# Auction design with shortlisting when value discovery is covert* 

Murali Agastya ${ }^{\dagger}$ Xin Feng ${ }^{\ddagger}$ Jingfeng Lu ${ }^{\S}$

April 2023


#### Abstract

This paper studies optimal auction design when buyers' value discovery investment is covert but essential for mutually beneficial trade between seller and buyers. Since selling mechanisms contingent on value discovery (e.g. ex-ante examination fees charged upon information acquisition) are not feasible, we focus mainly on second price auctions with reserves, which can be contingent on the number of actual bidders ex-post. Under a regularity condition of monotone hazard rate, we find that the optimal reserve depends on the number of shortlisted bidders, but for any given shortlist it does not depend on the number of actual bidders. Depending on the value discovery cost, the seller shortlists either the socially efficient number of buyers or one more bidder. The comparison between the two options of the seller is completely resolved. The optimal reserve depends discontinuously and non-monotonically on the value discovery cost. In the former case, equilibrium information acquisition is efficient but ex-post allocation is inefficient, while in the latter case, it is the opposite.


Keywords: Exclusive bidding; Covert information acquisition; Endogenous market size; Optimal auctions; Revenue maximization.

JEL Classification Numbers: D44, D45, D82.
*We are grateful to the editors in charge and two anonymous reviewers for their insightful comments and suggestions, which greatly improved the quality of the papers. Feng gratefully acknowledges financial support from the National Natural Science Foundation of China (Grant Nos. 72273063 and 71803019). Lu gratefully acknowledges financial support from the MOE of Singapore (Grant No. R122- 000-298-115). All remaining errors are our own.
${ }^{\dagger}$ Murali Agastya, Economics Discipline, H04 Merewether Building, University of Sydney, Sydney NSW 2006, AUSTRALIA. E-mail: m.agastya@econ.usyd.edu.au.
${ }^{\ddagger}$ Xin Feng, School of Economics, Nanjing University, 22 Hankou Road, Nanjing, Jiangsu 210093, CHINA. E-mail: a0078175@u.nus.edu.
§Jingfeng Lu, Department of Economics, National University of Singapore, SINGAPORE 117570. E-mail: ecsljf@nus.edu.sg.

## 1 Introduction

In many auction environments, there are no gains from trade unless buyers invest in costly value discovery. Dasgupta (1990), Tan (1992), Bag (1997), Fullerton and McAfee (1999), Che and Gale (2003), Lu (2010), Moreno and Wooders (2011), Jehiel and Lamy (2015), Sogo, Bernhardt and Liu (2016), Li (2019), Gershkov, Moldovanu, Strack and Zhang (2021) among others explore auctions with value discovery or endogenous values.

Since bidders' value discovery investment determines the total surplus that can be split between the seller and buyers, the seller's revenue maximizing selling procedure has to appropriately encourage the bidders to make such an investment. One effective policy to boost buyers' incentives to acquire information is for the seller to create a shortlist of eligible buyers.

This has been well explored in the literature $\dagger$ Under the assumption that the seller observes the buyers' value discovery decision, she can charge a shortlisting fee or an examination fee upon buyers' information acquisition. It is now well understood in the literature that the seller should hold an efficient auction, shortlist a socially optimal number of buyers and charge each of them an appropriate shortlisting fee or examination fee that extracts all surplus. ${ }^{2}$

In this paper, we instead assume that buyers' value discovery is covert and investigate the optimal design of the selling mechanism while allowing shortlisting $3^{3}$ By covert value discovery, we mean that the seller does not observe the buyers' information acquisition $\sqrt[4]{4}$ As a result, she can charge neither a shortlisting fee nor an examination fee upon buyers' value discovery. Indeed, if the seller charges these fees, the shortlisted bidders can opt to first covertly discover their values before paying the fees. A shortlisted buyer can also opt to participate in the auction as an uninformed bidder if he has a chance to win.

We consider a standard symmetric independent private value (IPV) setting for the sale of a single object to a population of initially uninformed buyers. We formulate the revenue-maximization problem as a multistage game in which the seller's choice is to first pre-select a number of eligible buyers, say $N$ of them, and commit to run

[^0]a second price auction with a reserve $r_{i}$ if $i$ of them participate in the auction. The eligible buyers take $\mathbf{r}=\left(r_{1}, \ldots, r_{i}, \ldots, r_{N}\right)$ as given and choose whether to invest in value discovery at a cost $c>0$, and then whether to participate in the auction. We focus on a symmetric information acquisition equilibrium across the shortlisted bidders. We vary $(N, \mathbf{r})$ to maximize the seller's equilibrium revenue.

To describe the solution, let $u_{n}$ denote the maximum investment cost at which it is socially optimal for exactly $n$ buyers to invest in value discovery ${ }^{5}$ We show that there exists a sequence of cost cutoffs $\left\{\hat{c}_{n}, n \geq 1\right\}$ with $\hat{c}_{n} \in\left(u_{n+1}, u_{n}\right]$, such that if $c \in\left(u_{n+1}, \hat{c}_{n}\right)$ it is optimal for the seller to shortlist $N=n+1$ buyers, one more than the socially optimal number $n$, and run an efficient auction, that is to set $r_{i} \equiv v_{0}, \forall i \leq N$, the seller's reservation value. This conclusion follows relatively easily from Levin and Smith (1994) with some caveats (described later in Section 3.1.1). The analysis is more intricate when $c \in\left(\hat{c}_{n}, u_{n}\right], \forall n \geq 1$. In Section 3.1.2 we show that for such a value of $c$, it is optimal to shortlist the socially optimal number $n$ followed by an auction with a reserve that is non-contingent on the actual number of bidders. This reserve lies between the Myerson's reserve and $v_{0}$. (See Figure 1 for an illustration.)

As investment cost $c$ increases, whenever the cost goes over one of these thresholds $\left\{\hat{c}_{N}, N \geq 1\right\}$, it becomes optimal for the seller to exclude an extra buyer. At any such threshold, the optimal reserve jumps above the seller's valuation and then decreases continuously to the seller's value and remains there until the next threshold. Furthermore, in between these thresholds, participation in the revenue-maximizing auction by the eligible buyers changes from pure strategy to mixed strategy. Our finding thus echoes that of Shi (2012) who shows that optimal reserve should lie between the ex-ante mean valuation of bidders and the Myerson reserve. ${ }^{6}$

The recent paper by Chen and Kominers (2021) contains results that resemble some of ours. In their model, the auction format is fixed. There is an exogenous pool of buyers who make costly entry decisions based on a private entry cost. The seller controls the entry only through the use of an ex-ante entry fee contingent on incurring the cost.7 In our paper, the entry/value discovery cost is common to all buyers. However, the seller directly controls the number of buyers using the shortlist of eligible buyers. She is also able to vary the auction (reserves) as well to induce value discovery. Just as in Chen and Kominers (2021) or Bulow and Klemperer (1996), our results also provide insight into the role of additional bidders. We discuss this further in Section 3.3.2.

[^1]The rest of the paper is organized as follows. Section 2 sets up the seller's problem. In Section 3.1 we derive the optimal auction for a given $N$. The optimal shortlisting policy is then completely characterized in Section 3.2. We also offer here a somewhat intuitive account based on the value of a bidder that may be of independent interest. Section 3.3 contains a discussion of several aspects of the model. This section also discusses the issues on ex-ante subsidies/fees, the implications of dropping the "serious bidder assumption". The conclusion is presented in Section 4. Proofs for all the formal claims are in the Appendix.

## 2 The seller's problem

A seller owns an indivisible object worth $v_{0}$ to her. There are a sufficiently large number of potential buyers, and each buyer $i$ initially knows only that his value for the object is a random variable $V_{i}$. It is common-knowledge that buyers' values are distributed identically and independently according to a continuous probability distribution function $F(\cdot)$ and a positive density $f(\cdot)$ on an interval $[\underline{v}, \bar{v}]$ where $\underline{v} \geq 0$. By incurring a cost $c(>0)$, any of these agents can discover his value of the object. We refer to $c$ as the value discovery cost or information acquisition cost following the literature. The seller and bidders are risk neutral. Bidder $i$ thus values the object at $E V_{i}$ if he does not invest to identify his true value.

Let $\eta=E V_{i}, \forall i$. Throughout the paper, we assume that there are no (expected) gains from trade if no buyer incurs the cost to discover his value, i.e.,

Assumption $10<\eta \leq v_{0}$.

The importance of this condition is that it forces a buyer to engage in value discovery for any trade to be mutually beneficial 团 To make mutually beneficial trade indeed feasible, we assume that it is socially desirable for at least one buyer to discover his value, i.e.

Assumption $2 E\left[\max \left\{V_{i}-v_{0}, 0\right\}\right] \geq c$.

A shortlist is an integer $N$, denoting that buyers $i=1, \ldots, N$ who are given exclusive rights to bid for the object. We call them eligible bidders. Eligible bidders choose whether to invest in value discovery and then whether to bid for the object. An eligible bidder is said to be an actual bidder or simply a bidder if she participates in the auction. We assume that the seller runs a second price auction, possibly with a reserve price, between the actual bidders.

[^2]A precise description of the stages of the multi-stage game being played is as follows. ${ }^{9}$ We use $\mathcal{S P} \mathcal{A}_{k}(r)$ to denote a second price auction with $k$ actual bidders and a reserve $r$.

Stage 1 The seller publicly commits to a selling procedure, which is a tuple $\mu_{N}:=$ $\left(N, \mathbf{r}_{N}\right)$ where $\mathbf{r}_{N}=\left(r_{1}, \ldots, r_{k}, \ldots, r_{N}\right)$. That is, she selects a shortlist $N$ and commits to running $\mathcal{S P} \mathcal{A}_{k}\left(r_{k}\right)$ in the event there are $k$ actual bidders, where $k$ will be determined in Stage 3 as below.

Stage 2 Each eligible bidder chooses a probability $p_{i}$ with which to invest in value discovery/information acquisition by incurring a cost $c>0$. This decision is covert, i.e. unobservable.

Stage 3 The values of those eligible bidders who chose to invest are privately revealed to them. Then, all eligible bidders simultaneously choose whether to participate in the seller's auction. The number $(k)$ of participants in the auction is publicly observable. $\sqrt{10}$

Stage 4 If $k$ of the eligible bidders chose to participate in Stage $3, \mathcal{S P} \mathcal{A}_{k}\left(r_{k}\right)$ is played out.

The seller has the option of setting lower reserves to provide the eligible buyers a higher incentive to invest in value discovery. Importantly, the reserves are allowed to be below $v_{0}$. Indeed, the seller can, in principle, set some of the $r_{k}$ s to lie even below $\eta$ to induce participation by uninformed buyers and ensure a certain sale. Moreover, we allow contingent reserves and investigate whether this would enhance the optimal design.

As stated in the Introduction, our focus here is on the case in which value discovery of the eligible bidders is covert ${ }^{11}$ This makes it infeasible for the seller to charge shortlisting fees in stage 1 or examination fees in stage 2 upon eligible bidders' information acquisition to fully extract their surplus, and thus further complicates the optimal design. Note that if the seller charges these fees, the shortlisted bidders can opt to first covertly discover their values before paying the fees.

[^3]
## Buyers' decisions on being an actual bidder

At stage 3, the game has many nodes at which a buyer is indifferent between participation decisions. The shortlisted bidders' endogenous participation decisions at stage 3 depend on how these are broken. We describe below the tie-breaking assumptions.

At stage 3, a buyer is either informed or uninformed. For an informed buyer of type $v_{i}>\min r_{k}$, participation leads to a positive payoff or a zero payoff depending on the actions of her rivals. Non-participation leads to a zero payoff. For such types therefore, participation is a weakly dominant action. On the other hand, a type $v_{i}<\min r_{k}$ is indifferent between participating and not doing so. We break this indifference with the assumption that every informed buyer type participates in the auction. When the covert value discovery is accompanied by hard evidence of investment, the seller could announce an $\epsilon>0$ subsidy to be paid on presenting the evidence. This gives every type of informed bidder a strict incentive to participate. Similarly, when $\min r_{k}<\eta$, participation is a weakly dominant action for an uninformed buyer ${ }^{12}$ Otherwise, an uninformed buyer is indifferent between participation and non-participation. In this case, we break the indifference with the assumption that uninformed buyers participate if and only if $\min r_{k}<\eta$. When $\min r_{k} \geq \eta$, an $\epsilon$ participation fee for those who cannot provide hard evidence of investment would strictly discourage the uninformed from participating.

## Information Acquisition Equilibrium

Consider, a selling procedure $\mu_{N}=\left(N, \mathbf{r}_{N}\right)$, where $r_{k} \geq \eta, \forall k$. Later, via Proposition 1 , we shall argue that this restriction on reserves is without loss of generality.

The information acquisition decision of eligible bidder $i=1, \ldots, N$ can be described by a number $p_{i} \in[0,1]$, namely the probability of investing in value discovery. Our interest is in (equilibrium) situations where this decision is symmetric across the agents, i.e. $p_{i} \equiv p$ for some $p \in[0,1]{ }^{13}$ Given that each of his rivals invests in value discovery with the probability $p$, when an eligible bidder incurs the cost and discovers his value, $\beta(k ; p, N-1)=\binom{N-1}{k} p^{k}(1-p)^{(N-1)-k}$ is the probability that there are $k$ other bidders who also discover their values and participate in the auction. Letting $u_{k}(r)$ denote a bidder's ex-ante payoff in $\mathcal{S} \mathcal{P} \mathcal{A}_{k}(r)$ with $k$ informed bidders, his expected payoff from

[^4]investing in value discovery is
\[

$$
\begin{equation*}
U_{N}\left(p, \mathbf{r}_{N}, c\right)=\sum_{k=0}^{N-1} \beta(k ; p, N-1) u_{k+1}\left(r_{k+1}\right)-c \tag{1}
\end{equation*}
$$

\]

His payoff is zero if he does not invest in value discovery. The symmetric information acquisition equilibrium is defined as follows:

Definition 1 (Information Acquisition Equilibrium) $p_{e}\left(\mu_{N} ; c\right) \in[0,1]$ is said to be a symmetric information acquisition equilibrium of a selling procedure $\mu_{N}=\left(N, \mathbf{r}_{N}\right)$ if at $p=p_{e}\left(\mu_{N} ; c\right)$ one of the following conditions hold: a) $\left.U_{N}\left(p, \mathbf{r}_{N}, c\right)=0 ; b\right)$ $U_{N}\left(p, \mathbf{r}_{N}, c\right)>0$ and $p=1$; or c) $U_{N}\left(p, \mathbf{r}_{N}, c\right)<0$ and $p=0$.

Note that $U_{N}\left(p, \mathbf{r}_{N}, c\right)$ is continuous in $p$. This ensures the existence of a symmetric information acquisition equilibrium. Letting $R_{k}(r)$ and $W_{k}(r)$ denote the expected revenue and expected welfare in $\mathcal{S P} \mathcal{A}_{k}(r)$, the expected revenue under $\mu_{N}=\left(N, \mathbf{r}_{N}\right)$ is

$$
\begin{equation*}
R_{N}\left(p, \mathbf{r}_{N}\right)=\sum_{k=0}^{N} \beta(k ; p, N) R_{k}\left(r_{k}\right) \tag{2}
\end{equation*}
$$

and the expected social surplus is

$$
\begin{equation*}
S_{N}\left(p, \mathbf{r}_{N}, c\right)=\sum_{k=0}^{N} \beta(k ; p, N)\left(W_{k}\left(r_{k}\right)-k c\right) \tag{3}
\end{equation*}
$$

## 3 Revenue-maximizing selling procedure

We first introduce some notations. Let $G_{k}(\cdot)$ denote the probability distribution of the random variable $Y_{k}=\max \left\{V_{1}, \ldots, V_{k}\right\}$, i.e. the highest value among $k$ informed bidders. The total welfare under $\mathcal{S} \mathcal{P} \mathcal{A}_{k}(r)$ with $k$ informed bidders is

$$
W_{k}(r)=E\left[\max \left\{Y_{k}, r\right\}\right]-\left(r-v_{0}\right) G_{k}(r)
$$

When $r=v_{0}$, auction $\mathcal{S P} \mathcal{A}_{k}(r)$ is the VCG mechanism with $k$ informed buyers. Therefore, the payoff of a typical buyer in $\mathcal{S P} \mathcal{A}_{k}\left(v_{0}\right)$ is the change in welfare resulting from his addition. Therefore, the payoff of a typical buyer in $\mathcal{S P} \mathcal{A}_{k}\left(v_{0}\right)$ is given by

$$
\begin{equation*}
u_{k}\left(v_{0}\right)=W_{k}\left(v_{0}\right)-W_{k-1}\left(v_{0}\right) . \tag{4}
\end{equation*}
$$

More generally, the payoff of a typical buyer in an auction $\mathcal{S P} \mathcal{A}_{k}(r)$ with $k$ actual
informed buyers is given by

$$
u_{k}(r)=E\left[\max \left\{Y_{k}, r\right\}\right]-E\left[\max \left\{Y_{k-1}, r\right\}\right] .
$$

Note that $u_{k+1}\left(v_{0}\right)<u_{k}\left(v_{0}\right)$, and $\lim _{k \rightarrow \infty} u_{k}\left(v_{0}\right)=0{ }^{14}$ From Assumption 2, $c \leq u_{1}\left(v_{0}\right)$. Therefore, there must exist a unique integer $N_{c} \geq 1$ such that

$$
\begin{equation*}
u_{N_{c}+1}\left(v_{0}\right)<c \leq u_{N_{c}}\left(v_{0}\right) \tag{5}
\end{equation*}
$$

$N_{c}$ is a decreasing step function of $c$ as $N_{c}=N$ if and only if $c \in\left(u_{N+1}\left(v_{0}\right), u_{N}\left(v_{0}\right)\right] .^{15}$
Our approach to characterize the revenue-maximizing selling procedure is as follows. We first ask what reserves maximize the seller's payoff for a fixed number $(N)$ of shortlisted bidders. Then we optimize over $N$. When $N \geq N_{c}+1$, the analysis largely follows Levin and Smith (1994), although there are some differences. The value discovery is partial, leaving a typical eligible bidder indifferent between investing in value discovery and remaining uninformed. Consequently, the entire social surplus accrues to the seller as her revenue. The analysis for the case where $N \leq N_{c}$ is significantly different, and begins in Section 3.1.2.

We begin by formally showing that it is sufficient to restrict attention to selling procedures where every reserve is at least $\eta$.
Proposition 1 (Sub-optimality of reserves lower than $\boldsymbol{\eta}$ ) Fix $N$ and consider a selling procedure $\mu_{N}=\left(N, \mathbf{r}_{N}\right)$ such that $\min \left\{r_{k}\right\}<\eta$. There exists a $\tilde{\mu}_{N}=\left(N, \tilde{\mathbf{r}}_{N}\right)$ with $\min \left\{\tilde{r}_{k}\right\} \geq \eta$ such that the expected revenue under $\tilde{\mu}_{N}$ is at least that under $\mu_{N}$.

The proof of the above proposition when $N>N_{c}$ involves arguments similar to those in Levin and Smith (1994) leading to their conclusion that an efficient auction is optimal, i.e. set $\tilde{r}_{k} \equiv v_{0}(>\eta)$. When $N \leq N_{c}$ and $\operatorname{since} \min \left\{r_{k}\right\}<\eta$, every eligible buyer participates and in stage 4 , at $\mu_{N}$ the only relevant reserve is $r_{N}$. If $r_{N} \geq \eta$, there is nothing to prove. Otherwise, we set $\tilde{r}_{N} \equiv \eta$. By Assumptions 1.2, $u_{N}(\eta) \geq u_{N}\left(v_{0}\right) \geq c$ and hence all $N$ eligible bidders acquire information at equilibrium under $\tilde{\mu}_{N}$. The stage 4 auction thus consists of $N$ informed buyers and a higher reserve than under $\mu_{N}$. Since both reserves are lower than the seller's value, more informed bidders and a higher reserve would result in higher revenue.
Remark 1 In view of Proposition 1, for the rest of the paper, we will only consider selling procedures with each reserve $r_{k} \in[\eta, \bar{v}]$.

[^5]
### 3.1 Revenue-maximizing selling procedure for given $N$

### 3.1.1 Optimal reserve when $N \geq N_{c}+1$

As mentioned earlier, when the shortlist $N \geq N_{c}+1$, we will show that the seller gets the entire social surplus as her expected revenue in equilibrium. The following Lemma isolates properties of the general social surplus that will prove useful throughout the paper.

Lemma 1 The following holds for all $N, p, c$ :

$$
\begin{align*}
S_{N}\left(p, \mathbf{r}_{N}, c\right) & =R_{N}\left(p, \mathbf{r}_{N}\right)+N p U_{N}\left(p, \mathbf{r}_{N}, c\right), \forall \mathbf{r}_{N}  \tag{6}\\
\frac{\partial S_{N}\left(p, \mathbf{v}_{0}, c\right)}{\partial p} & =N U_{N}\left(p, \mathbf{v}_{0}, c\right) \tag{7}
\end{align*}
$$

where $\mathbf{v}_{0}=\left(v_{0}, \ldots, v_{0}\right)$.
Recall that $U_{N}\left(p, \mathbf{r}_{N}, c\right)$ is a representative bidder's expected payoff upon information acquisition ${ }^{16}$ (6) is simply the statement that the total surplus is in general the sum of expected revenue and the eligible bidders' payoffs. (7) links the impact of information acquisition probability $p$ on social surplus and the bidders' payoff from value discovery, when the reservation prices are uniformly set at the efficient level of $v_{0}$. If $S_{N}\left(p, \mathbf{v}_{0}, c\right)$ is maximized at an interior $p_{N}^{*}$, then we have $U_{N}\left(p_{N}^{*}, \mathbf{v}_{0}, c\right)=\frac{\partial S_{N}\left(p_{N}^{*}, \mathbf{v}_{0}, c\right)}{\partial p} / N=0$. This means $p_{N}^{*}$ would be an information acquisition equilibrium in the efficient auction. By (6), we then conclude that a constant reserve set at the seller's value $v_{0}$ is both efficient and revenue-maximizing. ${ }^{17}$
Definition 2 (Standard Selling Procedure) A selling procedure $\mu_{N}:=\left(N, \mathbf{r}_{N}\right)$ is said to be standard if $\mathbf{r}_{N}=\left(r_{k}\right)$ is such that $r_{k} \equiv r, \forall k$ and will be denoted by $(N, r)$.

Recall that the seller can choose reserves that depend non-trivially on the number of eligible bidders that participate in the auction. The following shows that it is optimal to choose a reserve that is constant with respect to the number of participants.

Proposition 2 Fix $N \geq N_{c}+1$. Under Assumptions 1 and 2 , The standard selling procedure $\left(N, v_{o}\right)$ maximizes expected revenue across all selling procedures $\mu_{N}$. Moreover, at $\left(N, v_{0}\right)$, (i) Value discovery is partial; (ii) A buyer's ex-ante payoff is zero; and (iii) Expected revenue equals the social surplus.

The intuition behind Proposition 2 is as follows. Recall the definition of $N_{c}$ in (5), for $N \geq N_{c}+1$, we have $u_{N}\left(v_{0}\right)<0$, which means $U_{N}\left(p, \mathbf{v}_{0}, c\right)<0$ when $p=1$.

[^6]Therefore, (7) means $\frac{\partial S_{N}\left(p, \mathbf{v}_{0}, c\right)}{\partial p}<0$ when $p=1$, which entails an interior optimal value discovery that maximizes the total surplus. When $p^{*}=\arg \max _{p} S_{N}\left(p, \mathbf{v}_{0}, c\right)$ is an interior point, (7) means that buyers earn zero surplus from participation. As a result, the optimal seller's revenue $R_{N}\left(p^{*}, \mathbf{v}_{0}\right)$ must equal the maximal surplus.

### 3.1.2 Optimal reserve when $N \leq N_{c}$

The hazard rate of the prior $F$ at $v$ is $H(v)=\frac{f(v)}{1-F(v)}$. We introduce the following assumption:

Assumption $3 H(v)$ is increasing.

The increasing hazard rate is a sufficient condition to ensure that the prior is regular. Moreover, this assumption further means that we can search for the optimal $\mu_{N}$ for a given $N \leq N_{c}$ within standard selling procedures. To be precise, we have the following result:

Proposition 3 (Optimality of Uniform Reserve) Under Assumption 3, for any $N \leq N_{c}$, the equilibrium revenue in an arbitrary selling procedure ( $N, \mathbf{r}_{N}$ ) where $r_{k} \geq \eta$ is bounded above by the equilibrium revenue of a standard selling procedure ( $N, r$ ) for some $r \geq \eta$, which induces the same information acquisition equilibrium.

Proof of the above proposition is fairly technical and unfortunately, even after completing the proof, there does not appear to be an intuitive guide to the result. Nevertheless, given Proposition 3, we need only optimize with respect to the uniform reserve $r$, a scalar, to characterize the optimal selling procedure for a given $N \leq N_{c}$. To this end, we need the following upper bound on a bidder's payoff, expressed as a change in social surplus resulting from an additional bidder.

Lemma $2 u_{k}(r)<W_{k}(r)-W_{k-1}(r), \forall r>v_{0}, \forall k$.
Definition 3 Let $r_{N}^{c}$ and $r^{m}$ satisfy the following:

$$
\begin{align*}
u_{N}\left(r_{N}^{c}\right) & =c,  \tag{8}\\
r^{m}-\frac{1-F\left(r^{m}\right)}{f\left(r^{m}\right)} & =v_{0} \tag{9}
\end{align*}
$$

$$
\begin{equation*}
r_{N}^{*}=\min \left\{r_{N}^{c}, r^{m}\right\} . \tag{10}
\end{equation*}
$$

$r^{m}$ is the familiar optimal reserve in a standard IPV setting with any number of buyers, as shown in Myerson (1981). $r^{m}$ is the type of buyer whose virtual value equals the seller's value for the object. $r_{N}^{c}$ is the reserve that makes a typical bidder indifferent between incurring the cost of information acquisition and remaining uninformed if all the remaining $N-1$ eligible bidders are known to become informed and participate in the auction. Note when $N \leq N_{c}$, we have $r_{N}^{c} \geq v_{0}$, since $u_{N_{c}}\left(v_{0}\right) \geq u_{N_{c}}\left(v_{0}\right)=c$ and a buyer's ex-ante payoff $u_{N}(r)$ is decreasing in $r$.

Proposition 4 Under Assumption 3, for any $N \leq N_{c}$, the optimal uniform reserve $r_{N}^{*}$ is $\min \left\{r_{N}^{c}, r^{m}\right\}$, which is higher than $v_{0}$. This selling procedure induces a full value discovery equilibrium, i.e., all bidders acquire information with probability 1.

A sketch of the steps in the proof is as follows. Under Assumption 3, the Myerson reservation price $r^{m}$ is revenue-maximizing if it induces all bidders to acquire information with probability 1. Otherwise, we have $r_{N}^{c} \in\left[v_{0}, r^{m}\right)$. With Assumption 3, the seller revenue must increase with the reservation price provided it is lower than $r_{N}^{c}$, since such a reservation price induces all bidders to acquire information with probability 1. Note when the reservation price equals $r_{N}^{c}$, the seller revenue equals the total surplus since buyer payoffs are zero. When the reservation price goes beyond $r_{N}^{c}$, the information equilibrium becomes partial and the reservation price is further away from the efficient level of $v_{0}$. Clearly, the total surplus must drop, which entails that expected revenue must be lower than what it was at $r_{N}^{c}$.

### 3.2 Optimal shortlisting

We now are ready to construct the optimal shortlist. We first establish the following result, which says that the optimal number shortlisted must be either $N_{c}$ or $N_{c}+1$.

Proposition 5 If $N_{c} \geq 1$, (i) optimal revenue $R_{N}\left(p_{e}^{N}, \mathbf{v}_{0}\right)$ strictly decreases with $N$ when $N \geq N_{c}+1$; (ii) optimal revenue $R_{N}\left(r_{N}^{*}\right)$ increases with $N$ when $N \leq N_{c}$.

Levin and Smith (1994) prove part (i) for the case where $v_{0}=0$. It says when the number $N$ of shortlisted goes beyond the efficient level $N_{c}$, the seller revenue drops with $N$. In this case, seller revenue coincides with the total surplus, and the total surplus drops with $N$ due to coordination in information acquisition among bidders, as pointed out by Levin and Smith (1994). Part (ii) is one of our main contributions. It says that with the optimally designed reservation prices, the seller revenue increases in $N\left(\leq N_{c}\right)$, which is in contrast to the result of part (i). Given the optimal reservation price is in general lower with a higher $N\left(\leq N_{c}\right)$, this result is highly nontrivial. Without loss of generality, one may assume the reservation price is below the Myerson $r^{m}$ for an
$N \in\left\{2, \ldots, N_{c}\right\}{ }^{18}$ In this case, the buyer payoff is zero, and seller revenue equals total surplus. Relying on Lemma 2 and the monotonicity of reservation prices in $N\left(\leq N_{c}\right)$, we find that the total surplus must be lower with one less bidder. With $N$ bidders, seller revenue equals total surplus while seller revenue is weakly lower than the total surplus with $N-1$ bidders. It is thus clear that the seller revenue is higher with $N$ bidders.

## Value of a bidder

It is useful to offer a more intuitive interpretation of Proposition 5 in terms of the "value of an additional bidder". For any market size $N \geq N_{c}+1$, both at $N$ and $N-1$ the information acquisition equilibrium is necessarily random for the revenuemaximizing selling mechanism. Therefore, the "market tightness principle" of Levin and Smith (1994) is directly applicable and the seller gains from dropping a bidder.

When $N \leq N_{c}$, there is an $r \geq v_{0}$ such that $u_{N}(r)=c$. If $r>r^{m}$, then $u_{N}\left(r^{m}\right)>c$. The seller's revenue would increase if the number of bidders increases from $N-1$ to $N$ with $N \leq N_{c}$. Assume $r \leq r^{m}$. If the seller induces an equilibrium in which all $N-1$ buyers participate with probability 1 , the highest possible payoff of the seller is the generated total expected social surplus $W_{N-1}\left(r^{\prime}\right)-(N-1) c$, where $u_{N-1}\left(r^{\prime}\right) \geq c$ and $r^{\prime} \in\left(r, r^{m}\right]{ }^{19}$ By adding a bidder, the seller can get all the $N$ buyers to participate with probability 1 by setting $\mathbf{r}_{N}=(r, \ldots, r)$. This drives the entire rent of a buyer to zero and gets the entire social surplus $W_{N}(r)-N c$ as the seller's revenue. Therefore, the change in revenue is higher than

$$
\begin{aligned}
& {\left[W_{N}(r)-N c\right]-\left[W_{n}\left(r^{\prime}\right)-(N-1) c\right] } \\
= & W_{N}(r)-W_{N-1}\left(r^{\prime}\right)-c \\
= & W_{N}(r)-W_{N-1}\left(r^{\prime}\right)-u_{N}(r) \\
> & W_{N-1}(r)-W_{N-1}\left(r^{\prime}\right), \quad \text { (using Lemma 2) }
\end{aligned}
$$

which is positive as $v_{0}<r<r^{\prime}$. The above result shows that the "additional competition is valuable" principle of Bulow and Klemperer (1996) remains valid in our setting until $N=N_{c}$.

However, as Proposition 5(i) shows, the "additional competition is valuable" principle fails when the number of bidders goes beyond $N_{c}+1$. When $N \geq N_{c}+1$, information acquisition is partial, and just as in Levin and Smith (1994), social sur-

[^7]plus (and from Proposition 2) decreases with $N$.
From the foregoing arguments, the maximal revenue is first increasing in $N$ until $N_{c}$ and then is decreasing beyond $N_{c}+1$. Observe that Assumption 2 implies $c \leq$ $u_{1}\left(v_{0}\right)$. Moreover, since $u_{K}\left(v_{0}\right)$ monotonically converges to zero when $K$ approaches infinity, there exists a unique $K \geq 1$ such that $c \in\left(u_{K+1}\left(v_{0}\right), u_{K}\left(v_{0}\right)\right]$. For all such $c$, by definition, $N_{c}=K$. We are therefore left with determining whether $\left(K, r_{K}^{*}\right)$ or $\left(K+1, v_{0}\right)$ is the optimal selling procedure.

Proposition 6 Let $c \in\left(u_{K+1}\left(v_{0}\right), u_{K}\left(v_{0}\right)\right.$ ] for some $K \geq 1$. Under Assumptions 1 3. there exists a unique $\hat{c}_{K} \in\left(u_{K+1}\left(v_{0}\right), u_{K}\left(v_{0}\right)\right]$ such that:

1. If $c \in\left(\hat{c}_{K}, u_{K}\left(v_{0}\right)\right]$, then the optimal selling procedure is $\left(K, r_{K}^{*}\right)$.
2. If $c \in\left(u_{K+1}\left(v_{0}\right), \hat{c}_{K}\right)$, then optimal selling procedure is $\left(K+1, v_{0}\right)$.
3. Either of the above two selling procedures is optimal if (i) $K \geq 2$ and $c=\hat{c}_{K}$ or (ii) $K=1$ and $c \in\left\{\hat{c}_{1}, u_{0}\left(v_{0}\right)\right\}$.

The proof must distinguish between the case where $K=1$ and $K \geq 2$. A sketch of the proof when $K \geq 2$ is as follows ${ }^{20}$ The comparison is between revenue from a shortlist of $K+1$ bidders in an ex-post efficient auction versus a shortlist of $K$ bidders but a constant reserve $r_{K}^{*}$. That is, we compare the standard selling procedures $\hat{\mu}=\left(K+1, v_{0}\right)$ with $\mu^{c}=\left(K, r_{K}^{*}\right){ }^{21}$ Since $K+1 \geq N_{c}+1$, Proposition 2 applies and expected revenue from $\hat{\mu}$ equals the social surplus $S(c) \equiv S_{K+1}\left(p_{e}(\widehat{\mu} ; c), \mathbf{v}_{0}, c\right)$.

Under $\mu^{c}$, since $K \leq N_{c}$, there is full value discovery at equilibrium. If $c \leq u\left(r^{m}\right)$, then $r_{k}^{*}=r^{m}$ and the seller's revenue is $R(c)=R_{K}\left(r^{m}\right)$, the revenue from a standard optimal auction in an IPV setting. When $c \in\left[u_{K}\left(r^{m}\right), u_{K}\left(v_{0}\right)\right], r_{K}^{*}=r_{K}^{c}$, which leaves an eligible bidder with a zero ex-ante payoff. Hence for these values of $c$, the entire social surplus again goes to the seller, and hence the expected revenue is $R(c)=W_{K}\left(r_{K}^{c}\right)-K c$. The proof now proceeds to show $S(\cdot)$ is a decreasing convex function whereas $R(\cdot)$ is a decreasing concave function. Finally, we show that $R(\cdot)$ intersects $S(\cdot)$ from below at some $\hat{c}_{K} \in\left(u_{K+1}\left(v_{0}\right), u_{K}\left(v_{0}\right)\right)$ and such a $\hat{c}_{K}$ must be unique since $R(c)-S(c)$ is concave.

Showing that $R(c)$ and $S(c)$ intersect only once is a key element of the above proof. The single crossing property between $R(c)$ and $S(c)$ on $\left[u_{K+1}\left(v_{0}\right), u_{K}\left(v_{0}\right)\right]$ for $K \geq 2$ can be explained simply. When $c=u_{K+1}\left(v_{0}\right)$, we can show $R\left(u_{K+1}\left(v_{0}\right)\right)<$

[^8]$S\left(u_{K+1}\left(v_{0}\right)\right)$ due to the following arguments. By $\mathrm{Lu}(2008)$, when $c=u_{K+1}\left(v_{0}\right)$, $S\left(u_{K+1}\left(v_{0}\right)\right)$ is the maximum total social surplus, if there are $K+1$ eligible bidders and they are allowed to acquire information with different probabilities. Since each bidder's payoff is zero (i.e. $u_{K+1}\left(v_{0}\right)=c$ ), his contribution to the social surplus is zero in the VCG mechanism $\hat{\mu}=\left(K+1, \mathbf{v}_{0}\right)$. Note that $\hat{\mu}$ induces every bidder to acquire information with probability 1 . The above facts mean that $S\left(u_{K+1}\left(v_{0}\right)\right)=W_{K}\left(v_{0}\right)-K c$, i.e. eliminating one bidder would not change the total social surplus if the remaining $K$ bidders still acquire information with probability 1 . Since $\min \left\{r_{c}, r^{m}\right\}>v_{0}$, we further have $W_{K}\left(v_{0}\right)-K c>W_{K}\left(\min \left\{r_{c}, r^{m}\right\}\right)-K c$. The latter is the social surplus in a second price auction with $K$ informed bidders and reservation price $\min \left\{r_{c}, r^{m}\right\}$, which is clearly higher than the seller revenue $R\left(u_{K+1}\left(v_{0}\right)\right)$ in the same auction.

When $c=u_{K}\left(v_{0}\right)$, we have $N_{c}=K$. By Lu (2008), when there are $K+1$ eligible bidders, having exactly $K$ of them discovering their values for sure and participating in the auction would generate the highest social welfare $W_{K}\left(v_{0}\right)-K c$, which equals the seller revenue $R\left(u_{K}\left(v_{0}\right)\right)$ since bidder payoffs are zero. Recall $S\left(u_{K+1}\left(v_{0}\right)\right)$ is the total surplus generated with $K+1$ bidders discovering their values with an interior probability. Therefore, $S\left(u_{K+1}\left(v_{0}\right)\right)$ is strictly lower than the highest social welfare $W_{K}\left(v_{0}\right)-K c$ with $K \geq 2$. This leads to $S\left(u_{K+1}\left(v_{0}\right)\right)<R\left(u_{K}\left(v_{0}\right)\right)$. If one accepts the concavity and convexity of $R(c)$ and $S(c)$ respectively, the above comparisons show that the two revenue curves must single cross at an interior point in $\left[u_{K+1}\left(v_{0}\right), u_{K}\left(v_{0}\right)\right]$.

By Proposition 6, if $c \in\left(u_{K+1}\left(v_{0}\right), \hat{c}_{K}\right)$, the seller shortlists $K+1$ buyers and runs a second price auction that is ex-post efficient (i.e., the reserve is set at $v_{0}$ ). Each of these $K+1$ buyers invests in value discovery with equilibrium value discovery probability $p_{e}$. If $c \in\left(\hat{c}_{K}, u_{K}\left(v_{0}\right)\right)$, the seller shortlists $K$ buyers and runs a second price auction that is ex-post inefficient (i.e., the reserve is higher than $v_{0}$ ). Each of the $K$ buyers invests in value discovery with equilibrium probability 1.

There are two noteworthy features of the optimal selling procedure identified in Proposition 6. First, there is a trade-off between efficiency and value discovery.

## Corollary 1 (Optimal shortlisting \& efficient value discovery)

1. When $c \in\left(\hat{c}_{K}, u_{K}\left(v_{0}\right)\right)$, the equilibrium value discovery is socially efficient. However, the consequent allocation of the object among the informed buyers is ex-post inefficient.
2. When $c \in\left(u_{K+1}\left(v_{0}\right), \hat{c}_{K}\right)$, the equilibrium value discovery is socially inefficient. However, the consequent allocation of the object among the informed buyers is necessarily ex-post efficient.

Second, as illustrated in Figure 1, the reserve in the optimal selling procedure is a


Figure 1: Optimal Reserve for $c \in\left(u_{K+1}\left(v_{0}\right), u_{K}\left(v_{0}\right)\right]$, where $K \in\{1, \ldots, 5\}, v_{0}=0.5$ and uniform prior on $[0,1]$.
discontinuous function of the value discovery cost.
Corollary 2 (Comparative statics of the optimal reserve) As c varies in each interval $\left(u_{K+1}\left(v_{0}\right), u_{K}\left(v_{0}\right)\right], K \geq 1$, the optimal reserve price is first the constant $v_{0}$ to the right of $u_{K+1}\left(v_{0}\right)$, then jumps up (discontinuously) and then decreases continuously to $v_{0}$.

### 3.3 Discussion

### 3.3.1 Covert vs. observable information acquisition

In this section, we compare our results with the case where the value discovery investment is directly observable by the seller. In that case, the seller can create a shortlist of uninformed buyers and charge them a shortlisting fee in stage 1 . Using the fee, the seller can of course extract the entire ex-ante surplus. Thus, we have the following:

Proposition 7 Suppose the seller can observe the value discovery investment. The revenue-maximizing strategy for the seller is to shortlist $N_{c}$ uninformed buyers and charge each of them an ex-ante fee of $f=u_{N_{c}}\left(v_{0}\right)-c$. She commits to running an efficient auction among the shortlisted. Equilibrium value discovery is socially efficient and the seller gets the entire social surplus $W_{N_{c}}\left(v_{0}\right)-N_{c} c$.

The proof is clear - given the seller's strategy: $u_{N}\left(v_{0}\right)-c$ is the payoff of each of the eligible $N_{c}$ buyers from the procedure $\left(N_{c}, v_{0}\right)$. Therefore, upon agreeing to become an eligible buyer by paying the fee, the payoff is zero, which equals the payoff from refusing to be on the shortlist. The seller obviously cannot do any better since she receives the maximum social surplus as revenue.

When value discovery is covert, the above mechanism cannot work. In some cases, however, buyers can supply hard evidence of their investment. Even in this case, the seller cannot fully separate the informed from the uninformed. Upon being asked to pay the fee, a buyer may first engage in value discovery and pays the fee $f$ only if she knows that her value is sufficiently high. Therefore, given an $f$, the seller can only recover the fee from those types above a value threshold. A similar effect is in play if one were to consider charging examination fees upon value discovery at stage 2 .

The role of subsidies is in sharp contrast to fees. When the investment decision is directly observable, or an eligible buyer can supply hard evidence of his covert actions, the seller can pay only informed buyers. Subsidies in this case become a way of relaxing the value investment cost. Every eligible buyer has every incentive to supply the evidence and collect the subsidy $s \geq 0$. From a buyer's point of view, this has the same impact as reducing the value discovery cost from $c$ to $c-s$.

A selling procedure is now a triple: $\mu=\left(N, \mathbf{r}_{N}, s\right)$ where $s \geq 0$. At $\mu=\left(N, \mathbf{r}_{N}, s\right)$, if the buyers discover their values with a probability $p$, the payoff of the seller is $R_{N}(p, \mathbf{r})-N p s$ while a buyer will engage in value discovery only if $U_{N}\left(p, \mathbf{r}_{N}, c\right) \geq-s$. One might speculate that the relaxed incentives to engage in value discovery may enable the seller to set higher reserves at the optimum. However, we have the following result.

Proposition 8 In any revenue-maximizing selling procedure $\hat{\mu}=\left(\hat{N}, \hat{\mathbf{r}}_{N}, \hat{s}\right)$, the seller sets a zero subsidy on value discovery, i.e., $\hat{s}=0$. The choice of $\hat{N}$ and $\hat{\mathbf{r}}_{N}$ is thus exactly as in Proposition 6.

The reason why such subsidies cannot help is fairly intuitive. Given $N$, both subsidies and reserves may be used to influence participation. Setting a non-trivial reserve is distortive, it results in an ex-post inefficient allocation. To the extent that participation can be guaranteed with a lower reserve and a correspondingly lower subsidy, the seller gains from the increased efficiency.

### 3.3.2 On the role of Assumption 1

Assumption 1 is motivated by the fact that in many situations, a buyer needs to discover whether there are gains from trade. Larsen (2021) presents related empirical
evidence supporting Assumption 1. Assumption 1 also violates the "serious bidder" assumption in Bulow and Klemperer (1996). Nonetheless, we show that an additional informed bidder is valuable, in apparent contradiction to the counter-example they provide ${ }^{22}$ Note however, we are comparing an $N$-bidder optimal auction with an ( $N+1$ )-bidder optimal auction (subject to an entry constraint). Their method of proof involves a comparison of an optimal $N$-bidder auction with an efficient $(N+1)$ bidder auction.

## 4 Concluding remarks

Our paper represents a first step in analyzing auction design with covert information acquisition and shortlisting. In an environment where the creation of any gain from trade is only possible if buyers engage in costly information acquisition and these investments are covert, we have characterized the revenue-maximizing selling procedure. We obtained several interesting insights into the joint optimization of expected revenue with respect to shortlisting and reserve prices: a) A close connection is discovered between the social desirability of investment in value discovery of an additional bidder and the value of a bidder to the seller. b) The seller may prefer to induce random value discovery at the optimum. c) The optimal reserve as a function of the value discovery cost is non-monotonic and discontinuous.

There were two important assumptions in our setup: first, that trade is inefficient without value discovery; and second, that the value distributions satisfy the decreasing hazard rate condition. Dropping either of them will affect our results. If trade is efficient even without value discovery, a reserve equal to the seller's value would induce the uninformed bidder to make a bid. The monotone hazard rate condition was critical for deducing that a uniform reserve is optimal with fewer than the efficient number of bidders. Relaxing these assumptions will be a subject of future research.

[^9]
## 5 Appendix

### 5.1 Proof of Proposition 1

Proof. The detailed formal arguments for the above claim are as follows. For selling procedure $\left(N, \mathbf{r}_{N}\right)$ where $\min \left\{r_{k}\right\}<\eta$, all $N$ eligible bidders participate in the auction no matter they are informed or not. Therefore, an auction $\mathcal{S P} \mathcal{A}_{N}\left(r_{N}\right)$ prevails. It is a dominant strategy for each of the uninformed buyers to bid $\eta$.

We proceed by considering two cases, separately. In Case I, $N \leq N_{c}$; in Case II, $N \geq N_{c}+1$.

Case I: We first consider the case of $N \leq N_{c}$, i.e., $c \leq u_{N}\left(v_{0}\right) \leq u_{N}(\eta)$ using (5) and Assumption 1.

If $r_{N} \geq \eta$, then the selling procedure ( $N, \mathbf{r}_{N}$ ) is equivalent to the selling procedure $\left(N, \tilde{\mathbf{r}}_{N}\right)$ where $\tilde{r}_{k}=r_{N}, \forall k$ : an uninformed buyer never wins, each informed bidder's winning chance and payments are exactly the same across the two procedures no matter how many uninformed bidders submit bids. Therefore, the two selling procedures would induce the same value discovery decisions of bidders. The seller's revenue is also the same no matter how many uninformed bidders submit bids. As a result, the two procedures must be revenue equivalent.

Suppose $r_{N}<\eta$. Let $Z_{k}$ denote the second highest order statistic from the $k$ random variables $V_{1}, \ldots, V_{k}$. We are going to establish that the revenue from ( $N, \mathbf{r}_{N}$ ) is lower than that from the selling procedure $\left(N, \tilde{\mathbf{r}}_{N}\right)$ where $\tilde{r}_{k}=\eta, \forall k$.

When all $N$ buyers are uninformed, i.e., $k=0$, the seller's payoff equals $\eta$, which is less than $R_{N}(\eta)$.

In the event that there are $0<k \leq N-2$ informed buyers and at least two uninformed buyers, the seller's payoff in this auction is

$$
G_{k}(\eta) \eta+\left(1-G_{k}(\eta)\right) E\left[\max \left\{\eta, Z_{k}\right\} \mid Y_{k} \geq \eta\right]=R_{k}(\eta)+\left(\eta-v_{0}\right) G_{k}(\eta)
$$

In the event that there are $k=N-1$ informed buyers and one uninformed buyer, the seller's payoff in this auction is

$$
\begin{aligned}
& G_{k}(\eta) E\left[\max \left\{Y_{k}, r_{N}\right\} \mid Y_{k}<\eta\right]+\left(1-G_{k}(\eta)\right) E\left[\max \left\{\eta, Z_{k}\right\} \mid Y_{k} \geq \eta\right] \\
= & R_{k}(\eta)+\left(E\left[\max \left\{Y_{k}, r_{N}\right\} \mid Y_{k}<\eta\right]-v_{0}\right) G_{k}(\eta) \\
\leq & R_{k}(\eta)+\left(\max \left\{\eta, r_{N}\right\}-v_{0}\right) G_{k}(\eta) .
\end{aligned}
$$

From Assumption 1 and the above results, it follows that whenever there is at least
one uninformed buyer, the seller's revenue is no more than $R_{k}(\eta)$, which is smaller than $R_{N}(\eta)$ using $r_{N}<\eta \leq v_{0}$.

When all $N$ buyers are informed, the seller's revenue is exactly $R_{N}\left(r_{N}\right)$. Next, we show that $R_{N}\left(r_{N}\right) \leq R_{N}(\eta)$ when $r_{N}<\eta \leq v_{0}$. In the event that all the $N$ buyers are informed, when the reserve price is $r$, the expected payment of a bidder with value $V$ is given by

$$
m(V)=r G_{N-1}(r)+\int_{r}^{V} y g_{N-1}(y) d y
$$

The seller's expected payoff equals

$$
\begin{aligned}
R_{N}(r) & =N \int_{r}^{\bar{v}} m(V) f(V) d V+F^{N}(r) v_{0} \\
& =N r G_{N-1}(r)(1-F(r))+N \int_{r}^{\bar{v}} y g_{N-1}(y)(1-F(y)) d y+F^{N}(r) v_{0}
\end{aligned}
$$

Differentiating this with respect to $r$, we have

$$
\frac{d}{d r} R_{N}(r)=N G_{N-1}(r)(1-F(r))\left[1-\left(r-v_{0}\right) \frac{f(r)}{1-F(r)}\right]
$$

Note that $\frac{d}{d r} R_{N}(r)>0$, when $r<v_{0}$. This implies that an increase in $r$ always leads to a higher revenue whenever $r<v_{0}$. As a result, $R_{N}\left(r_{N}\right) \leq R_{N}(\eta)$, when $r_{N}<\eta \leq v_{0}$.

We are now ready to establish that the revenue from $\left(N, \mathbf{r}_{N}\right)$ is lower than that from selling procedure $\left(N, \tilde{\mathbf{r}}_{N}\right)$ where $\tilde{r}_{k}=\eta, \forall k$. Since $c \leq u_{N}(\eta), c \leq u_{k}(\eta), \forall k \leq N$. In other words, with $\left(N, \tilde{\mathbf{r}}_{N}\right)$, an informed bidder always receives a payoff which is higher than his value discovery cost no matter how many informed bidders are participating. It is thus a dominant strategy for each bidder to conduct value discovery. It entails that a second price auction with $N$ informed buyers and a reserve of $\eta$ would prevail. The seller's revenue is thus $R_{N}(\eta)$.

Case II: We now turn to the case of $N \geq N_{c}+1$. Following the same procedure of Levin and Smith (1994), one can show that a selling procedure ( $N, \tilde{\mathbf{r}}_{N}$ ) with $\tilde{r}_{k}=v_{0}$ $(>\eta), \forall k$ is efficient and revenue-maximizing among all mechanisms that implement symmetric value discovery across eligible bidders. The details will be provided when we formally present the revenue-maximizing selling procedure for $N \geq N_{c}+1$ in the proof of Proposition 2. We now have fully established Proposition 1.

1

### 5.2 Proof of Lemma 1

2 Proof. We note that

$$
\begin{aligned}
& \sum_{k=0}^{N} \beta(k ; p, N) k\left[u_{k}\left(r_{k}\right)-c\right] \\
= & \sum_{k=1}^{N} \frac{N!}{k!(N-k)!} p^{k}(1-p)^{N-k}\left[u_{k}\left(r_{k}\right)-c\right] \\
= & N p \sum_{k=1}^{N} \frac{(N-1)!}{(k-1)!(N-k)!} p^{k-1}(1-p)^{N-k}\left[u_{k}\left(r_{k}\right)-c\right] \\
= & N p U_{N}\left(p, \mathbf{r}_{N}, c\right) .
\end{aligned}
$$

Since $W_{k}\left(r_{k}\right)-k c=R_{k}\left(r_{k}\right)+k\left[u_{k}\left(r_{k}\right)-c\right]$, it readily follows that $S_{N}\left(p, \mathbf{r}_{N}, c\right)=$ $R_{N}\left(p, \mathbf{r}_{N}\right)+N p U_{N}(p, \mathbf{r}, \mathbf{c})$, namely (6), which is first shown by Moreno and Wooders (2011). (7) is shown in Levin and Smith (1994) for the case of $v_{0}=0$. (7) with a general $v_{0}$ can be verified directly using (4). The details are available from the authors.

### 5.3 Proof of Proposition 2

Proof. Given any exogenous symmetric information acquisition $p \in[0,1]$, the maximum total surplus achievable is $S_{N}\left(p, \mathbf{v}_{0}, c\right)$ where $\mathbf{v}_{0}=\left(v_{0}, \ldots, v_{0}\right)$. Let $p^{*}$ denote the maximum of $S_{N}\left(p, \mathbf{v}_{0}, c\right)$ with respect to $p$. From Lemma 1

$$
\partial S_{N}\left(p, \mathbf{v}_{0}, c\right) / \partial p=N\left(U_{N}\left(p, \mathbf{v}_{0}, c\right)\right)
$$

Since $U_{N}\left(1, \mathbf{v}_{0}, c\right)=u_{N}\left(v_{0}\right)-c<0$ and $U_{N}\left(0, \mathbf{v}_{0}, c\right)=u_{1}\left(v_{0}\right)-c>0$, it follows that $0<p^{*}<1$ and that $U_{N}\left(p^{*}, \mathbf{v}_{0}, c\right)=0$. Thus for $\hat{\mu}_{N}=\left(N, \mathbf{v}_{0}\right)$, we have that the seller's revenue is $R_{N}\left(p^{*}, \mathbf{v}_{0}\right)=S_{N}\left(p^{*}, \mathbf{v}_{0}, c\right)$ using (6). At any equilibrium $p_{e}$ of $\mu_{N}=\left(N, \mathbf{r}_{N}\right)$, we have

$$
R_{N}\left(p_{e}, \mathbf{r}_{N}\right) \leq S_{N}\left(p_{e}, \mathbf{v}_{0}, c\right) \leq S_{N}\left(p^{*}, \mathbf{v}_{0}, c\right)=R_{N}\left(p^{*}, \mathbf{v}_{0}\right) .
$$

The first inequality arises due to the fact that at equilibrium $p_{e}$, bidders' expected payoffs might be non-negative.

### 5.4 Proof of Proposition 3

Proof. Consider any given $\mu_{N}=\left(N, \mathbf{r}_{N}\right)$ where $r_{k} \in[\eta, \bar{v}], \forall k$, which induces information acquisition equilibrium $p_{e}\left(\mu_{N} ; c\right) \in[0,1]$. If $p_{e}\left(\mu_{N} ; c\right)=0$, it continues to be an
equilibrium of standard procedure $(N, \bar{v})$ and the revenue is the same. If $p_{e}\left(\mu_{N} ; c\right)=1$, it continues to be an equilibrium of standard selling procedure $\left(N, r_{N}\right)$ and the revenue is the same.

We now consider the remaining case with $p_{e}\left(\mu_{N} ; c\right) \in(0,1)$. Consider the program $\mathbb{P}$ where,

$$
\begin{aligned}
\mathbb{P}:=\quad & \max _{\tilde{\mathbf{r}}_{N} \in[\eta, \bar{v}]^{N}} S_{N}\left(p_{e}\left(\mu_{N} ; c\right), \tilde{\mathbf{r}}_{N}, c\right) \\
& \text { s.t. }: U_{N}\left(p_{e}\left(\mu_{N} ; c\right), \tilde{\mathbf{r}}_{N}, \mathbf{c}\right)=0 .
\end{aligned}
$$

This program solves the revenue-maximizing reserves which implement the given equilibrium $p_{e}\left(\mu_{N} ; c\right) \in(0,1)$. Let us then analyze $\mathbb{P}$. Clearly, it must have a solution $\tilde{\mathbf{r}}_{N}^{*}$, since all the involved functions are continuous and the domain is nonempty and compact.

Since $\tilde{\mathbf{r}}_{N}^{*}$ induces an interior value-discovery equilibrium and $N \leq N_{c}$, it must not be the case that $\tilde{r}_{k}^{*} \leq v_{0}, \forall k$. Otherwise, we must have $p_{e}\left(\mu_{N} ; c\right)=1$ as $u_{k}\left(\tilde{r}_{k}^{*}\right) \geq$ $u_{k}\left(v_{0}\right) \geq c, \forall k \leq N \leq N_{c}$.

For problem $\mathbb{P}$, set up the Lagrangian:

$$
L\left(\tilde{\mathbf{r}}_{N}, \lambda\right)=S_{N}\left(p_{e}\left(\mu_{N} ; c\right), \tilde{\mathbf{r}}_{N}, c\right)+\lambda\left(-U_{N}\left(p_{e}\left(\mu_{N} ; c\right), \tilde{\mathbf{r}}_{N}, c\right)\right)
$$

One can verify that $u_{k}^{\prime}(r)=-[1-F(r)] G_{k-1}(r)$ and

$$
\begin{equation*}
W_{k}^{\prime}(r)=\left(v_{0}-r\right) G_{k}^{\prime}(r)=k\left(v_{0}-r\right) G_{k-1}(r) f(r) \tag{11}
\end{equation*}
$$

using $\beta(k ; p, N)=\frac{N p}{k} \beta(k-1 ; p, N-1)$.
Note that $\varphi_{k}(\bar{v})<0$, it cannot be the case that $\tilde{r}_{k}^{*}=\bar{v}$ for any $k$. Based on the above results, there exists a $k_{0} \leq N$ such that $\tilde{r}_{k_{0}}^{*} \in\left(v_{0}, \bar{v}\right)$. For this $\tilde{r}_{k_{0}}^{*}$, the first order
condition for an interior optimum requires that $\varphi_{k_{0}}\left(\tilde{r}_{k_{0}}^{*}\right)=0$, i.e.

$$
N p_{e}\left(\mu_{N} ; c\right)\left(v_{0}-\tilde{r}_{k_{0}}^{*}\right) f\left(\tilde{r}_{k_{0}}^{*}\right)+\lambda\left[1-F\left(\tilde{r}_{k_{0}}^{*}\right)\right]=0 .
$$

Therefore $\lambda=\frac{N p_{e}\left(\mu_{N} ; c\right)\left(\tilde{r}_{0}^{*}-v_{0}\right) f\left(\tilde{r}_{k_{0}}^{*}\right)}{1-F\left(\tilde{r}_{k_{0}}^{*}\right)}>0$.
It remains to prove that $\tilde{r}_{k}^{*}=\tilde{r}_{k_{0}}^{*}, \forall k$. Using the identified value for $\lambda$, for any $\tilde{r}_{k} \in[\eta, \bar{v}), k \neq k_{0}$, we may rewrite $\varphi_{k}\left(\tilde{r}_{k}\right)$ as

$$
\begin{aligned}
\varphi_{k}\left(\tilde{r}_{k}\right)= & p_{e}\left(\mu_{N} ; c\right) N \beta\left(k-1 ; p_{e}\left(\mu_{N} ; c\right), N-1\right) G_{k-1}\left(\tilde{r}_{k}\right)\left[1-F\left(\tilde{r}_{k}\right)\right] \\
& \times\left(\frac{f\left(\tilde{r}_{0}^{*}\right)}{1-F\left(\tilde{r}_{k_{0}}^{*}\right)}\left(\tilde{r}_{k_{0}}^{*}-v_{0}\right)-\frac{f\left(\tilde{r}_{k}\right)}{1-F\left(\tilde{r}_{k}\right)}\left(\tilde{r}_{k}-v_{0}\right)\right) .
\end{aligned}
$$

The term in the large brackets is positive for $\tilde{r}_{k}<\tilde{r}_{k_{0}}^{*}$ and negative for $\tilde{r}_{k}>\tilde{r}_{k_{0}}^{*}$ due to the monotone hazard rate, which means that at optimum $\tilde{r}_{k}^{*}=\tilde{r}_{k_{0}}^{*}, \forall k$. We thus established the claim of Proposition 3.

### 5.5 Proof of Lemma 2

Proof. Recall that $u_{k}(r)=E\left[\max \left\{Y_{k}, r\right\}\right]-E\left[\max \left\{Y_{k-1}, r\right\}\right]$ and $W_{k}(r)=E\left[\max \left\{Y_{k}, r\right\}\right]-$ $\left(r-v_{0}\right) G_{k}(r)$. By direct calculation, $W_{k}(r)-W_{k-1}(r)=E\left[\max \left\{Y_{k}, r\right\}\right]-E\left[\max \left\{Y_{k-1}, r\right\}\right]+$ $\left(r-v_{0}\right)\left(G_{k-1}(r)-G_{k}(r)\right)>u_{k}(r)$ as $r>v_{0}$ and $G_{k-1}(r)>G_{k}(r)$.

### 5.6 Proof of Proposition 4

Proof. Under Assumption 3, virtual value function $J(\cdot)$ is increasing. Reserve $r^{m}$ is thus the Myerson optimal reserve in a standard IPV setting with $N$ buyers.

If $u_{N}\left(r^{m}\right) \geq c$, then at $r^{m}$, every bidder acquires information, and the maximum revenue is achieved by the optimal Myerson mechanism.

If $u_{N}\left(r^{m}\right)<c$, then $r_{N}^{c}<r^{m}$. It follows that $J(\cdot)<0$ for $v \leq r_{N}^{c}$. Therefore, any reserve $r \in\left[\eta, r_{N}^{c}\right)$ is dominated by $r_{N}^{c}$. A reserve $r\left(>r_{N}^{c}\right)$ must induce partial value discovery equilibrium $p_{e}<1$ as $u_{N}(r)<c$. Note that at equilibrium $p_{e}$ and reserve $r$, the revenue is bounded by the total surplus $\sum_{k=0}^{N} \beta_{k}\left(p_{e}, N\right)\left[W_{k}(r)-k c\right]$, which is smaller than $\sum_{k=0}^{N} \beta_{k}\left(p_{e}, N\right)\left[W_{k}\left(r_{N}^{c}\right)-k c\right]$ as $W_{k}(\cdot)$ decreases to the right of $v_{0}$. Note that $\forall k \leq N$, we have

$$
\begin{aligned}
& {\left[W_{k}\left(r_{N}^{c}\right)-k c\right]-\left[W_{k-1}\left(r_{N}^{c}\right)-(k-1) c\right] } \\
= & {\left[W_{k}\left(r_{N}^{c}\right)-W_{k-1}\left(r_{N}^{c}\right)-u_{k}\left(r_{N}^{c}\right)\right]+\left[u_{k}\left(r_{N}^{c}\right)-c\right]>0, }
\end{aligned}
$$

1 according to Lemma 2 and $u_{k}\left(r_{N}^{c}\right)>u_{N}\left(r_{N}^{c}\right)=c$. Therefore,

$$
\begin{aligned}
& \sum_{k=0}^{N} \beta_{k}\left(p_{e}, N\right)\left[W_{k}\left(r_{N}^{c}\right)-k c\right] \\
< & W_{N}\left(r_{N}^{c}\right)-N c=W_{N}\left(r_{N}^{c}\right)-N u_{N}\left(r_{N}^{c}\right),
\end{aligned}
$$

2 which is the seller's expected revenue when reserve is uniformly $r_{N}^{c}$.

### 5.7 Proof of Proposition 5

Proof. Part (i) is established by Propositions 1 and 9 in Levin and Smith (1994).
To show part (ii), it is sufficient to show that $R_{N-1}\left(r_{N-1}^{*}\right)<R_{N}\left(r_{N}^{*}\right), \forall N \leq N_{c}$. Note that $r_{N-1}^{*} \geq r_{N}^{*}$ as $r_{N-1}^{c}>r_{N}^{c}$. Recall that $r_{N}^{*}=\min \left\{r_{N}^{c}, r^{m}\right\}$ by Proposition 4 .

If $r_{N}^{*}=r^{m}$, then $r_{N-1}^{*}=r^{m}$. In this case, it must be true that $R_{N-1}\left(r_{N-1}^{*}\right)<$ $R_{N}\left(r_{N}^{*}\right)$.

If $r_{N}^{*}=r_{N}^{c}<r^{m}$, then $r_{N-1}^{*}>r_{N}^{c}$ and thus $u_{N}\left(r_{N-1}^{*}\right)<u_{N}\left(r_{N}^{c}\right)=c$. Note that

$$
\begin{aligned}
R_{N-1}\left(r_{N-1}^{*}\right) & \leq \sum_{k=0}^{N-1} \beta_{k}\left(p_{e}^{N-1}, N-1\right)\left[W_{k}\left(r_{N-1}^{*}\right)-k c\right] \\
& \leq \sum_{k=0}^{N-1} \beta_{k}\left(p_{e}^{N-1}, N-1\right)\left[W_{k}\left(r_{N}^{c}\right)-k c\right]
\end{aligned}
$$

The first inequality holds since the revenue is bounded by the total surplus, and the second inequality holds since $r_{N-1}^{*}>r_{N}^{c}>v_{0}$.

Moreover, $\forall k \leq N$, we have

$$
\begin{aligned}
& {\left[W_{k}\left(r_{N}^{c}\right)-k c\right]-\left[W_{k-1}\left(r_{N}^{c}\right)-(k-1) c\right] } \\
= & {\left[W_{k}\left(r_{N}^{c}\right)-W_{k-1}\left(r_{N}^{c}\right)-u_{k}\left(r_{N}^{c}\right)\right]+\left[u_{k}\left(r_{N}^{c}\right)-c\right]>0, }
\end{aligned}
$$

according to Lemma 2 and $u_{k}\left(r_{N}^{c}\right)>u_{N}\left(r_{N}^{c}\right)=c$. It implies that

$$
\begin{aligned}
R_{N-1}\left(r_{N-1}^{*}\right) & \leq \sum_{k=0}^{N-1} \beta_{k}\left(p_{e}, N-1\right)\left[W_{k}\left(r_{N}^{c}\right)-k c\right] \\
& <W_{N-1}\left(r_{N}^{c}\right)-(N-1) c \\
& <W_{N}\left(r_{N}^{c}\right)-N c \\
& =W_{N}\left(r_{N}^{c}\right)-N u_{N}\left(r_{N}^{c}\right) \\
& =R_{N}\left(r_{N}^{c}\right) \\
& =R_{N}\left(r_{N}^{*}\right) .
\end{aligned}
$$

The last equality holds since $r_{N}^{*}=r_{N}^{c}$ and the second last equality holds since all buyers participate with probability 1 and $u_{N}\left(r_{N}^{c}\right)=0$.

### 5.8 Proof of Proposition 6

Proof. The cases where $K=1$ and $K \geq 2$ are considered separately. We begin with the case $K \geq 2$.

Case $K \geq 2$. The proof consists of three steps. It uses Lemma 1. Consider the standard selling procedures $\hat{\mu}=\left(K+1, v_{0}\right)$ and $\mu^{c}=\left(K, r_{k}^{*}\right)$. The revenue under procedure $\hat{\mu}=\left(K+1, \mathbf{v}_{0}\right)$ is $S(c) \equiv S_{K+1}\left(p_{e}(\hat{\mu} ; c), \mathbf{v}_{0}, c\right)$, where equilibrium value discovery $p_{e}(\hat{\mu} ; c)$ is in $(0,1)$. The first step in the remainder of the proof consists of showing that $S(\cdot)$ is a decreasing convex function. Next, let $R(c)$ denote the seller's revenue under $\mu^{c}=\left(K, r_{K}^{*}\right)$. Since $(1-F(v)) / f(v)$ is assumed to be decreasing, under $\mu^{c}$, we have $r_{K}^{*}=\min \left\{r_{K}^{c}, r^{m}\right\}$. Since $r^{m}$ is the unconstrained Myerson reserve for an optimal auction, $R(c)=R_{K}\left(r^{m}\right)$ for all $c \leq u_{K}\left(r^{m}\right)$. When $c \in\left[u_{K}\left(r^{m}\right), u_{K}\left(v_{0}\right)\right]$, $r_{K}^{*}=r_{K}^{c}$ and therefore $R(c)=W_{K}\left(r_{K}^{c}\right)-K c$. The second step in the proof involves showing that $R(\cdot)$ is also decreasing in this region, just as $S(\cdot)$, but that it is concave. The final step involves showing that $R(\cdot)$ intersects $S(\cdot)$ from below at some $\hat{c}_{K} \in$ $\left(u_{K+1}\left(v_{0}\right), u_{K}\left(v_{0}\right)\right)$ and moreover such a $\hat{c}_{K}$ must be unique since $R(c)-S(c)$ is concave.
Step 1. Note that the expectation of the number of participants is

$$
\sum_{k=0}^{K+1} \beta\left(k, p_{e}(\hat{\mu} ; c), K+1\right) k=(K+1) p_{e}(\hat{\mu} ; c) .
$$

Recall that from Lemma 1. we have $\partial S_{N}\left(p, \mathbf{v}_{0}, c\right) / \partial p=N\left(U_{N}\left(p, \mathbf{v}_{0}, c\right)\right)$. Since $p_{e}(\hat{\mu} ; c)$ is an interior equilibrium for $N \geq N_{c}+1$, we have $U_{N}\left(p_{e}(\hat{\mu} ; c), \mathbf{v}_{0}, c\right)=0$,
which implies that $\partial S_{N}\left(p_{e}(\hat{\mu} ; c), \mathbf{v}_{0}, c\right) / \partial p=0$, i.e. $p_{e}(\hat{\mu} ; c)$ maximizes $S_{K+1}\left(\cdot, \mathbf{v}_{0}, c\right)$. It then follows from the envelope theorem that the derivative of $S(\cdot)$ is

$$
\frac{d S(c)}{d c}=\frac{\partial S_{K+1}\left(p_{e}(\hat{\mu} ; c), \mathbf{v}_{0}, c\right)}{\partial c}=-(K+1) p_{e}(\hat{\mu} ; c)<0
$$

Therefore, $S(\cdot)$ is decreasing in $c$. To see that it is convex, we need to verify that $\frac{d p_{e}(\hat{\mu} ; c)}{d c}<0$. By Milgrom-Shannon Theorem, it is equivalent to show that $S_{K+1}\left(p_{e}(c), c\right):=S_{K+1}\left(p_{e}(\hat{\mu} ; c), \mathbf{v}_{0}, c\right)$ obeys the single crossing condition. In particular, it suffices to show that $\frac{\partial^{2}}{\partial p_{e} \partial(-c)} S_{K+1}\left(p_{e}(c), c\right) \geq 0$, which holds as $\frac{\partial^{2}}{\partial p_{e} \partial(-c)} S_{K+1}\left(p_{e}(c), c\right)=$ $\frac{\partial}{\partial p_{e}}\left[(K+1) p_{e}(c)\right]=K+1 \geq 0 . \frac{d p_{e}(c)}{d c}<0$ follows.
Step 2. Without loss of generality, we assume $u_{K+1}\left(v_{0}\right) \leq u_{K}\left(r^{m}\right)$. On interval $\left[u_{K+1}\left(v_{0}\right), u_{K}\left(r^{m}\right)\right], R(c)=R_{K}\left(r^{m}\right)$ is constant.

On interval $\left[u_{K}\left(r^{m}\right), u_{K}\left(v_{0}\right)\right], R(c)=W_{K}\left(r_{K}^{c}\right)-K c$. We need to show that it is decreasing and concave. Taking $u_{K}\left(r_{K}^{c}\right)=c$ to be an identity and noting that $u_{k}^{\prime}(r)=-(1-F(r)) G_{k-1}(r)$ give

$$
\frac{d r_{K}^{c}}{d c}=\frac{1}{-\left(1-F\left(r_{K}^{c}\right)\right) G_{K-1}\left(r_{K}^{c}\right)}<0
$$

Recalling $W_{k}^{\prime}(r)=\left(v_{0}-r\right) G_{k}^{\prime}(r)=k\left(v_{0}-r\right) G_{k-1}(r) f(r)$ (eqn. 11) $)$, we have

$$
\begin{align*}
\frac{d R(c)}{d c} & =K \frac{\left(r_{K}^{c}-v_{0}\right) f\left(r_{K}^{c}\right)}{1-F\left(r_{N_{c}}^{c}\right)}-K  \tag{12}\\
& =K \frac{f\left(r_{K}^{c}\right)}{1-F\left(r_{K}^{c}\right)}\left[J\left(r_{K}^{c}\right)-v_{0}\right] \tag{13}
\end{align*}
$$

$J\left(r_{K}^{c}\right)<J\left(r^{m}\right)=v_{0}$ since $r_{K}^{c}<r^{m}$ and hence $\frac{d R(c)}{d c}<0$. Moreover, since $r_{N_{c}}^{c}>v_{0}$ and $f(\cdot) /(1-F(\cdot))$ is increasing, the fact that $r_{K}^{c}$ is decreasing implies that the first term in 12 is decreasing in $c$, i.e., $\frac{d^{2} R(c)}{d c^{2}}<0$. Hence, $R(\cdot)$ is concave.
Step 3. We will first argue that $R\left(u_{K+1}\left(v_{0}\right)\right)<S\left(u_{K+1}\left(v_{0}\right)\right)$ and $R\left(u_{K}\left(v_{0}\right)\right)>$ $S\left(u_{K}\left(v_{0}\right)\right)$. Recall that $R(c)=R_{K}\left(r^{m}\right)$ for all $c \leq u_{K}\left(r^{m}\right)$, and $R(c)=W_{K}\left(r_{c}\right)-K c$ for all $c \in\left[u_{K}\left(r^{m}\right), u_{K}\left(v_{0}\right)\right]$.

By (5), when $c=u_{K+1}\left(v_{0}\right), N_{c}=K+1$. In other words, $K+1$ bidders discover their values and participate in the auction, and the resulting total surplus is $S\left(u_{K+1}\left(v_{0}\right)\right)=$ $W_{K+1}\left(v_{0}\right)-(K+1) c$, which is the highest total welfare for $K+1$ bidders.

Using (4) and $c=u_{K+1}\left(v_{0}\right)$, we have

$$
\begin{aligned}
& S\left(u_{K+1}\left(v_{0}\right)\right) \\
= & W_{K+1}\left(v_{0}\right)-(K+1) c \\
= & E\left[\max \left\{Y_{K+1}, v_{0}\right\}\right]-(K+1) c \\
= & E\left[\max \left\{Y_{K}, v_{0}\right\}\right]-(K+1) c+E\left[\max \left\{Y_{K+1}, v_{0}\right\}\right]-E\left[\max \left\{Y_{K}, v_{0}\right\}\right] \\
= & E\left[\max \left\{Y_{K}, v_{0}\right\}\right]-K c-c+u_{K+1}\left(v_{0}\right) \\
= & W_{K}\left(v_{0}\right)-K c \\
> & W_{K}\left(r^{m}\right)-K c \\
> & W_{K}\left(r^{m}\right)-K\left(u_{K}\left(r^{m}\right)-c\right)-K c \\
= & R\left(u_{K+1}\left(v_{0}\right)\right)
\end{aligned}
$$

using $R(c)=R_{K}\left(r^{m}\right)$ and $R_{K}\left(r^{m}\right)+K\left(u_{K}\left(r^{m}\right)-c\right)=W_{K}\left(r^{m}\right)-K c$ for $c=u_{K+1}\left(v_{0}\right) \leq$ $u_{K}\left(r^{m}\right)$. Therefore, $R\left(u_{K+1}\left(v_{0}\right)\right)<S\left(u_{K+1}\left(v_{0}\right)\right)$.

Likewise, by (5), when $c=u_{K}\left(v_{0}\right)$, we have $N_{c}=K$. In other words, there are $K$ bidders who discover their values and participate in the auction, and the resulting expected revenue is $R\left(u_{K}\left(v_{0}\right)\right)=W_{K}\left(v_{0}\right)-K c$, which is the highest social welfare for $K$ bidders.

Consider $c=u_{K}\left(v_{0}\right)$. Let $\tilde{\mu}=\left(K, v_{0}\right)$. Then, from Proposition 2, we have $p_{e}(\hat{\mu} ; c)<1$. Therefore,

$$
\begin{aligned}
& S_{K+1}\left(u_{K}\left(v_{0}\right)\right) \\
= & S_{K+1}\left(p_{e}(\hat{\mu} ; c), \mathbf{v}_{0}, c\right) \\
= & S_{K}\left(p_{e}(\hat{\mu} ; c), \mathbf{v}_{0}, c\right) \\
< & S_{K}\left(p_{e}(\tilde{\mu} ; c), \mathbf{v}_{0}, c\right) \\
= & W_{K}\left(v_{0}\right)-K c \\
= & R\left(u_{K}\left(v_{0}\right)\right) .
\end{aligned}
$$

The first equality holds by definition. The second equality holds since given $K$ players acquire information with equilibrium probability $p_{e}(\hat{\mu} ; c)$, the information acquisition equilibrium condition means that the contribution of the $(K+1)$-th player is zero. The detail is provided in the proof of Proposition 9 in Levin and Smith (1994). The second to last equality holds since $p_{e}(\tilde{\mu} ; c)=1$ with $c=u_{K}\left(v_{0}\right)$. The last equality holds since the bidders' payoffs are zero at equilibrium. The strict inequality holds for $K \geq 2$, since in this case $p_{e}(\tilde{\mu} ; c)$ uniquely maximizes $S_{K}\left(\cdot, \mathbf{v}_{0}, c\right)$. Note that when $K=1$, we rather have $S_{K}\left(p_{e}(\hat{\mu} ; c), \mathbf{v}_{0}, c\right)=S_{K}\left(p_{e}(\tilde{\mu} ; c), \mathbf{v}_{0}, c\right)=v_{0}$, which is shown
below.
Therefore, $R\left(u_{K}\left(v_{0}\right)\right)=W_{K}\left(v_{0}\right)-K c$ is greater than the social welfare $S\left(u_{K}\left(v_{0}\right)\right)=$ $S_{K+1}\left(u_{K}\left(v_{0}\right)\right)$. From the above reasoning, the function $d(c)=R(c)-S(c)$ is such that $d\left(u_{K+1}\left(v_{0}\right)\right)<0$ and $d\left(u_{K}\left(v_{0}\right)\right)>0$. Since it is continuous, there is $\hat{c}_{K} \in$ $\left(u_{K+1}\left(v_{0}\right), u_{K}\left(v_{0}\right)\right)$ such that $d\left(\hat{c}_{K}\right)=0$. From Step 1 and Step 2, $d(\cdot)$ is concave and hence such a $\hat{c}_{K}$ must be unique.

Case $K=1$. The proof resembles that of Proposition 6. The only difference is that when $c=u_{1}\left(v_{0}\right)$, the two procedures $\left(1, r_{1}^{*}\right)$ and $\left(2, \mathbf{v}_{0}\right)$ generate the same revenue of $v_{0} 23$ It thus suffices to further show that $\left.\frac{d R_{1}\left(r_{1}^{*}\right)}{d c}\right|_{c=u_{1}\left(v_{0}\right)}<\left.\frac{d R_{2}\left(p_{c}^{2}, \mathbf{v}_{0}\right)}{d c}\right|_{c=u_{1}\left(v_{0}\right)} \leq 0$. We next establish this property.

When $c \in\left(\hat{c}_{1}, u_{1}\left(v_{0}\right)\right)$ is in a small neighborhood of $u_{1}\left(v_{0}\right)$, we have

$$
R_{1}\left(r_{1}^{*}\right)=v_{0} F\left(r_{1}^{*}\right)+r_{1}^{*}\left(1-F\left(r_{1}^{*}\right)\right),
$$

where $\int_{r_{1}^{*}}^{\bar{v}}\left(v-r_{1}^{*}\right) f(v) d v=c$. Note $r_{1}^{*}=v_{0}$ when $c=u_{1}\left(v_{0}\right)$.
We thus have $\left.\frac{d r_{1}^{*}}{d c}\right|_{c=u_{1}\left(v_{0}\right)}=-\frac{1}{1-F\left(v_{0}\right)}$, and $\left.\frac{d R_{1}\left(r_{1}^{*}\right)}{d c}\right|_{c=u_{1}\left(v_{0}\right)}=-1$.
When $c \in\left(\hat{c}_{1}, u_{1}\left(v_{0}\right)\right)$ is in a small neighborhood of $u_{1}\left(v_{0}\right)$, we have

$$
R_{2}\left(p_{e}^{2}, \mathbf{v}_{0}\right)=\left(p_{e}^{2}\right)^{2} R_{2}\left(v_{0}\right)+2 p_{e}^{2}\left(1-p_{e}^{2}\right) R_{1}\left(v_{0}\right)+\left(1-p_{e}^{2}\right)^{2} R_{0}\left(v_{0}\right),
$$

where $p_{e}^{2}$ is the entry equilibrium and $R_{k}\left(v_{0}\right)$ stands for the expected seller revenue in a standard second price auction with $k$ bidders and a reservation price $v_{0}$. Note $R_{1}\left(v_{0}\right)=R_{0}\left(v_{0}\right)=v_{0}$, and $p_{e}^{2}=0$ when $c=u_{1}\left(v_{0}\right)$.

The entry equilibrium is given by $p_{e}^{2} u_{2}\left(v_{0}\right)+\left(1-p_{e}^{2}\right) u_{1}\left(v_{0}\right)=c$. We thus have $\left.\frac{d p_{c}^{2}}{d c}\right|_{c=u_{1}\left(v_{0}\right)}=-\frac{1}{u_{1}\left(v_{0}\right)-u_{2}\left(v_{0}\right)}$, and $\left.\frac{d R_{2}\left(p_{e}^{2} \mathbf{v}_{0}\right)}{d c}\right|_{c=u_{1}\left(v_{0}\right)}=-\frac{2\left[R_{1}\left(v_{0}\right)-R_{0}\left(v_{0}\right)\right]}{u_{1}\left(v_{0}\right)-u_{2}\left(v_{0}\right)}=0$.

We thus have $\left.\frac{d R_{1}\left(r_{c}^{*}\right)}{d c}\right|_{c=u_{1}\left(v_{0}\right)}<\left.\frac{d R_{2}\left(p_{c}^{2}, \mathbf{v}_{0}\right)}{d c}\right|_{c=u_{1}\left(v_{0}\right)} \leq 0$, which further means $R_{1}\left(r_{1}^{*}\right)>$ $R_{2}\left(p_{e}^{2}, \mathbf{v}_{0}\right)$ when $c \in\left(\hat{c}_{1}, u_{1}\left(v_{0}\right)\right)$.

### 5.9 Proof of Proposition 8

Proof. Now suppose $p_{e}$ is in fact the equilibrium value discovery for $\hat{\mu}=\left(\hat{N}, \hat{\mathbf{r}}_{N}, \hat{s}\right)$. If $\hat{N} \geq N_{c}+1$, then by the proof of Proposition 2, the seller's revenue is dominated by the total surplus under standard procedure $\left(N, v_{0}\right)$, which is the highest possible revenue achieved by $\left(N, v_{0}\right)$. Reserves $\hat{\mathbf{r}}_{N}\left(\neq \mathbf{v}_{0}\right)$ or subsidy $\hat{s}(>0)$ will lead to a suboptimal

[^10]value discovery equilibrium or ex post inefficient allocations. In both cases, the highest revenue will not be achieved.

Now assume $\hat{N} \leq N_{c}$. We claim that $U_{\hat{N}}\left(p_{e}, \hat{\mathbf{r}}_{N}, c\right)+\hat{s}=0$ must hold if $\hat{s}>$ 0 . This clearly holds if $p_{e} \in(0,1)$. If $p_{e}=1, \hat{s}$ can be reduced and we still have $U_{\hat{N}}\left(1, \hat{\mathbf{r}}_{N}, c\right)+\hat{s}=0$ holds such that $p_{e}=1$ is still induced. But the revenue will be higher. This contradicts to the assumption that $\hat{\mu}$ is revenue-maximizing.

When $\hat{N} \leq N_{c}$, we know that $r_{\hat{N}}^{c} \geq v_{0}$ and also that $u_{k}\left(r_{\hat{N}}^{c}\right)>c$ for all $k<\hat{N}$. Should it be the case that $\hat{r}_{k} \leq r_{\hat{N}}^{c}$ for all $k \leq \hat{N}$, it would then mean $U_{\hat{N}}\left(p_{e}, \hat{\mathbf{r}}_{N}, c\right)>0$ and the only way for $U_{\hat{N}}\left(p_{e}, \hat{\mathbf{r}}_{N}, c\right)+\hat{s}=0$ to hold would be to have $\hat{s}<0$. Since $\hat{s}>0$, we must have $\hat{r}_{k}>r_{\hat{N}}^{c}$ for some $k$.

We can now prove that $\hat{s}=0$. Otherwise, pick one $\hat{r}_{k}>r_{\hat{N}}^{c}$ and reduce it slightly, which in turn increases $U_{\hat{N}}\left(p_{e}, \hat{\mathbf{r}}_{N}, c\right)$, but since $\hat{s}>0$, it can also be reduced to offset this increase and keep the information acquisition equilibrium intact. Since $S_{N}\left(p_{e},\left(\cdot, \hat{\mathbf{r}}_{-k}\right), c\right)$ is decreasing to the right of $v_{0}$, the above change immediately contradicts that $\left(\hat{\mathbf{r}}_{N}, \hat{s}\right)$ is revenue-maximizing. Note the expected payoffs of bidders are zero, thus revenue and total surplus coincide.

## References

Bag, Parimal Kanti, "Optimal auction design and R\&D," European Economic Review, 1997, 41 (9), 1655-1674.

Bergemann, Dirk and Juuso Välimäki, "Information Acquisition and Efficient Mechanism Design," Econometrica, 2002, 70 (3), 1007-1033.

Bhattacharya, Vivek, James Roberts, and Andrew Sweeting, "Regulating bidder participation in auctions, " The RAND Journal of Economics, 2014, 45, 675-704.

Bulow, Jeremy and Paul Klemperer, "Auctions vs Negotiations," American Economic Review, 1996, 86, 180-194.

Che, Yeon-Koo and Ian Gale, "Optimal Design of Research Contests," American Economic Review, 2003, 93 (3), 646-671.

Chen, Jiafeng and Scott Duke Kominers, Auctioneers Sometimes Prefer Entry Fees to Extra Bidders, International Journal of Industrial Organization, 2021, 79, 102737.

Coey, Dominic, Larsen, Bradley and Sweeney, Kane, "The Bidder Exclusion Effect," 2019, The Rand Journal of Economics, 50(1), 93-120..

Crèmer, Jacques, Yossi Spiegel, and Charles Zhoucheng Zheng, "Auctions with Costly Information Acquisition," Economic Theory, 2009, 38 (1), 41-72,

Dasgupta, S., "Competition for procurement contracts and underinvestment," International Economics Review, 1990, 31 (4), 841-865.

Doval, L., "Whether or not to open Pandora's box," Journal of Economic Theory, 2018, 175, 127-158.

Fullerton, Richard L. and R. Preston McAfee, "Auctioning Entry into Tournaments," Journal of Political Economy, 1999, 107 (3), 573-605.

Gershkov, Alex, Benny Moldovanu, Philipp Strack and Mengxi Zhang, "A theory of auctions with endogenous valuations," Journal of Political Economy, 2021, 129(4), pp.1011-1051.

Jehiel, Philippe and Laurent Lamy, "On discrimination in auctions with endogenous entry," American Economic Review 2015, 105(8): 2595-2643.

Larsen, Bradley, "The Efficiency of Real-World Bargaining: Evidence from Wholesale Used-Auto Auctions," Review of Economic Studies, 2021, 88(2), 851-882.

Lee, Joosung and Li, Daniel, "Seller Compound Search for Bidders," , Journal of Industrial Economics, forthcoming, 2022.

Levin, Dan and James L. Smith, "Equilibrium in Auctions with Entry," American Economic Review, 1994, 83 (3), 585-99.

Li, Yunan, "Efficient mechanisms with information acquisition," Journal of Economic Theory, 2019, 182, 279-329.

Li, Tong and Xiaoyong Zheng, "Entry and Competition Effects in First-Price Auctions: Theory and Evidence from Procurement Auctions," Review of Economic Studies, 2009, 76, 1397-1429.

Lu, Jingfeng, "Optimal Entry in Auctions with Valuation Discovery Costs," Applied Economics Research Bulletin, 2008, 1 (2).

Lu, Jingfeng,"Entry Coordination and Auction Design with Private Costs of Information-Acquisition," Economic Inquiry, 48 (2), 2010, 274-289.

Lu, Jingfeng and Lixin Ye, "Efficient and Optimal Mechanisms with Private Information Acquisition Costs," Journal of Economic Theory, 2013, 148, 393-408.

Lu, Jingfeng, Lixin Ye, and Xin Feng. "Orchestrating Information Acquisition." American Economic Journal: Microeconomics, 13(4), 420-465, 2021.

Moreno, Diego and John Wooders, "Auctions with heterogeneous entry costs," The RAND Journal of Economics, 2011, 42(2), 313-336, 80-112.

Persico, Nicola, "Information Acquisition in Auctions," Econometrica, 2000, 68 (1), 135-148.

Quint, Daniel and Kenneth Hendricks, "A theory of indicative bidding," American Economic Journal: Microeconomics, 2018, 10 (2), 118-151.

Shi, Xianwen, "Optimal Auctions with Information Acquisition," Games and Economic Behavior, 2012, 74 (2), 666-686.

Sogo, Takeharu, Dan Bernhardt, and Tingjun Liu, "Endogenous entry to security-bid auctions," American Economic Review, 2016, 106 (11), 3577-3589.

Sweeting, Andrew and Bhattacharya, Vivek, "Selective Entry and Auction Design, " International Journal of Industrial Organization, 2015, 43, 189-207.

Szech, Nora, "Optimal advertising of auctions, " Journal of Economic Theory, 2011, 146 (6), 2596-2607.

Taylor, Curtis, "Digging for Golden Carrots: An Analysis of Research Tournaments,"American Economic Review, 1995, 85, 872-890.

Tan, Guofu, "Entry and R \& D in procurement contracting," Journal of Economic Theory, 1992, 58 (1), 41-60.

Ye, Lixin, "Indicative Bidding and A Theory of Two-Stage Auctions," Games and Economic Behavior, 2007, 58 (1): 181-207.


[^0]:    ${ }^{1}$ This literature includes Levin and Smith (1994), Taylor (1995), Ye (2007), Li and Zheng (2009), Lu and Ye (2013), Bhattacharya, Roberts and Sweeting (2014), Sweeting and Bhattacharya (2015), Quint and Hendricks (2018), and Lu, Ye and Feng (2021) among others. In a closely related literature, the seller also needs to determine the optimal number of bidders, since the seller needs to incur costs to search for bidders. See Crémer, Spiegel and Zheng (2009), Szech (2011), Doval (2018), and Lee and Li (2022).
    ${ }^{2}$ Please refer to Lu (2008) for detail.
    ${ }^{3}$ This is in the spirit of Persico (2000), Bergemann and Välimäki (2002), Shi (2012), Li (2019), Gershkov, Moldovanu, Strack and Zhang (2021).
    ${ }^{4}$ Nevertheless, informed buyers may have hard evidence for their value discovery.

[^1]:    ${ }^{5} u_{n}$ decreases with $n$.
    ${ }^{6}$ Our Assumption 1 means that the ex-ante expected valuation of bidders is lower than the seller's value. Shi (2012) allows cost of information acquisition to be continuous but does not consider shortlisting of bidders.
    ${ }^{7}$ Such entry fees are infeasible in our paper since value discovery is covert.

[^2]:    ${ }^{8} \mathrm{~A}$ further discussion on the role of Assumption 1 is deferred to Section 3.3.2.

[^3]:    ${ }^{9}$ The timing of the game is consistent with Assumption 4 in Levin and Smith (1994).
    ${ }^{10}$ This ensures that the seller can credibly commit to a reserve that is contingent on the number of participants.
    ${ }^{11}$ For the case that the seller could observe the bidders' value discovery and charge a stage- 1 shortlisting fee or an examination fee upon their information acquisition in stage 2 to extract their surplus, the optimal design will be discussed in Section 3.3.1.

[^4]:    ${ }^{12}$ It is a dominant strategy for an uninformed bidder to bid his expected value $\eta$ if he participates.
    ${ }^{13}$ We assume that the bidders play a symmetric information acquisition equilibrium, for the tractability of the analysis. There are many asymmetric information acquisition equilibria, which are far less convenient for the analysis. It is a focal point for the symmetric bidders to play a symmetric equilibrium. This assumption is common in the literature.

[^5]:    ${ }^{14}$ With one more bidder, a representative bidder's winning chance drops. Moreover, in the event he wins, he pays (weakly) more. Thus we have $u_{k+1}\left(v_{0}\right)<u_{k}\left(v_{0}\right)$. When $k \rightarrow \infty$, a representative bidder's winning chance converges to zero, which means $u_{k}\left(v_{0}\right)$ must converge to zero.
    ${ }^{15} \mathrm{Lu}(2008)$ notes that, $N_{c}$ represents the efficient amount of value discovery even if each potential bidder's choice to invest in discovery is not necessarily symmetric.

[^6]:    ${ }^{16}$ His payoff is zero if he does not acquire information.
    ${ }^{17}$ In this case, the total surplus and the seller revenue coincide since the bidders' payoffs are zero.

[^7]:    ${ }^{18}$ Otherwise, the optimal reservation price is the Myerson $r^{m}$ for both $N$ and $N-1$ bidders. It is then clear that a higher number of bidders leads to higher revenue.
    ${ }^{19} \mathrm{~A}$ reserve $r^{\prime}$ lower than $r$ cannot lead to higher revenue with one less bidder.

[^8]:    ${ }^{20}$ The proof for $K=1$ is slightly different, as the two procedures generate the same revenue at $c=\hat{c}_{1}$ and $u_{0}\left(v_{0}\right)$.
    ${ }^{21}$ Note that $r_{K}^{*}$ depends on $c$, although the notation does not make this explicit. Therefore, standard selling procedure $\mu^{c}$ is a function of $c$.

[^9]:    ${ }^{22}$ Coey, Larsen and Sweeney (2019) also drop the "serious bidder assumption" in their empirical analysis of the bidder exclusion effect.

[^10]:    ${ }^{23}$ When $c=u_{1}\left(v_{0}\right), r_{1}^{*}=v_{0}$, the selling procedures $\left(1, r_{1}^{*}\right)=\left(1, v_{0}\right)$ generates revenue of $v_{0}$. For the selling procedure $\left(2, \mathbf{v}_{0}\right)$, the entry equilibrium is given by $p_{e} u_{2}\left(v_{0}\right)+\left(1-p_{e}\right) u_{1}\left(v_{0}\right)=c=u_{1}\left(v_{0}\right)$, which implies that $p_{e}=0$ and the revenue thus equals the seller's reserve value $v_{0}$.

