_i;\boldsymbol{\alpha}_{-i})] \sum_{g_i} [A^{g_i}(\hat{\alpha}_i,\boldsymbol{\alpha}_{-i}) u_1(\alpha_i,s_i) P_i^{g_i}(\boldsymbol{\alpha}_{-i},\hat{\alpha}_i,\sigma_i(y,\hat{\alpha}_i,s_i))] \end{array} \right\} dG_i(s_i),
\end{aligned}
$$

which gives the next lemma immediately.

Lemma 7. Suppose $\boldsymbol{\alpha}_{-i}$ is truthfully revealed from the first stage and the second-stage mechanism is incentive-compatible given a truthfully revealed $\boldsymbol{\alpha}$. If buyer i with type α_i reports $\hat{\alpha}_i$ in the first stage, then i's first-stage expected payoff can be expressed as

$$
\begin{aligned}
&\pi_i(\alpha_i, \hat{\alpha}_i) - \pi_i(\hat{\alpha}_i, \hat{\alpha}_i) \qquad (53) \\
&= \int_{\hat{\alpha}_i}^{\alpha_i} E_{s_i} E_{\boldsymbol{\alpha}_{-i}} \left\{ \begin{array}{c} p_i(\hat{\alpha}_i;\boldsymbol{\alpha}_{-i}) u_1(y, s_i) + \\ [1-\sum_j p_j(\hat{\alpha}_i;\boldsymbol{\alpha}_{-i})] \sum_{g_i} [A^{g_i}(\hat{\alpha}_i,\boldsymbol{\alpha}_{-i}) u_1(y,s_i) P_i^{g_i}(\hat{\alpha}_i,\boldsymbol{\alpha}_{-i},\sigma_i(y,\hat{\alpha}_i,s_i))] \end{array} \right\} dy.
\end{aligned}
$$

Applying the envelop theorem and using (52), we have

$$
\begin{aligned}
\frac{d\pi_i(\alpha_i, \alpha_i)}{d\alpha_i} &= \frac{\partial \pi_i(\alpha_i, \hat{\alpha}_i)}{\partial \alpha_i}\Big|_{\hat{\alpha}_i = \alpha_i} \\
&= \int E_{\boldsymbol{\alpha}_{-i}} \left\{ \begin{array}{c} p_i(\boldsymbol{\alpha}) u_1(\alpha_i, s_i) + \\ [1-\sum_j p_j(\boldsymbol{\alpha})] \sum_{g_i} [A^{g_i}(\boldsymbol{\alpha}) u_1(\alpha_i,s_i) P_i^{g_i}(\boldsymbol{\alpha}, s_i)] \end{array} \right\} dG_i(s_i), \qquad (54)
\end{aligned}
$$

which leads to the next lemma.

Lemma 8. If the two-stage mechanism is incentive compatible, then buyer i's expected payoff (as a function of her pre-entry type) can be expressed as

$$
\begin{aligned}
&\pi_i(\alpha_i, \alpha_i) - \pi_i(\underline{\alpha}, \underline{\alpha}) \qquad (55) \\
&= \int_{\underline{\alpha}}^{\alpha_i} \int E_{\boldsymbol{\alpha}_{-i}} \left\{ \begin{array}{c} p_i(y;\boldsymbol{\alpha}_{-i}) u_1(y, s_i) + \\ [1-\sum_j p_j(y;\boldsymbol{\alpha}_{-i})] \sum_{g_i} [A^{g_i}(y,\boldsymbol{\alpha}_{-i}) u_1(y,s_i) P_i^{g_i}(y,\boldsymbol{\alpha}_{-i},s_i)] \end{array} \right\} dG_i(s_i) dy.
\end{aligned}
$$

As shown by (42) and (55), bidders' first-stage expected payoffs do not depend on whether information $\mathbf{s}$ is public or private. Moreover, with truthful revelation, the total expected surplus TS from the two-stage mechanism is given by (45). The seller's expected revenue is thus given by

$$\begin{aligned} & ER \\ = \; & TS - \sum_{i=1}^{N} E\pi_i(\alpha_i, \alpha_i) \\ = \; & E_{\boldsymbol{\alpha}} \left\{ \begin{array}{c} \sum_i \left[p_i(\boldsymbol{\alpha}) E_{s_i} \left(u(\alpha_i, s_i) - \frac{1-F(\alpha_i)}{f(\alpha_i)} u_1(\alpha_i, s_i) \right) \right] + \\ [1 - \sum_i p_i(\boldsymbol{\alpha})] \sum_g A^g(\boldsymbol{\alpha}) E_{\mathbf{s}} \left[\sum_{i \in g} p_i^g(\boldsymbol{\alpha}, \mathbf{s}^g) \left(u(\alpha_i, s_i) - \frac{1-F(\alpha_i)}{f(\alpha_i)} u_1(\alpha_i, s_i) \right) - |g|c \right] \end{array} \right\} \\ & - \sum_{i=1}^{N} \pi_i(\underline{\alpha}, \underline{\alpha}). \end{aligned} \tag{56}$$

which coincides with the seller expected revenue with public $\mathbf{s}$, i.e. the expression in (46).

It is clear that if allocation rule $(p_i^*(\boldsymbol{\alpha}), A^{*g}(\boldsymbol{\alpha}), p_i^{*g}(\boldsymbol{\alpha}, \mathbf{s}^g))$ defined in (49), (6), and (4) can be supported by some appropriately defined payment rule $(\tilde{x}_i^{*g}(\boldsymbol{\alpha}), \tilde{t}_i^{*g}(\boldsymbol{\alpha}, \mathbf{s}^g))$ which also ensures $\pi_i(\underline{\alpha}, \underline{\alpha}) = 0$, then the revenue bound ER^{**} in (50) can be achieved. As a result, these allocation and payment rules constitute a revenue-maximizing two-stage selling mechanism in the original setting.

We next proceed to show such payment rule $(\tilde{x}_i^{*g}(\boldsymbol{\alpha}), \tilde{t}_i^{*g}(\boldsymbol{\alpha}, \mathbf{s}^g))$ exists. To this end, we need to utilize the properties of the allocation rule $(p_i^*(\boldsymbol{\alpha}), A^{*g}(\boldsymbol{\alpha}), p_i^{*g}(\boldsymbol{\alpha}, \mathbf{s}^g))$, which are revealed by Corollaries 1, 2 and Lemma 6.

Note that $u(\alpha_i, s_i)$ increases in s_i and by Assumption 1, $u_1(\alpha_i, s_i)$ (weakly) decreases with s_i. This implies that $w(\alpha_i, s_i)$ increases with s_i. By the final good allocation rule (4), the winning probability $P_i^{*g}(\boldsymbol{\alpha}, s_i)$ is weakly increasing in s_i. By Lemma 2 in Myerson (1981), the second-stage mechanism is incentive compatible (given $\boldsymbol{\alpha}$ and g). Thus, given the truthfully revealed $\boldsymbol{\alpha}$ and shortlisted group g, a second-stage payment rule, say, $\tilde{t}_i^{*g}(\boldsymbol{\alpha}, \mathbf{s}^g), \forall i \in g, \forall g$, can be constructed to truthfully implement the second-stage allocation rule $p_i^{*g}(\boldsymbol{\alpha}, \mathbf{s}^g), \forall i \in g, \forall g$ while maintaining the second-stage IR constraints (to participate in the second-stage mechanism), i.e. $\tilde{\pi}_i^g(\boldsymbol{\alpha}, \alpha_i; s_i, s_i) \geq 0$ on equilibrium path. This resembles the Myerson (1981) setting with asymmetric bidders. Note $\tilde{t}_i^{*g}(\boldsymbol{\alpha}, \mathbf{s}^g)$ coincides with $t_i^{*g}(\boldsymbol{\alpha}, \mathbf{s}^g)$ for the case without first-stage sale.

Recall that we use $\tilde{\pi}_i^{*g_i}(\alpha_i, \hat{\alpha}_i; \boldsymbol{\alpha}_{-i})$ to denote the second-stage expected payoff to buyer i of type α_i if she announces $\hat{\alpha}_i$ and is shortlisted in group g_i, given that everyone else announces $\boldsymbol{\alpha}_{-i}$ truthfully at the first stage. Lemma 7 must hold given the second stage mechanism is IC upon truthful revelation in the first stage.

Construct the first-stage payment rule as follows:

$$\tilde{x}_i^*(\boldsymbol{\alpha}) \;=\; p_i^*(\boldsymbol{\alpha}) E_{s_i} u(\alpha_i, s_i) + [1 - \sum_j p_j^*(\boldsymbol{\alpha})] \sum_{g_i} A^{*g_i}(\alpha_i, \boldsymbol{\alpha}_{-i}) [\tilde{\pi}_i^{*g_i}(\alpha_i, \alpha_i; \boldsymbol{\alpha}_{-i}) - c] \tag{57}$$

$$-\int_{\underline{\alpha}}^{\alpha_i}\int u_1(y,s_i)\left\{\begin{array}{c} p_i^*(y;\boldsymbol{\alpha}_{-i})+ \\ [1-\sum_j p_j^*(y;\boldsymbol{\alpha}_{-i})]\sum_{g_i}\left[A^{*g_i}(y,\boldsymbol{\alpha}_{-i})P_i^{*g_i}(y,\boldsymbol{\alpha}_{-i},s_i)\right]\end{array}\right\}dG_i(s_i)dy.$$

Substituting (57) into (51), we can verify that

$$\begin{aligned}&\pi_i^*(\alpha_i,\alpha_i) \qquad (58)\\ =\ &\int_{\underline{\alpha}}^{\alpha_i}\int u_1(y,s_i)\cdot E_{\boldsymbol{\alpha}_{-i}}\left\{\begin{array}{c} p_i^*(y;\boldsymbol{\alpha}_{-i})+ \\ [1-\sum_j p_j^*(y;\boldsymbol{\alpha}_{-i})]\sum_{g_i}\left[A^{*g_i}(y,\boldsymbol{\alpha}_{-i})P_i^{*g_i}(y,\boldsymbol{\alpha}_{-i},s_i)\right]\end{array}\right\}dG_i(s_i)dy,\end{aligned}$$

which is precisely equation (55) with $\pi_i^*(\underline{\alpha},\underline{\alpha})=0$. Note that $\pi_i^*(\alpha_i,\alpha_i)\geq 0$, so IR is satisfied in the first stage.

Proposition 6. Under Assumptions 1 and 2, the optimal first-stage selling probabilities (49), the first-stage optimal shortlisting rules (6) and the second-stage optimal final good allocation (4) are IR and IC implementable by payments $(\tilde{x}_i^*(\boldsymbol{\alpha}),\ \tilde{t}_i^{*g}(\boldsymbol{\alpha},\mathbf{s}^g))$. Moreover, $\pi_i^*(\underline{\alpha},\underline{\alpha})=0$.

Proof of Proposition 6: Following the above discussions, it remains to show the first-stage IC. Suppose that all buyers except i report their types $\boldsymbol{\alpha}_{-i}$ truthfully. Consider buyer i with α_i contemplating to misreport $\hat{\alpha}_i<\alpha_i$. The deviation payoff is

$$\Delta=\pi_i^*(\alpha_i,\hat{\alpha}_i)-\pi_i^*(\alpha_i,\alpha_i)=[\pi_i^*(\alpha_i,\hat{\alpha}_i)-\pi_i^*(\hat{\alpha}_i,\hat{\alpha}_i)]+[\pi_i^*(\hat{\alpha}_i,\hat{\alpha}_i)-\pi_i^*(\alpha_i,\alpha_i)]. \qquad (59)$$

Since (55) is satisfied by the construction of $x_i^*(\boldsymbol{\alpha})$, we have

$$\begin{aligned}&\pi_i^*(\hat{\alpha}_i,\hat{\alpha}_i)-\pi_i^*(\alpha_i,\alpha_i)\\ =\ &-\int_{\hat{\alpha}_i}^{\alpha_i}\int u_1(y,s_i)\cdot E_{\boldsymbol{\alpha}_{-i}}\left\{\begin{array}{c} p_i^*(y;\boldsymbol{\alpha}_{-i})+[1-\sum_j p_j^*(y;\boldsymbol{\alpha}_{-i})] \\ \cdot\sum_{g_i}\left[A^{*g_i}(y,\boldsymbol{\alpha}_{-i})P_i^{*g_i}(y,\boldsymbol{\alpha}_{-i},s_i)\right]\end{array}\right\}dG_i(s_i)dy.\end{aligned}$$

By Lemma 7,

$$\begin{aligned}&\pi_i^*(\alpha_i,\hat{\alpha}_i)-\pi_i^*(\hat{\alpha}_i,\hat{\alpha}_i)\\ =\ &\int_{\hat{\alpha}_i}^{\alpha_i}E_{s_i}E_{\boldsymbol{\alpha}_{-i}}\left\{\begin{array}{c} p_i^*(\hat{\alpha}_i;\boldsymbol{\alpha}_{-i})u_1(y,s_i)+[1-\sum_j p_j^*(\hat{\alpha}_i;\boldsymbol{\alpha}_{-i})] \\ \cdot\sum_{g_i}\left[A^{*g_i}(\hat{\alpha}_i,\boldsymbol{\alpha}_{-i})u_1(y,s_i)P_i^{*g_i}(\hat{\alpha}_i,\boldsymbol{\alpha}_{-i},\sigma_i(y,\hat{\alpha}_i,s_i))\right]\end{array}\right\}dy.\end{aligned}$$

Therefore, we have

$$\begin{aligned}\Delta\ =\ &\int_{\hat{\alpha}_i}^{\alpha_i}\int E_{s_i}\left[u_1(y,s_i)E_{\boldsymbol{\alpha}_{-i}}\left[\begin{array}{c} p_i^*(\hat{\alpha}_i;\boldsymbol{\alpha}_{-i})+(1-\sum_j p_j^*(\hat{\alpha}_i;\boldsymbol{\alpha}_{-i})) \\ \cdot\sum_{g_i}\left[A^{*g_i}(\hat{\alpha}_i,\boldsymbol{\alpha}_{-i})P_i^{*g_i}(\hat{\alpha}_i,\boldsymbol{\alpha}_{-i},\sigma_i(y,\hat{\alpha}_i,s_i))\right]\end{array}\right]\right]dy\\ &-\int_{\hat{\alpha}_i}^{\alpha_i}\int E_{s_i}\left[u_1(y,s_i)E_{\boldsymbol{\alpha}_{-i}}\left[\begin{array}{c} p_i^*(y;\boldsymbol{\alpha}_{-i})+[1-\sum_j p_j^*(y;\boldsymbol{\alpha}_{-i})] \\ \cdot\sum_{g_i}\left[A^{*g_i}(y,\boldsymbol{\alpha}_{-i})P_i^{*g_i}(y,\boldsymbol{\alpha}_{-i},s_i)\right]\end{array}\right]\right]dy\end{aligned}$$

$$
= \underbrace{\left\{ \begin{array}{c} \int_{\hat{\alpha}_i}^{\alpha_i} E_{s_i} \left[u_1(y,s_i) E_{\boldsymbol{\alpha}_{-i}} \left[\begin{array}{c} p_i^*(\hat{\alpha}_i;\boldsymbol{\alpha}_{-i}) + (1-\sum_j p_j^*(\hat{\alpha}_i;\boldsymbol{\alpha}_{-i})) \\ \cdot \sum_{g_i} A^{*g_i}(\hat{\alpha}_i,\boldsymbol{\alpha}_{-i}) P_i^{*g_i}(\hat{\alpha}_i,\boldsymbol{\alpha}_{-i},\sigma_i(y,\hat{\alpha}_i,s_i)) \end{array} \right] \right] dy \\ - \int_{\hat{\alpha}_i}^{\alpha_i} E_{s_i} \left[u_1(y,s_i) \cdot E_{\boldsymbol{\alpha}_{-i}} \left[\begin{array}{c} p_i^*(\hat{\alpha}_i;\boldsymbol{\alpha}_{-i}) + (1-\sum_j p_j^*(\hat{\alpha}_i;\boldsymbol{\alpha}_{-i})) \\ \cdot \sum_{g_i} A^{*g_i}(\hat{\alpha}_i,\boldsymbol{\alpha}_{-i}) P_i^{*g_i}(y,\boldsymbol{\alpha}_{-i},s_i) \end{array} \right] \right] dy \end{array} \right\}}_{\text{First Term}}
$$

$$
+ \underbrace{\left\{ \begin{array}{c} \int_{\hat{\alpha}_i}^{\alpha_i} E_{s_i} \left[u_1(y,s_i) \cdot E_{\boldsymbol{\alpha}_{-i}} \left[\begin{array}{c} p_i^*(\hat{\alpha}_i;\boldsymbol{\alpha}_{-i}) + (1-\sum_j p_j^*(\hat{\alpha}_i;\boldsymbol{\alpha}_{-i})) \\ \cdot \sum_{g_i} A^{*g_i}(\hat{\alpha}_i,\boldsymbol{\alpha}_{-i}) P_i^{*g_i}(y,\boldsymbol{\alpha}_{-i},s_i) \end{array} \right] \right] dy \\ - \int_{\hat{\alpha}_i}^{\alpha_i} E_{s_i} \left[u_1(y,s_i) \cdot E_{\boldsymbol{\alpha}_{-i}} \left[\begin{array}{c} p_i^*(y;\boldsymbol{\alpha}_{-i}) + (1-\sum_j p_j^*(y;\boldsymbol{\alpha}_{-i})) \\ \cdot \sum_{g_i} A^{*g_i}(\hat{\alpha}_i,\boldsymbol{\alpha}_{-i}) P_i^{*g_i}(y,\boldsymbol{\alpha}_{-i},s_i) \end{array} \right] \right] dy \end{array} \right\}}_{\text{Second Term}}
$$

$$
+ \underbrace{\left\{ \begin{array}{c} \int_{\hat{\alpha}_i}^{\alpha_i} E_{s_i} \left[u_1(y,s_i) \cdot E_{\boldsymbol{\alpha}_{-i}} \left[\begin{array}{c} p_i^*(y;\boldsymbol{\alpha}_{-i}) + (1-\sum_j p_j^*(y;\boldsymbol{\alpha}_{-i})) \\ \cdot \sum_{g_i} A^{*g_i}(\hat{\alpha}_i,\boldsymbol{\alpha}_{-i}) P_i^{*g_i}(y,\boldsymbol{\alpha}_{-i},s_i) \end{array} \right] \right] dy \\ - \int_{\hat{\alpha}_i}^{\alpha_i} E_{s_i} \left[u_1(y,s_i) \cdot E_{\boldsymbol{\alpha}_{-i}} \left[\begin{array}{c} p_i^*(y;\boldsymbol{\alpha}_{-i}) + (1-\sum_j p_j^*(y;\boldsymbol{\alpha}_{-i})) \\ \cdot \sum_{g_i} A^{*g_i}(y,\boldsymbol{\alpha}_{-i}) P_i^{*g_i}(y,\boldsymbol{\alpha}_{-i},s_i) \end{array} \right] \right] dy \end{array} \right\}}_{\text{Third Term}}.
$$

For any s_i and any $y \in [\hat{\alpha}_i, \alpha_i]$, $\sigma_i(y,\hat{\alpha}_i,s_i) \geq s_i$, where $\sigma_i(y,\hat{\alpha}_i,s_i)$ is the lie correction strategy. From Corollary 1 (ii), we have $P_i^{*g_i}(\hat{\alpha}_i,\boldsymbol{\alpha}_{-i},\sigma_i(y,\hat{\alpha}_i,s_i)) - P_i^{*g_i}(y,\boldsymbol{\alpha}_{-i},s_i) \leq 0$, which implies that the first term in Δ is nonpositive.

We now consider the second term in Δ when $y > \hat{\alpha}_i$. If $p_i^*(\hat{\alpha}_i;\boldsymbol{\alpha}_{-i}) = 1$, we must have $p_i^*(y;\boldsymbol{\alpha}_{-i}) = 1$ by Lemma 6. In this case, the two terms in the square brackets are identical. If $p_i^*(\hat{\alpha}_i;\boldsymbol{\alpha}_{-i}) = 0$, we must have $p_i^*(y;\boldsymbol{\alpha}_{-i}) = 0$ or 1. If $p_i^*(y;\boldsymbol{\alpha}_{-i}) = 0$, the two terms in the two pairs of square brackets are identical. If $p_i^*(y;\boldsymbol{\alpha}_{-i}) = 1$, then the term in the first pair of square brackets must be smaller than the term in the second pair of square brackets. Thus, the second term in Δ must be nonpositive.

We now consider the third term in Δ when $y > \hat{\alpha}_i$. By Corollary 2, the optimal shortlisting rule implies that given $\boldsymbol{\alpha}_{-i}$, when buyer i is admitted with a higher α_i, she must be admitted to a group with a weakly smaller size. If y and $\hat{\alpha}_i$ are admitted in the same group, then $A^{*g_i}(\hat{\alpha}_i,\boldsymbol{\alpha}_{-i}) = A^{*g_i}(y,\boldsymbol{\alpha}_{-i})$ and this term in Δ is zero.

We now turn to the case where $g^*(\hat{\alpha}_i,\boldsymbol{\alpha}_{-i}) \supset g^*(y,\boldsymbol{\alpha}_{-i}) \supset \{i\}$. Note that $A^{*g_i}(\cdot,\boldsymbol{\alpha}_{-i})$ is 1 for the shortlisted group, and 0 for all other groups. Therefore,

$$
\begin{aligned}
& \sum_{g_i} [A^{*g_i}(\hat{\alpha}_i,\boldsymbol{\alpha}_{-i}) - A^{*g_i}(y,\boldsymbol{\alpha}_{-i})] P_i^{*g_i}(y,\boldsymbol{\alpha}_{-i},s_i) \\
= \; & P_i^{*g^*(\hat{\alpha}_i,\boldsymbol{\alpha}_{-i})}(y,\boldsymbol{\alpha}_{-i},s_i) - P_i^{*g^*(y,\boldsymbol{\alpha}_{-i})}(y,\boldsymbol{\alpha}_{-i},s_i) \\
\leq \; & 0,
\end{aligned}
$$

which implies that the third term in Δ is nonpositive. Since $g^*(\hat{\alpha}_i, \boldsymbol{\alpha}_{-i}) \supset g^*(y, \boldsymbol{\alpha}_{-i}) \supset \{i\}$, we must have $P_i^{*g^*(\hat{\alpha}_i, \boldsymbol{\alpha}_{-i})}(y, \boldsymbol{\alpha}_{-i}, s_i) \leq P_i^{*g^*(y, \boldsymbol{\alpha}_{-i})}(y, \boldsymbol{\alpha}_{-i}, s_i)$, i.e. entrant i wins with a smaller probability if a strictly bigger group is shortlisted.

A similar argument can be used to rule out deviating to $\hat{\alpha}_i > \alpha_i$. $\square$

Proposition 6 reveals that allocation rule $(p_i^*(\boldsymbol{\alpha}), A^{*g}(\boldsymbol{\alpha}), p_i^{*g}(\boldsymbol{\alpha}, \mathbf{s}^g))$ and payment rule $(\tilde{x}_i^{*g}(\boldsymbol{\alpha}), \tilde{t}_i^{*g}(\boldsymbol{\alpha}, \mathbf{s}^g))$ constitute a feasible (both IC and IR) two-stage mechanism and entail $\pi_i(\underline{\alpha}, \underline{\alpha}) = 0$. Clearly, by (56) this mechanism achieves the revenue bound in ER^{**} (50). Therefore, these rules constitute the revenue-maximizing two-stage selling mechanism in the original setting.

Proposition 7. Under Assumptions 1 and 2, allocation rule $(p_i^*(\boldsymbol{\alpha}), A^{*g}(\boldsymbol{\alpha}), p_i^{*g}(\boldsymbol{\alpha}, \mathbf{s}^g))$ and payment rule $(\tilde{x}_i^{*g}(\boldsymbol{\alpha}), \tilde{t}_i^{*g}(\boldsymbol{\alpha}, \mathbf{s}^g))$ constitute a revenue-maximizing two-stage selling mechanism in the original setting, which achieves revenue bound ER^{**} in (50).

B Sequential Shortlisting

Now we move to the setting where the seller may conduct sequential shortlisting. The mechanism is specified in the same way as in Section 4 except that in the first stage the selling probability to buyer i is $p_i(\mathbf{m}_1)$, where $\sum_{i \in \mathbf{N}} p_i(\mathbf{m}_1) \leq 1$; and only if the object is unsold in the first stage, each subgroup $g_1 \in 2^{\mathbf{N}}$ would be shortlisted with probability $A^{g_1}(\mathbf{m}_1|g_0)$ for information acquisition, where $\sum_{g \in 2^{\mathbf{N}}} A^g(\mathbf{m}_1|g_0) = 1$. Here, we follow the same notation as in Section 4.

Our analysis proceeds as follows. We first consider a relaxed environment where the agents are only endowed with private information $\boldsymbol{\alpha}$, where s_i's become known to bidders once they are discovered. The optimal solution for this relaxed environment provides an upper bound for the seller's expected revenue in the original environment where the discovered s_i's are private information to the shortlisted bidders. We will establish that this upper bound is actually achievable in the original environment.

B.1 The Relaxed Environment

For a given mechanism and message sequence $(\mathbf{m}_k, k = 1, 2, ..., M)$, the probability of a shortlisting outcome $\mathbf{g} = (g_1, g_2, ..., g_M)$ is given by

$$\Pr(\mathbf{g}|(\mathbf{m}_i)_{i=1}^M) = \Pi_{k=1}^M A^{g_k}(\mathbf{m}_1, \mathbf{m}_2, ..., \mathbf{m}_k|g_0, g_1, g_2, ..., g_{k-1}).$$

As s_i becomes known to bidder i once discovered in the relaxed environment, we have that for $k \geq 2$, $m_{k,i} = s_i$, $i \in g_{k-1}$,and $m_{k,i} = \phi$, $i \notin g_{k-1}$. We use $\mathbf{m}_k^s$, $k \geq 2$ to denote these true types from stages 2 to $M+1$. Agent i's expected payoff when i is endowed with α_i but announces $\hat{\alpha}_i$ is given by:

$$\pi_i(\alpha_i, \hat{\alpha}_i)$$

$$
\begin{aligned}
&= E_{\boldsymbol{\alpha}_{-i}} E_{\mathbf{s}} \left\{ \begin{array}{c} p_i(\hat{\alpha}_i;\boldsymbol{\alpha}_{-i}) E_{s_i} u(\alpha_i, s_i) + [1 - \sum_j p_j(\hat{\alpha}_i;\boldsymbol{\alpha}_{-i})] \\ \cdot \left[\begin{array}{c} \sum_{\forall \mathbf{g} \text{ s.t. } i \in G_{\mathbf{g}}} \left[\begin{array}{c} \Pr(\mathbf{g}|(\hat{\alpha}_i, \boldsymbol{\alpha}_{-i}), \mathbf{m}_2^{\mathbf{s}}, ..., \mathbf{m}_{M+1}^{\mathbf{s}}) \\ \cdot \left[u(\alpha_i, s_i) p_i^{G_{\mathbf{g}}}((\hat{\alpha}_i, \boldsymbol{\alpha}_{-i}), \mathbf{m}_2^{\mathbf{s}}, ..., \mathbf{m}_{M+1}^{\mathbf{s}}) - c \right] \end{array} \right] \\ - \sum_{\forall \mathbf{g}} \left[\Pr(\mathbf{g}|(\hat{\alpha}_i, \boldsymbol{\alpha}_{-i}), \mathbf{m}_2^{\mathbf{s}}, ..., \mathbf{m}_{M+1}^{\mathbf{s}}) \sum_{k=1}^{M} t_{k+1,i}((\hat{\alpha}_i, \boldsymbol{\alpha}_{-i}), \mathbf{m}_2^{\mathbf{s}}, ..., \mathbf{m}_{k+1}^{\mathbf{s}}) \right] \end{array} \right] \end{array} \right\} \\
&\quad - E_{\boldsymbol{\alpha}_{-i}} [t_{1,i}((\hat{\alpha}_i, \boldsymbol{\alpha}_{-\mathbf{i}}))].
\end{aligned}
$$

Incentive compatibility together with the envelop theorem gives:

$$
\begin{aligned}
&\frac{d\pi_i(\alpha_i, \alpha_i)}{d\alpha_i} \\
&= E_{\boldsymbol{\alpha}_{-i}} E_{\mathbf{s}} \left\{ \left[\begin{array}{c} p_i(\alpha_i;\boldsymbol{\alpha}_{-i}) E_{s_i} u_1(\alpha_i, s_i) + [1 - \sum_j p_j(\alpha_i;\boldsymbol{\alpha}_{-i})] \cdot \\ \sum_{\forall \mathbf{g}, i \in G_{\mathbf{g}}} \left[\Pr(\mathbf{g}|(\alpha_i, \boldsymbol{\alpha}_{-i}), \mathbf{m}_2^{\mathbf{s}}, ..., \mathbf{m}_{M+1}^{\mathbf{s}}) p_i^{G_{\mathbf{g}}}((\alpha_i, \boldsymbol{\alpha}_{-i}), \mathbf{m}_2^{\mathbf{s}}, ..., \mathbf{m}_{M+1}^{\mathbf{s}}) u_1(\alpha_i, s_i) \right] \end{array} \right] \right\}.
\end{aligned} \tag{60}
$$

Thus, we have

$$
\begin{aligned}
&\pi_i(\alpha_i, \alpha_i) = \pi_i(\underline{\alpha}, \underline{\alpha}) + \\
&E_{\boldsymbol{\alpha}_{-i}} \int_{\underline{\alpha}}^{\alpha_i} E_{\mathbf{s}} \left\{ \left[\begin{array}{c} p_i(y;\boldsymbol{\alpha}_{-i}) E_{s_i} u_1(y, s_i) + [1 - \sum_j p_j(y;\boldsymbol{\alpha}_{-i})] \\ \cdot \sum_{\forall \mathbf{g} \text{ s.t. } i \in G_{\mathbf{g}}} \left[\Pr(\mathbf{g}|(y, \boldsymbol{\alpha}_{-i}), \mathbf{m}_2^{\mathbf{s}}, ..., \mathbf{m}_{M+1}^{\mathbf{s}}) p_i^{G_{\mathbf{g}}}((y, \boldsymbol{\alpha}_{-i}), \mathbf{m}_2^{\mathbf{s}}, ..., \mathbf{m}_{M+1}^{\mathbf{s}}) u_1(y, s_i) \right] \end{array} \right] \right\} dy.
\end{aligned} \tag{61}
$$

The expected social surplus given $\boldsymbol{\alpha}$ is as follows:

$$
TS = E_{\boldsymbol{\alpha}} \left\{ \begin{array}{c} \sum_i [p_i(\boldsymbol{\alpha}) E_{s_i} u(\alpha_i, s_i)] + [1 - \sum_i p_i(\boldsymbol{\alpha})] \\ \cdot \sum_{\forall \mathbf{g}} E_{\mathbf{s}} \left[\Pr(\mathbf{g}|\boldsymbol{\alpha}, \mathbf{m}_2^{\mathbf{s}}, ..., \mathbf{m}_{M+1}^{\mathbf{s}}) \sum_{i \in G_{\mathbf{g}}} \left[p_i^{G_{\mathbf{g}}}(\boldsymbol{\alpha}, \mathbf{m}_2^{\mathbf{s}}, ..., \mathbf{m}_{M+1}^{\mathbf{s}}) u(\alpha_i, s_i) - c \right] \right] \end{array} \right\}. \tag{62}
$$

The seller seeks to maximize the expected revenue:

$$
ER = TS - \sum_i E_{\boldsymbol{\alpha}_i} \left[\pi_i(\alpha_i, \alpha_i) \right].
$$

By the standard procedure, we can rewrite the seller's objective as follows.

Lemma 9. The seller's objective is to maximize:

$$
\begin{aligned}
ER &= E_{\boldsymbol{\alpha}} \left\{ \begin{array}{c} \sum_i [p_i(\boldsymbol{\alpha}) E_{s_i} w(\alpha_i, s_i)] + [1 - \sum_i p_i(\boldsymbol{\alpha})] \\ \cdot \sum_{\forall \mathbf{g}} E_{\mathbf{s}} \left[\Pr(\mathbf{g}|\boldsymbol{\alpha}, \mathbf{m}_2^{\mathbf{s}}, ..., \mathbf{m}_{M+1}^{\mathbf{s}}) \sum_{i \in G_{\mathbf{g}}} \left[p_i^{G_{\mathbf{g}}}(\boldsymbol{\alpha}, \mathbf{m}_2^{\mathbf{s}}, ..., \mathbf{m}_{M+1}^{\mathbf{s}}) w(\alpha_i, s_i) - c \right] \right] \end{array} \right\} \\
&\quad - \sum_i \pi_i(\underline{\alpha}, \underline{\alpha}),
\end{aligned} \tag{63}
$$

where $w(\alpha_i, s_i) = u(\alpha_i, s_i) - u_1(\alpha_i, s_i)\frac{1-F(\alpha_i)}{f(\alpha_i)}$.

Define $\Pr(\mathbf{g}|\boldsymbol{\alpha}, \mathbf{s}) = \Pr(\mathbf{g}|\boldsymbol{\alpha}, \mathbf{m}_2^{\mathbf{s}}, ..., \mathbf{m}_{M+1}^{\mathbf{s}})$, and for any $G \in 2^N$, define

$$\Pr(G|\boldsymbol{\alpha}, \mathbf{s}) = \sum_{\forall \mathbf{g} \text{ s.t. } G_{\mathbf{g}}=G} \Pr(\mathbf{g}|\boldsymbol{\alpha}, \mathbf{s}),$$

where, as before, $G_{\mathbf{g}}$ denotes the set of all agents shortlisted in sequence $\mathbf{g}$.

Note we have

$$\sum_{\forall \mathbf{g} \text{ s.t. } i \in G_{\mathbf{g}}} \Pr(\mathbf{g}|\boldsymbol{\alpha}, \mathbf{s}) = \sum_{\forall G \text{ s.t. } i \in G} \Pr(G|\boldsymbol{\alpha}, \mathbf{s}).$$

To maximize the expected revenue ER, at the final allocation stage, given the revealed $\boldsymbol{\alpha}$ and the shortlisted group G, $\forall \mathbf{s}^G$, the optimal allocation rule is given by[39]

$$p_i^{*G}(\boldsymbol{\alpha}, \mathbf{s}^G) = \begin{cases} 1 & \text{if } i = \arg\max_{j \in G}\{w(\alpha_j, s_j)\} \text{ and } w(\alpha_i, s_i) \geq 0, \\ 0 & \text{otherwise,} \end{cases} \quad \forall G, \forall i \in G, \tag{64}$$

which maximizes the virtual value among the bidders within the shortlisted group G.

Analogously to Corollary 1, under Assumptions 1 and 2, we can establish the following properties of the optimal final-stage allocation rule:

Corollary 4. (i) $p_i^{*G_i}(\boldsymbol{\alpha}, \mathbf{s}^{G_i})$ increases in both α_i and s_i, $\forall i \in G_i$, $\forall g_i$, $\boldsymbol{\alpha}_{-i}$, and $\mathbf{s}_{-i}^{G_i}$, which implies that $P_i^{*G_i}(\alpha_i, \boldsymbol{\alpha}_{-i}, s_i) := E_{\boldsymbol{\alpha}_{-i}}[p_i^{*G_i}(\boldsymbol{\alpha}, \mathbf{s}^{G_i})]$ increases in both α_i and s_i, $\forall g_i$, $\boldsymbol{\alpha}_{-i}$; (ii) If $\alpha_i > \hat{\alpha}_i$, $s_i < \hat{s}_i$ and $u(\alpha_i, s_i) \geq u(\hat{\alpha}_i, \hat{s}_i)$, then $p_i^{*G_i}(\alpha_i, \boldsymbol{\alpha}_{-i}, s_i, \mathbf{s}_{-i}^{G_i}) \geq p_i^{*G_i}(\hat{\alpha}_i, \boldsymbol{\alpha}_{-i}, \hat{s}_i, \mathbf{s}_{-i}^{G_i})$, which implies $P_i^{*G_i}(\alpha_i, \boldsymbol{\alpha}_{-i}, s_i) \geq P_i^{*G_i}(\hat{\alpha}_i, \boldsymbol{\alpha}_{-i}, \hat{s}_i), \forall g_i$, $\boldsymbol{\alpha}_{-i}$.

By substituting (64) into ER in (63), we have the following result.

Lemma 10. For any $\{\Pr(G), \forall G \in 2^{\Omega}\}$ derived from any shortlisting rule, to maximize the expected revenue ER, the seller sets $\pi_i(\underline{\alpha}, \underline{\alpha}) = 0$ and allocates the object to the shortlisted bidder whose virtual value is the highest, provided that it is positive. Ties are randomly broken. In this case,

$$ER = E_{\boldsymbol{\alpha}} \left\{ \begin{array}{c} \sum_i p_i(\boldsymbol{\alpha}) E_{s_i} w(\alpha_i, s_i) \\ + [1 - \sum_i p_i(\boldsymbol{\alpha})] E_{\mathbf{s}} \left[\sum_{G \in 2^{\mathbf{N}}} \Pr(G|\boldsymbol{\alpha}, \mathbf{s}) \left[\max\{w_i^+(\alpha_i, s_i)\}_{i \in G} - \sum_{i \in G} c \right] \right] \end{array} \right\}. \tag{65}$$

B.2 Optimal Shortlisting

Lemma 4 and Proposition 3 still hold. Therefore, the same optimal sequential shortlisting rule of Proposition 3 in Section 4.1.1 remains valid.

[39] Ties occur with probability zero and are hence ignored.

B.3 Optimal Selling at The First Stage

For any shortlisting procedure, we define the expected *virtual surplus* (the virtual value less the entry cost) as follows

$$w^*(\boldsymbol{\alpha}) = E_{\mathbf{s}}\left[\sum_{G\in 2^{\mathbf{N}}} Pr(G|\boldsymbol{\alpha},\mathbf{s})\left(\max\{w_i^+(\alpha_i,s_i)\}_{i\in G} - \sum_{i\in G} c\right)\right].$$

For the optimal shortlisting rule described in Proposition 3, we define

$$R_2^*(\boldsymbol{\alpha}) = E_{\mathbf{s}}\left[\sum_{G\in 2^{\mathbf{N}}} \Pr{}^*(G|\boldsymbol{\alpha},\mathbf{s})\left(\max\{w_i^+(\alpha_i,s_i)\}_{i\in G} - \sum_{i\in G} c\right)\right], \tag{66}$$

where $\Pr^*(G|\boldsymbol{\alpha},\mathbf{s})$ denote the probability that the set of bidders G is admitted under the optimal shortlisting rule.

Recall that we let

$$\begin{aligned} R_1^*(\boldsymbol{\alpha}) &= \max_{\{i=1,2,..,N\}} E_{s_i} w(\alpha_i, s_i) \\ &= E_s w(\alpha_{(1)}, s), \end{aligned} \tag{67}$$

where $\alpha_{(1)}$ denotes the highest first-stage type among all agents, and s is uniformly distributed over [0,1].

It is clear that the optimal first-stage selling probabilities are:

$$\tilde{p}_i^*(\boldsymbol{\alpha}) = \begin{cases} 1 & \text{if } \alpha_i \geq \alpha_j, \forall j \text{ and } R_1^*(\boldsymbol{\alpha}) \geq R_2^*(\boldsymbol{\alpha}), \\ 0 & \text{otherwise,} \end{cases} \forall i. \tag{68}$$

In other words, given the first-stage type profile $\boldsymbol{\alpha}$, the object would be sold in the first stage if and only if expected revenue generated from the first-stage sale to the buyer with the highest first-stage type is higher than that from the optimal sequential shortlisting rule and final stage optimal selling mechanism.

For the first-stage selling probabilities, we have the following property.

Lemma 11. For given $\boldsymbol{\alpha}_{-i}$, if $\tilde{p}_i^*(\alpha_i, \boldsymbol{\alpha}_{-i}) = 1$, then $\tilde{p}_i^*(\tilde{\alpha}_i, \boldsymbol{\alpha}_{-i}) = 1$ for $\tilde{\alpha}_i > \alpha_i$.

Proof of Lemma 11: Note $\tilde{p}_i^*(\alpha_i, \boldsymbol{\alpha}_{-i})$ is either zero or one, and it can be one only if α_i is the highest first stage signal among all buyers. To establish the wanted result, it suffices to show that if α_i is the highest first stage signal among all buyers, and $R_1^*(\alpha_i, \boldsymbol{\alpha}_{-i}) \geq R_2^*(\alpha_i, \boldsymbol{\alpha}_{-i})$, then we would have $R_1^*(\tilde{\alpha}_i, \boldsymbol{\alpha}_{-i}) \geq R_2^*(\tilde{\alpha}_i, \boldsymbol{\alpha}_{-i})$, for any $\tilde{\alpha}_i > \alpha_i$.

Recall that $R_1^*(\boldsymbol{\alpha}) = \max_{\{i=1,2,..,N\}} E_{s_i} w(\alpha_i, s_i)$ and

$$R_2^*(\alpha_i, \boldsymbol{\alpha}_{-i}) = E_{\mathbf{s}} \left[\sum_{G \in 2^{\mathbf{N}}} \Pr^*(G|\boldsymbol{\alpha}, \mathbf{s}) \left(\max\{w_i^+(\alpha_i, s_i)\}_{i \in G} - \sum_{i \in G} c \right) \right].$$

We use G_i to denote a non-empty shortlisted group. Note that G_i must contain buyer i. Moreover, G_i must consist of a group of buyers with the highest first stage types.

For any $\boldsymbol{\alpha}_{-i}$,

$$\begin{aligned} & R_2^*(\alpha_i, \boldsymbol{\alpha}_{-i}) \\ = \; & E_{\mathbf{s}} \left[\Pr^*(G_i = \{i\}|\alpha_i, \boldsymbol{\alpha}_{-i}, \mathbf{s}) \left[w_i^+(\alpha_i, s_i) - c \right] \right] \\ & + E_{\mathbf{s}} \left[\sum_{k=2}^{N} \Pr^*(|G_i| = k|\alpha_i, \boldsymbol{\alpha}_{-i}, \mathbf{s}) \left[\begin{array}{c} \max\{w_i^+(\alpha_i, s_i), w_j^+(\alpha_j, s_j)\}_{j \in G_i \setminus \{i\}} \\ -|G|c \end{array} \right] \right], \end{aligned}$$

and

$$\begin{aligned} & R_2^*(\tilde{\alpha}_i, \boldsymbol{\alpha}_{-i}) \\ = \; & E_{\mathbf{s}} \left[\Pr^*(G_i = \{i\}|\tilde{\alpha}_i, \boldsymbol{\alpha}_{-i}, \mathbf{s}) \left[w_i^+(\tilde{\alpha}_i, s_i) - c \right] \right] \\ & + E_{\mathbf{s}} \left[\sum_{k=2}^{N} \Pr^*(|G_i| = k|\tilde{\alpha}_i, \boldsymbol{\alpha}_{-i}, \mathbf{s}) \left[\begin{array}{c} \max\{w_i^+(\tilde{\alpha}_i, s_i), w_j^+(\alpha_j, s_j)\}_{j \in G_i \setminus \{i\}} \\ -|G|c \end{array} \right] \right]. \end{aligned}$$

Denote $\xi_{G_i \setminus \{i\}} = \max_{j \in G_i \setminus \{i\}} \{w_j^+(\alpha_j, s_j)\}$. We have

$$\begin{aligned} & R_2^*(\alpha_i, \boldsymbol{\alpha}_{-i}) \\ = \; & E_{\mathbf{s}} \left[\Pr^*(G_i = \{i\}|\alpha_i, \boldsymbol{\alpha}_{-i}, \mathbf{s}) \left[w_i^+(\alpha_i, s_i) - c \right] \right] \\ & + E_{\mathbf{s}} \left[\sum_{k=2}^{N} \Pr^*(|G_i| = k|\alpha_i, \boldsymbol{\alpha}_{-i}, \mathbf{s}) \left[\max\{w_i^+(\alpha_i, s_i), \xi_{G_i \setminus \{i\}}\} - |G|c \right] \right], \end{aligned}$$

and

$$\begin{aligned} & R_2^*(\tilde{\alpha}_i, \boldsymbol{\alpha}_{-i}) \\ = \; & E_{\mathbf{s}} \left[\Pr^*(G_i = \{i\}|\tilde{\alpha}_i, \boldsymbol{\alpha}_{-i}, \mathbf{s}) \left[w_i^+(\tilde{\alpha}_i, s_i) - c \right] \right] \\ & + E_{\mathbf{s}} \left[\sum_{k=2}^{N} \Pr^*(|G_i| = k|\tilde{\alpha}_i, \boldsymbol{\alpha}_{-i}, \mathbf{s}) \left[\max\{w_i^+(\tilde{\alpha}_i, s_i), \xi_{G_i \setminus \{i\}}\} - |G|c \right] \right]. \end{aligned}$$

Define

$$
\begin{aligned}
& R_2^*(\tilde{\alpha}_i, \boldsymbol{\alpha}_{-i}; \mathbf{s}) \\
= \; & \Pr^*(G_i = \{i\} | \tilde{\alpha}_i, \boldsymbol{\alpha}_{-i}, \mathbf{s}) \left[w_i^+(\tilde{\alpha}_i, s_i) - c \right] \\
& + \sum_{k=2}^{N} \Pr^*(|G_i| = k | \tilde{\alpha}_i, \boldsymbol{\alpha}_{-i}, \mathbf{s}) \left[\max\{w_i^+(\tilde{\alpha}_i, s_i), \xi_{G_i \setminus \{i\}}\} - |G|c \right],
\end{aligned}
$$

and

$$
\begin{aligned}
& R_2^*(\alpha_i, \boldsymbol{\alpha}_{-i}; \mathbf{s}) \\
= \; & \Pr^*(G_i = \{i\} | \alpha_i, \boldsymbol{\alpha}_{-i}, \mathbf{s}) \left[w_i^+(\alpha_i, s_i) - c \right] \\
& + \sum_{k=2}^{N} \Pr^*(|G_i| = k | \alpha_i, \boldsymbol{\alpha}_{-i}, \mathbf{s}) \left[\max\{w_i^+(\alpha_i, s_i), \xi_{G_i \setminus \{i\}}\} - |G|c \right].
\end{aligned}
$$

We next show that we have

$$
R_2^*(\tilde{\alpha}_i, \boldsymbol{\alpha}_{-i}) - R_2^*(\alpha_i, \boldsymbol{\alpha}_{-i}) \leq E_{\mathbf{s}} \left\{ w_i^+(\tilde{\alpha}_i, s_i) - w_i^+(\alpha_i, s_i) \right\}. \tag{69}
$$

Note that $\forall \mathbf{s}$, there exists one and only one group that can be shortlisted with probability 1. Note $\tilde{p}_i^*(\alpha_i, \boldsymbol{\alpha}_{-i}) = 1$ means that at least buyer i should be shortlisted when buyer i's first stage type is α_i. Give this, we have that at least buyer i should be shortlisted when buyer i's first stage type is $\tilde{\alpha}_i > \alpha_i$. We use $G_i(\alpha_i, \boldsymbol{\alpha}_{-i}; \mathbf{s})$ and $G_i(\tilde{\alpha}_i, \boldsymbol{\alpha}_{-i}; \mathbf{s})$ to denote the shortlisted groups, respectively. We must have $G_i(\tilde{\alpha}_i, \boldsymbol{\alpha}_{-i}; \mathbf{s}) \subseteq G_i(\alpha_i, \boldsymbol{\alpha}_{-i}; \mathbf{s})$. Note that the optimal shortlisting rule and selling rule mean that

$$
\begin{aligned}
& E_{\mathbf{s}} \max \left[\{w_i^+(\alpha_i, s_i), \xi_{G_i(\alpha_i, \boldsymbol{\alpha}_{-i}; \mathbf{s}) \setminus \{i\}}\} - |G_i(\alpha_i, \boldsymbol{\alpha}_{-i}; \mathbf{s})|c \right] \\
\geq \; & E_{\mathbf{s}} \max \left[\{w_i^+(\alpha_i, s_i), \xi_{G_i(\tilde{\alpha}_i, \boldsymbol{\alpha}_{-i}; \mathbf{s}) \setminus \{i\}}\} - |G_i(\tilde{\alpha}_i, \boldsymbol{\alpha}_{-i}; \mathbf{s})|c \right].
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
& R_2^*(\tilde{\alpha}_i, \boldsymbol{\alpha}_{-i}) - R_2^*(\alpha_i, \boldsymbol{\alpha}_{-i}) \\
= \; & E_{\mathbf{s}} R_2^*(\tilde{\alpha}_i, \boldsymbol{\alpha}_{-i}; \mathbf{s}) - E_{\mathbf{s}} R_2^*(\alpha_i, \boldsymbol{\alpha}_{-i}; \mathbf{s}) \\
= \; & E_{\mathbf{s}} \left[\max\{w_i^+(\tilde{\alpha}_i, s_i), \xi_{G_i(\tilde{\alpha}_i, \boldsymbol{\alpha}_{-i}; \mathbf{s}) \setminus \{i\}}\} - |G_i(\tilde{\alpha}_i, \boldsymbol{\alpha}_{-i}; \mathbf{s})|c \right] \\
& - E_{\mathbf{s}} \left[\max\{w_i^+(\alpha_i, s_i), \xi_{G_i(\alpha_i, \boldsymbol{\alpha}_{-i}; \mathbf{s}) \setminus \{i\}}\} - |G_i(\alpha_i, \boldsymbol{\alpha}_{-i}; \mathbf{s})|c \right] \\
\leq \; & E_{\mathbf{s}} \left[\max\{w_i^+(\tilde{\alpha}_i, s_i), \xi_{G_i(\tilde{\alpha}_i, \boldsymbol{\alpha}_{-i}; \mathbf{s}) \setminus \{i\}}\} - |G_i(\tilde{\alpha}_i, \boldsymbol{\alpha}_{-i}; \mathbf{s})|c \right] \\
& - E_{\mathbf{s}} \left[\max\{w_i^+(\alpha_i, s_i), \xi_{G_i(\tilde{\alpha}_i, \boldsymbol{\alpha}_{-i}; \mathbf{s}) \setminus \{i\}}\} - |G_i(\tilde{\alpha}_i, \boldsymbol{\alpha}_{-i}; \mathbf{s})|c \right]
\end{aligned}
$$

$$
\begin{aligned}
&= E_{\mathbf{s}}\left\{\begin{array}{c}\left[\max\{w_i^+(\tilde{\alpha}_i,s_i),\xi_{G_i(\tilde{\alpha}_i,\boldsymbol{\alpha}_{-i};\mathbf{s})\setminus\{i\}}\}-|G_i(\tilde{\alpha}_i,\boldsymbol{\alpha}_{-i};\mathbf{s})|c\right]\\ -\left[\max\{w_i^+(\alpha_i,s_i),\xi_{G_i(\tilde{\alpha}_i,\boldsymbol{\alpha}_{-i};\mathbf{s})\setminus\{i\}}\}-|G_i(\tilde{\alpha}_i,\boldsymbol{\alpha}_{-i};\mathbf{s})|c\right]\end{array}\right\}\\
&= E_{\mathbf{s}}\left(\max\{w_i^+(\tilde{\alpha}_i,s_i),\xi_{G_i(\tilde{\alpha}_i,\boldsymbol{\alpha}_{-i};\mathbf{s})\setminus\{i\}}\}-\max\{w_i^+(\alpha_i,s_i),\xi_{G_i(\tilde{\alpha}_i,\boldsymbol{\alpha}_{-i};\mathbf{s})\setminus\{i\}}\}\right)\\
&\leq E_{\mathbf{s}}\left(w_i^+(\tilde{\alpha}_i,s_i)-w_i^+(\alpha_i,s_i)\right),
\end{aligned}
$$

which gives (69).

We thus have that if $\tilde{p}_i^*(\alpha_i,\boldsymbol{\alpha}_{-i})=1$, then $\tilde{p}_i^*(\tilde{\alpha}_i,\boldsymbol{\alpha}_{-i})=1$ for $\tilde{\alpha}_i>\alpha_i$. □

B.4 Incentive Compatibility in the Original Setting

We are now ready to show that the optimal first-stage selling rule (68), the optimal sequential shortlisting procedure described in Proposition 3, and the final-stage optimal allocation rule (64) are truthfully implementable by some well constructed payment rules.

We use $(\hat{\boldsymbol{\alpha}},\mathbf{m}_2,...,\mathbf{m}_{M+1})$ to denote the announcements of agents at different stages. We denote the shortlisting rule of Proposition 3 by $\mathbf{A}^*=\{A^{*g_k}(\hat{\boldsymbol{\alpha}},\mathbf{m}_2,...,\mathbf{m}_{k-1};g_1,g_2,...,g_{k-1}),\ k=1,2,...,M,\ \forall\mathbf{g}=(g_1,g_2,...,g_M)\}$, and denote the allocation rule of (64) by $\mathbf{p}^*=\{p_i^{*G_{\mathbf{g}}}(\hat{\boldsymbol{\alpha}},\mathbf{m}_2,...,\mathbf{m}_{M+1}),i\in\mathbf{N},\ \forall\mathbf{g}=(g_1,g_2,...,g_M)\}$. In addition,

$$
\Pr{}^*(\mathbf{g}|(\mathbf{m}_i)_{i=1}^M)=\Pi_{k=1}^M A^{*g_k}(\mathbf{m}_1,\mathbf{m}_2,...,\mathbf{m}_k|g_0,g_1,g_2,...,g_{k-1}),
$$

which is the probability that sequence $\mathbf{g}$ is shortlisted given messages reported $(\mathbf{m}_i)_{i=1}^M$.

The analysis on IC and IR for stage $k\in\{2,...,N+1\}$ are identical to those of Section 4.2. We now focus on IC and IR in stage 1.

By the same logic as in Lemma 4 of Esö and Szentes (2007) and Lemma 2 of Liu, Liu, and Lu (2020), when agent i reports $\hat{\alpha}_i$ at stage 1, she will report lie correction $\sigma_i(\alpha_i,\hat{\alpha}_i,s_i)$. In the proof of Corollary 1 in Esö and Szentes (2007) and Corollary 2 in Liu, Liu, and Lu (2020), they show that $w_i(\alpha_i,s_i)\lesseqgtr w_i(\hat{\alpha}_i,\sigma_i(\alpha_i,\hat{\alpha}_i,s_i))$ if and only if $\alpha_i\lesseqgtr\hat{\alpha}_i$.

Let $r(\hat{\alpha}_i,\boldsymbol{\alpha}_{-i})$ denote the rank of $\hat{\alpha}_i$ in $(\hat{\alpha}_i,\boldsymbol{\alpha}_{-i})$, and $\mathbf{m}^{\boldsymbol{\sigma}_i}_{r(\hat{\alpha}_i,\boldsymbol{\alpha}_{-i})+1}$ denote the stage $r(\hat{\alpha}_i,\boldsymbol{\alpha}_{-i})$ reports in which agent i's report is $\sigma_i(\alpha_i,\hat{\alpha}_i,s_i)$. Further assume that all shortlisted agents receive a subsidy of c from the seller besides the stage-1 transfer $t_{1,i}(\cdot)$ to make sure that they have the incentive to conduct the due diligence. At stage 1, agent i's expected payoff when i is of type α_i but announces $\hat{\alpha}_i$ is:

$$
\pi_i(\alpha_i,\hat{\alpha}_i) \tag{70}
$$

$$
= \quad E_{\boldsymbol{\alpha}_{-i}} E_{\mathbf{s}} \left[\begin{array}{c} \tilde{p}_i^*(\hat{\alpha}_i;\boldsymbol{\alpha}_{-i}) E_{s_i} u(\alpha_i, s_i) + \left[1 - \sum_j \tilde{p}_j^*(\hat{\alpha}_i;\boldsymbol{\alpha}_{-i})\right] \\ \cdot \left[\begin{array}{c} \sum\limits_{h=r(\hat{\alpha}_i,\boldsymbol{\alpha}_{-i})}^{N} \left[\begin{array}{c} \Pr{}^*(\mathbf{g}_{1,h}|(\hat{\alpha}_i,\boldsymbol{\alpha}_{-i}), \mathbf{m}_2^{\mathbf{s}}, ..., \mathbf{m}_{r(\hat{\alpha}_i,\boldsymbol{\alpha}_{-i})+1}^{\sigma_i(\alpha_i,\hat{\alpha}_i,s_i)}, ..., \mathbf{m}_{M+1}^{\mathbf{s}}) \\ \cdot \left[u(\alpha_i, s_i) p_i^{*G_{\mathbf{g}_{1,h}}}((\hat{\alpha}_i,\boldsymbol{\alpha}_{-i}), \mathbf{m}_2^{\mathbf{s}}, ..., \mathbf{m}_{r(\hat{\alpha}_i,\boldsymbol{\alpha}_{-i})+1}^{\sigma_i(\alpha_i,\hat{\alpha}_i,s_i)}, ..., \mathbf{m}_{M+1}^{\mathbf{s}}) - c \right] \end{array} \right] \\ - \sum\limits_{h=r(\hat{\alpha}_i,\boldsymbol{\alpha}_{-i})}^{N} \left[\begin{array}{c} \Pr{}^*(\mathbf{g}_{1,h}|(\hat{\alpha}_i,\boldsymbol{\alpha}_{-i}), \mathbf{m}_2^{\mathbf{s}}, ..., \mathbf{m}_{r(\hat{\alpha}_i,\boldsymbol{\alpha}_{-i})+1}^{\sigma_i(\alpha_i,\hat{\alpha}_i,s_i)}, ..., \mathbf{m}_{M+1}^{\mathbf{s}}) \\ \cdot\, t_{r(\hat{\alpha}_i,\boldsymbol{\alpha}_{-i})+1,i}^*((\hat{\alpha}_i,\boldsymbol{\alpha}_{-i}), \mathbf{m}_2^{\mathbf{s}}, ..., \mathbf{m}_{r(\hat{\alpha}_i,\boldsymbol{\alpha}_{-i})+1}^{\sigma_i(\alpha_i,\hat{\alpha}_i,s_i)}) \end{array} \right] \end{array} \right] \\ -t_{1,i}((\hat{\alpha}_i,\boldsymbol{\alpha}_{-i})) \end{array} \right],
$$

where $\mathbf{g}_{k,h}$ is defined in the proof of Proposition 4.

By similar arguments in establishing (52), we have

$$
\frac{d\pi_i(\alpha_i,\hat{\alpha}_i)}{d\alpha_i} =
$$

$$
E_{\boldsymbol{\alpha}_{-i}} E_{\mathbf{s}} \left\{ \begin{array}{c} \tilde{p}_i^*(\hat{\alpha}_i;\boldsymbol{\alpha}_{-i}) E_{s_i} u_1(\alpha_i, s_i) + \left[1 - \sum_j \tilde{p}_j^*(\hat{\alpha}_i;\boldsymbol{\alpha}_{-i})\right] \\ \cdot \sum\limits_{h=r(\hat{\alpha}_i,\boldsymbol{\alpha}_{-i})}^{N} \left[\begin{array}{c} \Pr{}^*(\mathbf{g}_{1,h}|(\hat{\alpha}_i,\boldsymbol{\alpha}_{-i}), \mathbf{m}_2^{\mathbf{s}}, ..., \mathbf{m}_{r(\hat{\alpha}_i,\boldsymbol{\alpha}_{-i})+1}^{\sigma_i(\alpha_i,\hat{\alpha}_i,s_i)}, ..., \mathbf{m}_{M+1}^{\mathbf{s}}) \\ \cdot \left(u_1(\alpha_i, s_i) p_i^{*G_{\mathbf{g}_{1,h}}}((\hat{\alpha}_i,\boldsymbol{\alpha}_{-i}), \mathbf{m}_2^{\mathbf{s}}, ..., \mathbf{m}_{r(\hat{\alpha}_i,\boldsymbol{\alpha}_{-i})+1}^{\sigma_i(\alpha_i,\hat{\alpha}_i,s_i)}, ..., \mathbf{m}_{M+1}^{\mathbf{s}}) \right) \end{array} \right] \end{array} \right\}. \quad (71)
$$

We are now ready to pin down the transfer $t_{1,i}^*(\cdot)$ (net of the entry subsidy c) that induces truthful revelation in stage 1. By the envelop theorem, optimality of truthful revelation requires

$$
\begin{aligned}
&\frac{d\pi_i(\alpha_i,\alpha_i)}{d\alpha_i} \qquad (72) \\
&= \quad E_{\boldsymbol{\alpha}_{-i}} E_{\mathbf{s}} \left\{ \begin{array}{c} \tilde{p}_i^*(\alpha_i,\boldsymbol{\alpha}_{-i}) E_{s_i} u_1(\alpha_i, s_i) + \left[1 - \sum_j \tilde{p}_j^*(\alpha_i;\boldsymbol{\alpha}_{-i})\right] \\ \cdot \sum\limits_{h=r(\alpha_i,\boldsymbol{\alpha}_{-i})}^{N} \left[\begin{array}{c} \Pr{}^*(\mathbf{g}_{1,h}|(\alpha_i,\boldsymbol{\alpha}_{-i}), \mathbf{m}_2^{\mathbf{s}}, ..., \mathbf{m}_{r(\alpha_i,\boldsymbol{\alpha}_{-i})+1}^{\mathbf{s}}, ..., \mathbf{m}_{M+1}^{\mathbf{s}}) \\ \cdot \left(u_1(\alpha_i, s_i) p_i^{*G_{\mathbf{g}_{1,h}}}((\alpha_i,\boldsymbol{\alpha}_{-i}), \mathbf{m}_2^{\mathbf{s}}, ..., \mathbf{m}_{r(\alpha_i,\boldsymbol{\alpha}_{-i})+1}^{\mathbf{s}}, ..., \mathbf{m}_{M+1}^{\mathbf{s}}) \right) \end{array} \right] \end{array} \right\}.
\end{aligned}
$$

Recall that we set $\pi_i(\underline{\alpha},\underline{\alpha}) = 0$. We thus have

$$
\begin{aligned}
&\pi_i(\alpha_i,\alpha_i) \qquad (73) \\
&= \quad \int_{\underline{\alpha}}^{\alpha_i} E_{\boldsymbol{\alpha}_{-i}} E_{\mathbf{s}} \left\{ \begin{array}{c} \tilde{p}_i^*(y,\boldsymbol{\alpha}_{-i}) E_{s_i} u_1(y, s_i) + \left[1 - \sum_j \tilde{p}_j^*(y,\boldsymbol{\alpha}_{-i})\right] \\ \cdot \sum\limits_{h=r(y,\boldsymbol{\alpha}_{-i})}^{N} \left[\begin{array}{c} \Pr{}^*(\mathbf{g}_{1,h}|(y,\boldsymbol{\alpha}_{-i}), \mathbf{m}_2^{\mathbf{s}}, ..., \mathbf{m}_{r(y,\boldsymbol{\alpha}_{-i})+1}^{\mathbf{s}}, ..., \mathbf{m}_{M+1}^{\mathbf{s}}) \\ \cdot \left(u_1(y, s_i) p_i^{*G_{\mathbf{g}_{1,h}}}((y,\boldsymbol{\alpha}_{-i}), \mathbf{m}_2^{\mathbf{s}}, ..., \mathbf{m}_{r(y,\boldsymbol{\alpha}_{-i})+1}^{\mathbf{s}}, ..., \mathbf{m}_{M+1}^{\mathbf{s}}) \right) \end{array} \right] \end{array} \right\} dy.
\end{aligned}
$$

By (70) and (73), we set

$$
t_{1,i}^*(\hat{\boldsymbol{\alpha}}) \qquad (74)
$$

$$
\begin{aligned}
&= E_{\mathbf{s}}\left\{ \begin{array}{c} \tilde{p}_i^*(\hat{\alpha}_i, \hat{\boldsymbol{\alpha}}_{-i}) E_{s_i} u(\alpha_i, s_i) + \left[1 - \sum_j \tilde{p}_j^*(\hat{\alpha}_i, \hat{\boldsymbol{\alpha}}_{-i})\right] \\ \cdot \left[\begin{array}{c} \sum\limits_{h=r(\hat{\alpha}_i, \boldsymbol{\alpha}_{-i})}^{N} \left[\begin{array}{c} \Pr{}^*(\mathbf{g}_{1,h} | (\hat{\alpha}_i, \hat{\boldsymbol{\alpha}}_{-i}), \mathbf{m}_2^{\mathbf{s}}, ..., \mathbf{m}_{r(\hat{\alpha}_i, \hat{\boldsymbol{\alpha}}_{-i})+1}^{\mathbf{s}}, ..., \mathbf{m}_{M+1}^{\mathbf{s}}) \\ \cdot \left(u(\hat{\alpha}_i, s_i) p_i^{*G_{\mathbf{g}_{1,h}}}((\hat{\alpha}_i, \hat{\boldsymbol{\alpha}}_{-i}), \mathbf{m}_2^{\mathbf{s}}, ..., \mathbf{m}_{r(\hat{\alpha}_i, \hat{\boldsymbol{\alpha}}_{-i})+1}^{\mathbf{s}}, ..., \mathbf{m}_{M+1}^{\mathbf{s}}) - c \right) \end{array} \right] \\ - \sum\limits_{h=r(\hat{\alpha}_i, \boldsymbol{\alpha}_{-i})}^{N} \left[\begin{array}{c} \Pr{}^*(\mathbf{g}_{1,h} | (\hat{\alpha}_i, \hat{\boldsymbol{\alpha}}_{-i}), \mathbf{m}_2^{\mathbf{s}}, ..., \mathbf{m}_{r(\hat{\alpha}_i, \hat{\boldsymbol{\alpha}}_{-i})+1}^{\mathbf{s}}, ..., \mathbf{m}_{M+1}^{\mathbf{s}}) \\ \cdot\, t^*_{r(\hat{\alpha}_i, \hat{\boldsymbol{\alpha}}_{-i})+1, i}((\hat{\alpha}_i, \hat{\boldsymbol{\alpha}}_{-i}), \mathbf{m}_2^{\mathbf{s}}, ..., \mathbf{m}_{r(\hat{\alpha}_i, \hat{\boldsymbol{\alpha}}_{-i})+1}^{\mathbf{s}}) \end{array} \right] \end{array} \right] \end{array} \right\} \\
&\quad - \int_{\underline{\alpha}}^{\hat{\alpha}_i} E_{\mathbf{s}} \left\{ \begin{array}{c} \tilde{p}_i^*(y, \hat{\boldsymbol{\alpha}}_{-i}) E_{s_i} u_1(y, s_i) + \left[1 - \sum_j \tilde{p}_j^*(y, \hat{\boldsymbol{\alpha}}_{-i})\right] \\ \cdot \sum\limits_{h=r(y, \hat{\boldsymbol{\alpha}}_{-i})}^{N} \left[\begin{array}{c} \Pr{}^*(\mathbf{g}_{1,h} | (y, \hat{\boldsymbol{\alpha}}_{-i}), \mathbf{m}_2^{\mathbf{s}}, ..., \mathbf{m}_{r(y, \hat{\boldsymbol{\alpha}}_{-i})+1}^{\mathbf{s}}, ..., \mathbf{m}_{M+1}^{\mathbf{s}}) \\ \cdot \left(u_1(y, s_i) p_i^{*G_{\mathbf{g}_{1,h}}}((y, \hat{\boldsymbol{\alpha}}_{-i}), \mathbf{m}_2^{\mathbf{s}}, ..., \mathbf{m}_{r(y, \hat{\boldsymbol{\alpha}}_{-i})+1}^{\mathbf{s}}, ..., \mathbf{m}_{M+1}^{\mathbf{s}}) \right) \end{array} \right] \end{array} \right\} dy.
\end{aligned}
$$

The following proposition establishes IC in stage 1.

Proposition 8. We have $\pi_i(\alpha_i, \alpha_i) \geq \pi_i(\alpha_i, \hat{\alpha}_i), \forall \alpha_i, \hat{\alpha}_i$.

Proof of Proposition 8: Without loss of generality, we consider $\hat{\alpha}_i < \alpha_i$. By (71), we have

$$
\begin{aligned}
&\pi_i(\alpha_i, \hat{\alpha}_i) - \pi_i(\hat{\alpha}_i, \hat{\alpha}_i) \\
&= \int_{\hat{\alpha}_i}^{\alpha_i} E_{\boldsymbol{\alpha}_{-i}} E_{\mathbf{s}} \left\{ \begin{array}{c} \tilde{p}_i^*(\hat{\alpha}_i; \boldsymbol{\alpha}_{-i}) E_{s_i} u_1(y, s_i) + \left[1 - \sum_j \tilde{p}_j^*(\hat{\alpha}_i; \boldsymbol{\alpha}_{-i})\right] \times \\ \sum\limits_{h=r(\hat{\alpha}_i, \boldsymbol{\alpha}_{-i})}^{N} \left[\begin{array}{c} \Pr{}^*(\mathbf{g}_{1,h} | (\hat{\alpha}_i, \boldsymbol{\alpha}_{-i}), \mathbf{m}_2^{\mathbf{s}}, ..., \mathbf{m}_{r(\hat{\alpha}_i, \boldsymbol{\alpha}_{-i})+1}^{\sigma_i(y, \hat{\alpha}_i, s_i)}, ..., \mathbf{m}_{M+1}^{\mathbf{s}}) \\ \cdot \left(u_1(y, s_i) p_i^{*G_{\mathbf{g}_{1,h}}}((\hat{\alpha}_i, \boldsymbol{\alpha}_{-i}), \mathbf{m}_2^{\mathbf{s}}, ..., \mathbf{m}_{r(\hat{\alpha}_i, \boldsymbol{\alpha}_{-i})+1}^{\sigma_i(y, \hat{\alpha}_i, s_i)}, ..., \mathbf{m}_{M+1}^{\mathbf{s}}) - c \right) \end{array} \right] \end{array} \right\} dy.
\end{aligned}
$$

By (74) and (70), we have (73). Therefore,

$$
\begin{aligned}
&\pi_i(\alpha_i, \alpha_i) - \pi_i(\hat{\alpha}_i, \hat{\alpha}_i) \\
&= \int_{\hat{\alpha}_i}^{\alpha_i} E_{\boldsymbol{\alpha}_{-i}} E_{\mathbf{s}} \left\{ \begin{array}{c} \tilde{p}_i^*(y, \boldsymbol{\alpha}_{-i}) E_{s_i} u_1(y, s_i) + \left[1 - \sum_j \tilde{p}_j^*(y, \boldsymbol{\alpha}_{-i})\right] \\ \cdot \sum\limits_{h=r(y, \boldsymbol{\alpha}_{-i})}^{N} \left[\begin{array}{c} \Pr{}^*(\mathbf{g}_{1,h} | (y, \boldsymbol{\alpha}_{-i}), \mathbf{m}_2^{\mathbf{s}}, ..., \mathbf{m}_{r(y, \boldsymbol{\alpha}_{-i})+1}^{\mathbf{s}}, ..., \mathbf{m}_{M+1}^{\mathbf{s}}) \\ \cdot \left(u_1(y, s_i) p_i^{*G_{\mathbf{g}_{1,h}}}((y, \boldsymbol{\alpha}_{-i}), \mathbf{m}_2^{\mathbf{s}}, ..., \mathbf{m}_{r(y, \boldsymbol{\alpha}_{-i})+1}^{\mathbf{s}}, ..., \mathbf{m}_{M+1}^{\mathbf{s}}) \right) \end{array} \right] \end{array} \right\} dy.
\end{aligned}
$$

Consider

$$
\begin{aligned}
\Delta &= \pi_i(\alpha_i, \alpha_i) - \pi_i(\alpha_i, \hat{\alpha}_i) \\
&= [\pi_i(\alpha_i, \alpha_i) - \pi_i(\hat{\alpha}_i, \hat{\alpha}_i)] + [\pi_i(\hat{\alpha}_i, \hat{\alpha}_i) - \pi_i(\alpha_i, \hat{\alpha}_i)] \\
&= \int_{\hat{\alpha}_i}^{\alpha_i} E_{\boldsymbol{\alpha}_{-i}} E_{\mathbf{s}} \left\{ P(y, \boldsymbol{\alpha}_{-i}, \mathbf{s}) \right\} dy - \int_{\hat{\alpha}_i}^{\alpha_i} E_{\boldsymbol{\alpha}_{-i}} E_{\mathbf{s}} \left\{ P(\hat{\alpha}_i, \boldsymbol{\alpha}_{-i}, \mathbf{s}) \right\} dy,
\end{aligned}
$$

where

$$P(y,\boldsymbol{\alpha}_{-i},\mathbf{s}) = \tilde{p}_i^*(y,\boldsymbol{\alpha}_{-i}) + \left[\underbrace{\begin{array}{c} \left(1-\sum_j \tilde{p}_j^*(y,\boldsymbol{\alpha}_{-i})\right) \\ \cdot \sum_{\forall h=r(y,\boldsymbol{\alpha}_{-i})}^{N} \left[\begin{array}{c} \Pr{}^*(\mathbf{g}_{1,h}|(y,\boldsymbol{\alpha}_{-i}),\mathbf{m}_2^{\mathbf{s}},...,\mathbf{m}_{r(y,\boldsymbol{\alpha}_{-i})+1}^{\mathbf{s}},...,\mathbf{m}_{M+1}^{\mathbf{s}}) \\ \cdot p_i^{*G_{\mathbf{g}_{1,h}}}((y,\boldsymbol{\alpha}_{-i}),\mathbf{m}_2^{\mathbf{s}},...,\mathbf{m}_{r(y,\boldsymbol{\alpha}_{-i})+1}^{\mathbf{s}},...,\mathbf{m}_{M+1}^{\mathbf{s}}) \end{array} \right] \end{array}}_{\tilde{P}(y,\boldsymbol{\alpha}_{-i},\mathbf{s})} \right]$$

is agent i's winning probability given her type y and that she reports truthfully; and

$$P(\hat{\alpha}_i,\boldsymbol{\alpha}_{-i},\mathbf{s}) = \tilde{p}_i^*(\hat{\alpha}_i;\boldsymbol{\alpha}_{-i}) + \left[\underbrace{\begin{array}{c} \left(1-\sum_j \tilde{p}_j^*(\hat{\alpha}_i;\boldsymbol{\alpha}_{-i})\right) \\ \cdot \sum_{\forall h=r(\hat{\alpha}_i,\boldsymbol{\alpha}_{-i})}^{N} \left[\begin{array}{c} \Pr{}^*(\mathbf{g}_{1,h}|(\hat{\alpha}_i,\boldsymbol{\alpha}_{-i}),\mathbf{m}_2^{\mathbf{s}},...,\mathbf{m}_{r(\hat{\alpha}_i,\boldsymbol{\alpha}_{-i})+1}^{\sigma_i(y,\hat{\alpha}_i,s_i)},...,\mathbf{m}_{M+1}^{\mathbf{s}}) \\ \cdot p_i^{*G_{\mathbf{g}_{1,h}}}((\hat{\alpha}_i,\boldsymbol{\alpha}_{-i}),\mathbf{m}_2^{\mathbf{s}},...,\mathbf{m}_{r(\hat{\alpha}_i,\boldsymbol{\alpha}_{-i})+1}^{\sigma_i(y,\hat{\alpha}_i,s_i)},...,\mathbf{m}_{M+1}^{\mathbf{s}}) \end{array} \right] \end{array}}_{\tilde{P}(\hat{\alpha}_i,\boldsymbol{\alpha}_{-i},\mathbf{s})} \right]$$

is agent i's winning probability given her type y and that she reports $\hat{\alpha}_i$ in stage 1 and corrects her lie when shortlisted.

By Lemma 11, we always have $\tilde{p}_i^*(\hat{\alpha}_i;\boldsymbol{\alpha}_{-i}) \leq \tilde{p}_i^*(y;\boldsymbol{\alpha}_{-i})$. Moreover, $\tilde{p}_i^*(\cdot;\cdot) \in \{0,1\}$ by the optimal first-stage selling rule. Fix $\boldsymbol{\alpha}_{-i}$ and $\mathbf{s}$. We consider the following cases.

Case I: If $\tilde{p}_i^*(\hat{\alpha}_i;\boldsymbol{\alpha}_{-i}) = 1$, then $\tilde{p}_i^*(y;\boldsymbol{\alpha}_{-i}) = 1$ by Lemma 11. This implies $P(y,\boldsymbol{\alpha}_{-i},\mathbf{s}) = P(\hat{\alpha}_i,\boldsymbol{\alpha}_{-i},\mathbf{s})$, $\forall y$.

Case II: $\tilde{p}_i^*(\hat{\alpha}_i;\boldsymbol{\alpha}_{-i}) = 0$ and $\tilde{p}_i^*(y;\boldsymbol{\alpha}_{-i}) = 1$. This implies $P(y,\boldsymbol{\alpha}_{-i},\mathbf{s}) = 1 \geq P(\hat{\alpha}_i,\boldsymbol{\alpha}_{-i},\mathbf{s})$, $\forall y$.

Case III: $\tilde{p}_i^*(\hat{\alpha}_i;\boldsymbol{\alpha}_{-i}) = 0$ and $\tilde{p}_i^*(y;\boldsymbol{\alpha}_{-i}) = 0$. In this case, we have

$$\begin{aligned} P(y,\boldsymbol{\alpha}_{-i},\mathbf{s}) &= \left(1-\sum_j \tilde{p}_j^*(y;\boldsymbol{\alpha}_{-i})\right)\tilde{P}(y,\boldsymbol{\alpha}_{-i},\mathbf{s}), \\ P(\hat{\alpha}_i,\boldsymbol{\alpha}_{-i},\mathbf{s}) &= \left(1-\sum_j \tilde{p}_j^*(\hat{\alpha}_i;\boldsymbol{\alpha}_{-i})\right)\tilde{P}(\hat{\alpha}_i,\boldsymbol{\alpha}_{-i},\mathbf{s}). \end{aligned} \tag{75}$$

By the same arguments as in the proof of Proposition 4, we have

$$\tilde{P}(y,\boldsymbol{\alpha}_{-i},\mathbf{s}) \geq \tilde{P}(\hat{\alpha}_i,\boldsymbol{\alpha}_{-i},\mathbf{s}). \tag{76}$$

By our first-stage selling rule, if buyer i does not obtain the object, the other buyers' first-stage winning chances become smaller when buyer i's first-stage type becomes higher. The reason is that the change improves the expected revenue generated from optimal shortlisting, but does not change the expected revenue from the first-stage sale. We thus have $\sum_j \tilde{p}_j^*(y,\boldsymbol{\alpha}_{-i}) \leq \sum_j \tilde{p}_j^*(\hat{\alpha}_i;\boldsymbol{\alpha}_{-i})$, which

means

$$\left(1-\sum_j \tilde{p}_j^*(y,\boldsymbol{\alpha}_{-i})\right) \geq \left(1-\sum_j \tilde{p}_j^*(\hat{\alpha}_i;\boldsymbol{\alpha}_{-i})\right). \tag{77}$$

Therefore, by (75), (76), and (77), we also have $P(y,\boldsymbol{\alpha}_{-i},\mathbf{s}) \geq P(\hat{\alpha}_i,\boldsymbol{\alpha}_{-i},\mathbf{s})$, $\forall y$, for case III.

Aggregating all cases, we always have $P(y,\boldsymbol{\alpha}_{-i},\mathbf{s}) \geq P(\hat{\alpha}_i,\boldsymbol{\alpha}_{-i},\mathbf{s}), \forall y > \hat{\alpha}_i$, which immediately means $\Delta \geq 0$ when $\hat{\alpha}_i < \alpha_i$. The case of $\hat{\alpha}_i > \alpha_i$ can be similarly demonstrated. We have thus established IC for the first stage. The first-stage IR holds by construction since we set $\pi_i(\underline{\alpha},\underline{\alpha}) = 0$ and π_i is increasing in the first-stage type by (72). $\square$