Optimal Disclosure of Value Distribution Information in All-pay Auctions

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October 6, 2019

Abstract

We study the auction organizer's optimal policy of disclosing information about players' value distribution in an all-pay auction setting. There are two symmetric players whose values (either high v_h or low v_l) are independently distributed following an identical distribution. There are two possibilities for the value distribution. The players know their values privately, but they are uncertain about the value distribution. The organizer precommits to disclose information about the value distribution with a public signal and updates players' beliefs about opponents' value. With this indirect belief update, endogeneity is created and the conventional concavification approach is no longer workable. We adopt a new approach and find that when $v = v_h/v_l$ is sufficiently high, it is optimal for the organizer to adopt an uninformative disclosure policy. Otherwise, an informative partial disclosure policy is optimal.

Keywords: All-pay auction; Information disclosure; Public signal.

JEL classification: D44, D74, D82

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1 Introduction

All-pay auction, as a form of auction which requires all its bidding participants to pay their bid amounts regardless of whether they win or not, has been a useful tool to model competitions in many real-life environments. R&D races, sport competitions, lobbying, and election campaigns are just a few examples. As is well documented in the literature, auction participants may have private information, like abilities, values, and costs, and they need to form a belief about opponents' private information before making a move. The auction organizer can influence participants' beliefs through provision of information and induce them to behave in his favored way. For example, in the bidding for a procurement contract, the organizer can decide whether to reveal the list of competing suppliers. Such revelation of information about competitors on the list may enable a supplier to update his belief about competitors' provision costs and profitabilities, and lead to more aggressive biddings behaviors.

In this paper, we investigate the optimal design of information disclosure in an all-pay auction setting with two symmetric players competing for a single indivisible object. Players' private values are affiliated with a common unknown state of the world, which determines the distribution of values. The organizer precommits to disclose a public signal about the state before the auction starts.

The issue of information disclosure in contests has been studied extensively in literature, for example, Fu, Qiao, and Lu (2014), Denter, Morgan, and Sisak (2012), Lu, Ma, and Wang (2018), and Chen, Kuang, and Zheng (2019). However, to the best of our knowledge, most of the existing studies in this literature mainly focus on the comparison between no disclosure and full disclosure. From the contest organizer's perspective, the no-or-full disclosure comparison seems to be too restrictive. He also has the option to paritally disclose information. Pioneered by Kamenica and Gentzkow (2011), the Bayesian persuasion approach serves as a useful tool to model partial disclosure in several recent studies (Zhang and Zhou (2016), Chen, Z. C. (2019), Chen, Kuang, and Zheng (2019)). A disclosure signal can be seen as a conditional distribution, and the organizer's problem of choosing the optimal discloure rule can be transformed as a problem of choosing the optimal distribution of posterior beliefs.

In this paper, we also allow partial disclosure in the framework of Bayesian persuasion. However, unlike those recent studies (Zhang and Zhou (2016), Chen, Z. C. (2019), Chen, Kuang, and Zheng (2019)) which directly disclose information about players' values, this paper focus on disclosure policies which is indirect in the sense that the revealed information is on true state but players update belief about opponents' values with that information. We cannot simply view this indirect signal as a garbling of the direct signal, since players' private values are affiliated by the common state of world which would create endogeneity in the indirect belief updating process. It means that conventional concavification approach introduced by Kamenica and Gentzkow (2011) doesn't apply to our model. We thus adopt a new approach to solve the optimal design problem. We find that the optimal disclosure policy for the organizer is uninformative (i.e., no disclosure) when the two possible values are sufficiently different ($v = v_h/v_l \geq p_{\mu_0}(v_l|v_l)/p_{\mu_0}(v_l|v_h)$) such that a monotone equilibrium exists in the original game, otherwise it is informative as a partial disclosure.

Our paper is closely related to the literature on equilibrium characterization in all-pay auctions. Hillman and Riley (1989) first characterizes the unique mixed strategy of two-player all-pay auction with complete information. Baye, Kovenock, and de Vries (1996) extends the set of equilibria to include asymmetric ones for the class of all-pay auction games with complete information. Amann and Leininger (1996) shows the existence and uniqueness of equilibrium for asymmetric two-player all-pay auction with incomplete information. Siegel (2014) studies the monotone equilibria in a two-bidder all-pay auction with multiple types. The closest one is Liu and Chen (2016). They consider a model setting which shares the feature of posterior game in our paper, i.e., two-player all-pay auction with binary types and correlated information structure, and characterize both monotonic and non-monotonic symmetric Bayesian Nash equilibrium.

Another strand of closely related literature is on information disclosure in contests. Zhou and Zhang (2016) examines the type-dependent probabilistic disclosure policies (i.e., Bayesian persuasion approach) in a two-player siultaneous contest. Chen, Kuang, and Zheng (2017a) investigate the type-dependent probabilistic disclosure policies in a two-player sequential contest. However, in these papers, one player's value is commonly observed while the other's is private information. This one-sided asymmetric information feature differs from that of our paper in which both players' values are private information. For the all-pay auction scenario, Lu, Ma, and Wang (2018) compares four different type-dependent non-probabilistic disclosure policies in a two-player all-pay auction. Kuang, Zhao, and Zheng (2019) completely characterizes the optimal type-dependent probabilistic disclosure policy in a two-player all-pay auction with correlated values. Z.C. Chen (2019) consider the disclosure policy in a two-player all-pay auction with independent private values. He characterizes a public disclosure policy which is better than no disclosure. Our paper differs from all of these papers in that the disclosure policy is indirect and state-dependent, rather than type dependent. This indirect feature creates endogeneous terms which result in the failure of conventional concavification procedure in our paper. We proposed a new approach and identified the optimal rule of information disclosure.

The rest of this paper is organized as follows. In Section 2 we set up the model. In Section 3 we analyze the equilibrium in posterior all-pay auction game. In Section 4, we solve the organizer's optimal design of information disclosure. We conclude in Section 5. The Appendix collects the technical proofs.

2 Model

Consider the following all-pay auction with incomplete information. There are two risk-neutral players competing for an object by submitting their bids simultaneously. The winning probability of player $i \in \{1, 2\}$ under bid portfolio (x_1, x_2) is given by

$$p_i(x_1, x_2) = \begin{cases} 1 & \text{if } x_i > x_{-1}; \\ 0 & \text{if } x_i < x_{-1}. \end{cases}$$

If there is a tie, i.e., $x_1 = x_2$, the object is randomly allocated between the two players.

Prior to bidding, each player $i \in \{1, 2\}$ privately learns his value of the object $v_i \in \{v_h, v_l\}$. The two possible values are ordered as $v_h > v_l$ to capture the idea that a player with v_h has a higher value for the object than a player with v_l. The two players' values are affiliated with a common unknown state of world, $\omega \in \Omega = \{G, B\}$. Let $\mu_0 \in \Delta(\Omega)$ denote the common prior over states. Specifically, the two players' values are independently drawn from the same binary distribution conditioned on the prevailing state ω, and we represent the conditional distributions with vector (α, β), where $\alpha = p(v = v_h|G)$ and $\beta = p(v = v_l|B)$. We assume that $\alpha \geq 1 - \beta$ with the idea that G is a good state under which it's more likely for a player to draw high value v_h compared to B (bad state).

The auction organizer precommits to a signal before the auction starts with the intention to maximize the expected total bids collected from both players. A signal consists of a finite realisation space $\mathcal{S}$ and a family of distributions $\{\pi(\cdot|\omega)\}_{\omega\in\Omega}$. That is, conditioning on the prevailing state ω, the signal is realized according to distribution $\pi(\cdot|\omega) \in \Delta(\mathcal{S})$. If for any realisation s and s', $\pi(s|\omega) = \pi(s'|\omega)$, then the signal is *uninformative* since the updated belief about state under any realisation stays the same as the prior. If there exist two realisations s and s' such that $\pi(s|G) = 1$ and $\pi(s'|B) = 1$, then the signal is *fully informative* as the state can be directly inferred from the realized signal. Note that while the signal is conditional on the common state, eventually the players have to update beliefs about their opponents' private values. Thus, when a specific signal $s \in \mathcal{S}$ is realized, a player first updates his belief about state and then forms a belief about his opponent's value with the posterior belief about state using Bayes' rule. Denote the posterior belief about state as $\mu_s \in \Delta(\Omega)$. Due to the binary structure of value distribution, we write μ_s instead of $\mu_s(G)$ for notation simplicity in some scenarios.

The timing of the game is as follows:

1. The auction organizer precommits to a signal π.

2. Nature moves and the state of world is determined, say ω.

3. A signal realisation $s \in \mathcal{S}$ is generated according to $\pi(\cdot|\omega)$, and players' private values are generated according to $p(\cdot|\omega) \in \Delta(\Omega)$.

4. The signal realisation s is publically observed, and each player privately learns his own value for the object. With observed s and privately learned value, a player forms a posterior belief μ_s, which leads to a new belief about his opponent's value.

5. The auction takes place, and the players place their bids simultaneously .

The game in stage 5 is an all-pay auction with affiliated private values, and the affiliation between players' values is determined by the posterior belief from signal realisation in stage 3. The results in Chi, Murto, and Välimäki (2019) suggest that the level of affiliation in players' value in an all-pay auction may affect the structure of equilibrium, thus affecting the expected total bids. With the aim to maximize expected total bids, the orgnaizer needs to choose his precommitted signal optimally in stage 1. In the following, we first examine the posterior all-pay auction game in stage 5, and then proceed to solve the organizer's optimal design of signal in stage 1.

3 The Posterior All-pay Auction Game

In the posterior all-pay auction game, a player $i \in \{1,2\}$ privately learns his value v_i and has a belief about his opponent's value, which is formed from posterior μ_s as following:

$$p_s(v|v_i) = \frac{\sum_{\omega\in\Omega} p(v|\omega)p(v_i|\omega)\mu_s(\omega)}{\sum_{\omega\in\Omega} p(v_i|\omega)\mu_s(\omega)},\ \forall v \in \{v_l, v_h\}. \tag{3.1}$$

Since players' private values are independently drawn from the same distribution determined by the prevailing state, their values are affiliated, i.e., they co-move postively. This means that a player with high value (low value) is more likely to expect his opponent with high value (low value) than a player with low value (high value).

Claim 1. *In the posterior all-pay auction game, players' private values are affiliated, i.e.,*

$$p_s(v_i|v_i) \geq p_s(v_i|v_j). \tag{3.2}$$

Proof. See Appendix. □

From equation (3.1), a player's belief about his opponent's value is a function of μ_s. Due to the binary structure of value distribution, it's without loss of generality to denote μ_s by $\mu_s(G)$. Then it's easy to demonstrate that $p_s(v_h|v_h)$ increases in $\mu_s(G)$ and $p_s(v_l|v_l)$ decreases in $\mu_s(G)$.Thus, it's ambiguous how the affiliation[1] changes with $\mu_s(G)$.

To characterize the equilibrium in the posterior all-pay auction game, we define following monotonicity condition which would affect the structure[2] of equilibrium.

Condition M: For $i \in \{1,2\}$, $v_i p_s(v|v_i)$ increases in v_i for every $v \in \{v_h, v_l\}$.

That is, for each player i, the product of his value and his belief about opponent's value increases in his own value. Then we have $v_h p_s(v|v_h) \geq v_l p_s(v|v_l)$ for $\forall v \in \{v_h, v_l\}$. Because of the affiliation shown in Claim 1, the inequality holds automatically when $v = v_h$. But for $v = v_l$, the inequality holds only when the difference between v_h and v_l is large or the affiliation between low values is small (i.e., $p_s(v_l|v_h)$ is very close to $p_s(v_l|v_l)$).

Let $v = v_h/v_l$. We define real-valued function

$$\phi(\mu_s(G)) = v \cdot \underbrace{\frac{\alpha(1-\alpha)\mu_s(G) + \beta(1-\beta)(1-\mu_s(G))}{\alpha\mu_s(G) + (1-\beta)(1-\mu_s(G))}}_{p_s(v_l|v_h)} - \underbrace{\frac{(1-\alpha)^2\mu_s(G) + \beta^2(1-\mu_s(G))}{(1-\alpha)\mu_s(G) + \beta(1-\mu_s(G))}}_{p_s(v_l|v_l)}.$$

[1] We use $p_s(v_i|v_i)/p_s(v_i|v_j)$ to measure the affiliation level of v_i

[2] From the following analysis, the structure refers to monotone or monotone.

The Condition M is equivalent to requiring $\phi(\mu_s(G)) \geq 0$, which implies the increment in $v_i p_s(v_l|v_i)$ in response to player i's value change from v_l to v_h is positive.

Since the actual bid depends on a player's private value and belief about his opponent's value, we represent a strategy of player $i \in \{1,2\}$ by a pair of cumulative distribution functions $F_i^s = (F_i^s(\cdot|v_h), F_i^s(\cdot|v_l))$, where $F_i^s(x|v)$ is the probability that player i bids at most x when his value is v and his belief about opponent's value is formed with μ_s. As a bid more than the value definitely generates a negative payoff in an all-pay auction, a player would never make that bid. Thus, without loss of generality we restrict our attention to strategies with $\text{supp}[F_i^s(\cdot|v_i)] \in [0, v_i]$. Given a strategy profile $F^s = (F_1^s, F_2^s)$, player i's expected payoff conditional on his private value v_i is

$$u^s(v_i) = \int_0^{v_i} \left\{ v_i \underbrace{\left[p_s(v_h|v_i)F_{-i}^s(x|v_h) + p_s(v_l|v_i)F_{-i}^s(x|v_l)\right]}_{\text{expected winning probability}} - x \right\} dF_i^s(x|v_i) \tag{3.3}$$

We analyze Bayesian Nash Equilibria. That is, if strategy profile F^s is an equilibrium, for each player i with private value v_i, $x \in \text{supp}[F_i^s(\cdot|v_i)]$ implies $x \in \arg\max u^s(v_i)$. Throughout this work, we focus our attention on *symmetric* equilibria in which players with the same private value employ the same bidding strategy, i.e., $F_i^s = F^s = (F^s(\cdot|v_h), F^s(\cdot|v_l))$. A symmetric equilibrium is *monotone* if for any $x \in \text{supp}(F(v_h, s))$ and $y \in \text{supp}(F(v_l, s))$, we have $y \leq x$. Otherwise, it's non-monotone[3]. We establish the uniqueness of a symmetric equilibrium and characterize it in the following proposition.

Proposition 1. *In the posterior all-pay auction game with distribution of value distribution μ_s, there exists a unique symmetric equilibrium. Specifically,*

1. *if $\phi(\mu_s(G)) \geq 0$, the equilibrium is monotone, and players' equilibrium strategies are*

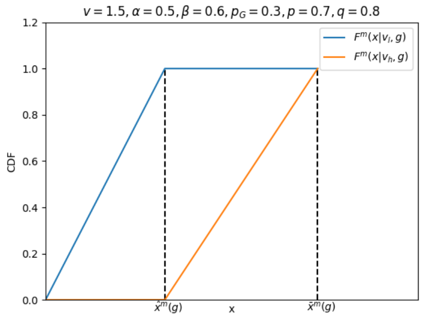
$$F^{s,m}(x|v_l) = \frac{x}{v_l p_s(v_l|v_l)} \quad \text{on } [0, v_l p_s(v_l|v_l)],$$

$$F^{s,m}(x|v_h) = \frac{x - v_l p_s(v_l|v_l)}{v_h p_s(v_h|v_h)} \quad \text{on } [v_l p_s(v_l|v_l), v_l p_s(v_l|v_l) + v_h p_s(v_h|v_h)];$$

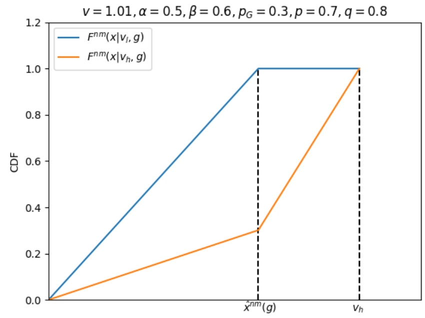
2. *if $\phi(\mu_s(G)) < 0$, the equilibrium is non-monotone, and players' equilibrium strategies are*

$$F^{s,nm}(x|v_l) = x \cdot \frac{v_h p_s(v_h|v_h) - v_l p_s(v_h|v_l)}{v_h v_l [p_s(v_h|v_h) - p_s(v_h|v_l)]} \quad \text{on } [0, \underline{x}(s)],$$

[3] It's impossible for the low value playe to win against the high value player with probability one in equilibrium. We would never have $y \geq x$ when $x \in \text{supp}(F(v_h, s))$ and $y \in \text{supp}(F(v_l, s))$. Thus, the equilibrium can't be monotone in an inversed pattern.



(a) monotone equilibrium when $\phi(\mu_s(G)) \geq 0$



(b) non-monotone equilibrium when $\phi(\mu_s(G)) < 0$

Figure 3.1: Symmetric equilibrium

$$F^{s,nm}(x|v_h) = \begin{cases} x \cdot \frac{v_l p_s(v_l|v_l) - v_h p_s(v_l|v_h)}{v_h v_l [p_s(v_h|v_h) - p_s(v_h|v_l)]} & \textit{on } [0, \underline{x}(s)] \\ \frac{x - v_h p_s(v_l|v_h)}{v_h p_s(v_h|v_h)} & \textit{on } [\underline{x}(s), v_h], \end{cases}$$

where $\underline{x}(s) = \frac{v_h v_l [p_s(v_h|v_h) - p_s(v_h|v_l)]}{v_h p_s(v_h|v_h) - v_l p_s(v_h|v_l)}$.

Proposition 1 tells us that it dependes on the sign of $\phi(\mu_s(G))$ whether the unique equilibrium in the posterior all-pay auction game is monotone or non-monotone. When $\phi(\mu_s(G)) \geq 0$, the equilibrium is monotone, in which the low value type uniformly randomizes its bid on a lower interval (i.e., $[0, v_l p_s(v_l|v_l)]$) while the high value type uniformly randomizes on an upper interval (i.e., $[v_l p_s(v_l|v_l), v_h p_s(v_h|v_h)]$). Specifically, the two intervals are connected at $v_l p_s(v_l|v_l)$. The equilibrium is separating since players' value types can be inferred from almost any pair of bids. Notice that the highest possible bid the low value type $v_l p_s(v_l|v_l)$ is smaller than his value v_l. This is because that in a montone equilibrium , a low value type, who makes his highest bid, wins only when his opponent is also low value type (i.e., probability $p_s(v_l|v_l)$). Since makes a zero expected payoff, his expected gain from the highest bid, $v_l p_s(v_l v_l)$ must equal the bid. The highest bid for a high value type player is also small than v_h, since he make a positive payoff but his gain is exactly v_h.

When $\phi(\mu_s(G)) < 0$, i.e., the monotoncity condition is violated, the unique equilibrium is non-monotone. The low value type still makes zero payoff by uniformly randomizing on an interval starting from 0. The high value type's strategy now takes the form of piecewise-uniform randonmization on interval $[0, v_h]$, which implies that he makes zero payoff as well.

The highest bid for a high value type in the non-monotone equilibrium is unambiguously larger than that in a monotone equilibrium, which implies a tougher competition for the object when $\phi(\mu_s(G)) < 0$. We learn that $\phi(\mu_s(G)) < 0$ holds when the two value types are close or the affiliation is large. Either scenario would lead to a tougher competition, which consists with the intuition from the characterized equilibrium in Proposition 1.

Corollary 1. *In the posterior all-pay auction game with μ_s,*

1. *if $\phi(\mu_s(G)) \geq 0$, the expected total bids in equilibrium is*

$$R^m(\mu_s) = v_l p_s(v_l|v_l) + \Big(v_h p_s(v_h|v_h) + v_l p_s(v_l|v_l)\Big) \sum_{\omega \in \{G,B\}} \mu_s(\omega) p(v_h|\omega).$$

The low value type makes zero payoff. The high value type's expected payoff is $v_l \phi(\mu_s(G)) = v_h p_s(v_l|v_h) - v_l p_s(v_l|v_l)$.

2. *if $\phi(\mu_s(G)) < 0$, the expected total bids in equilibrium is*

$$R^{nm}(\mu_s) = \underline{x}(s) + \frac{v_h(v_h - v_l)}{v_h p_s(v_h|v_h) - v_l p_s(v_h|v_l)} \cdot \sum_{\omega \in \{G,B\}} \mu_s(\omega) p(v_h|\omega).$$

Both value types make zero payoff.

Proof. See Appendix. □

Corollary 1 summarizes the expected total bids and palyers' payoffs in the posterior all-pay auction no matter the equilibrium is monotone ($\phi(\mu_s(G)) \geq 0$)or non-monotone ($\phi(\mu_s(G)) < 0$). In a monotone equilibrium, the high value type's payoff is $v_l\phi(\mu_s(G))$, which is positive only when $\phi(\mu_s(G)) > 0$. To see the intuition behind the high value type's positive payoff, we need to inspect the effect a value change from v_l to v_h. First of all, this value change is good news for a player since his gain from winning the object will be larger. But because of affiliation, this change also implies that his chance of facing a high value opponent higher, which implies a tougher competition. In the case where $\phi(\mu_s(G)) \geq 0$, $v = v_h/v_l$ is larger in comparison to $p_s(v_l|v_l)/p_s(v_l|v_h)$. That is, the value effect of a change from v_l to v_h is larger than the affiliation effect of that change. Therefore, the high value type enjoys a positive payoff in a monotone equiibrium.

When the $\phi(\mu_s(G)) < 0$, the high value type no longer has a positive payoff, which implies that the monotone equilibrium no longer holds. When $\phi(\mu_s(G)) < 0$, the equilibrium is non-monotone, and both types have zero payoff since the value effect of a type change in this case is smaller than the affiliation effect of the change. The high value type at most is as well off as the low value typle, which implies zero payoff for high value type. At first glance, it seems that the organizer can collect more revenue in a non-monotone equilibrium compared to a monotone equilibrium since players' rents are squeezed to zero. However, there exists efficiency loss in a non-monotone equilibrium as the low value may obtain the object when faced with a high value type. This efficiency loss in a non-monotone equilibrium restricts the organizer's ability of extracting surplus. Therefore, the effect of non-monotonicity of equilibrium on the organizer's expected total revenue is ambiguous.

4 Information Disclosure

Now we move to information disclosure problem for the auction organizer. In Stage 5, the organizer's problem is to maximize the ex ante expected total revenue from the all-pay auction by designing the signal about state (or value distribution) optimally. Given signal π, for each signal realisation s, a posterior about state is generated. Thus, a signal π actually can be viewed as a distribution τ of posteriors. And the probability of posterior μ_s in τ equals the probability of signal realisation s in π. Following the conventional approach introduced by Kamenica and Gentzkow (2011), we transform the organizer's signal choice problem into the problem of choosing distribution of posteriors. Given disclosure policy π, let τ be the distribution of posteriors induced by π. The organizer's problem is transformed as

$$\begin{aligned} \max_{\tau} \quad & \sum_{\mu_s} \tau(\mu_s) R(\mu_s) \\ \text{s.t.} \quad & \sum_{\mu_s} \tau(\mu_s)\mu_s(\omega) = \mu_0(\omega). \end{aligned}$$

$R(\mu_s)$ is the organizer's expected revenue in the posterior game induced by μ. The constraint is required by Bayes' plausible condition.

Given a signal realization s, a posterior about state (i.e., value distribution) μ_s is generated, and the risk-neutral auction organizer collects revenue $R(\mu_s)$ in the posterior all-pay auction game. By Corollary 1, we have either $R(\mu_s) = R^m(\mu_s)$ or $R(\mu_s) = R^{nm}(\mu_s)$, depending on whether $\phi(\mu_s(G)) \geq 0$ or not. Before proceeding to investigate the organizer's optimal information disclosure issue, we first explore the expression of $R(\mu_s)$ by examining the property of $\phi(\mu_s(G))$.

Lemma 1. *Define* $v_0 = 1 + \frac{(\sqrt{\alpha}-\sqrt{1-\beta})^2}{(1-\alpha)\beta}$. *Given posterior* μ,

1. *if* $v \geq v_0$, $\phi(\mu_s(G)) \geq 0$ *for* $\forall \mu_s(G) \in [0,1]$;
2. *if* $v < v_0$, *there exists an interval* $(x_1, x_2) \subset [0,1]$ *such that* $\phi(\mu_s(G)) < 0$ *for* $\forall \mu_s(G) \in (x_1, x_2)$.

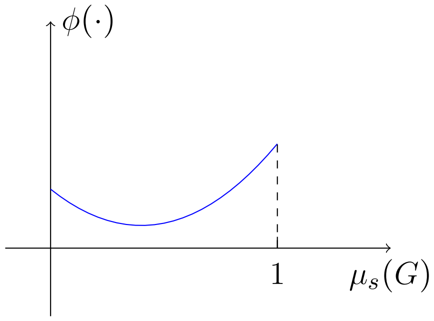


Figure 4.1: $v \geq v_0$

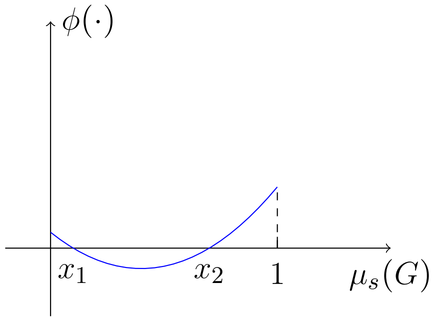


Figure 4.2: $v < v_0$

Lemma 1 tells us that when the two value types are sufficiently different such that $v \geq v_0$, we have $\phi(\mu_s(G)) = \frac{v_h}{v_l} \cdot p_s(v_l|v_h) - p_s(v_l|v_l) \geq 0$ holds for all $\mu_s(G)$. By mathematical computation, we obtain that the affiliation level $p_s(v_l|v_l)/p_s(v_l|v_h)$ is single-peaked in $\mu_s(G)$ on interval $[0,1]$ and capped at v_0. Thus, when $v \geq v_0$, we always have $\phi(\mu_s(G)) \geq 0$, that is, the monotonicity condition is always satisfied. In any posterior all-pay auction game induced by such a μ_s, the equilibrium is monotone, and the organizer extracts surplus $R(\mu_s) = R^m(\mu_s)$. However, when $v < v_0$, i.e., the two possible values are sufficiently close, since $p_s(v_l|v_l)/p_s(v_l|v_h)$ is single-peaked with peak at v_0, $\phi(\mu_s(G))$ changes its sign twice as $\mu_s(G)$ moves along $[0,1]$. For a posterior game induced by μ_s with $\phi(\mu_s(G)) < 0$, we have $R(\mu_s) = R^{nm}(\mu_s)$ in equilibrium.

4.1 Sufficiently different types: $v \geq v_0$

Consider the scenario in which the two possible types, v_h and v_l, are sufficiently different, i.e., $v \geq v_0$. By Lemma 1, the equilibrium in a posterior game induced by any $\mu_s \in \Delta(\Omega)$ is monotone, which implies that $R(\mu_s) = R^m(\mu_s)$ for any μ_s. Thus, the organizer's problem can be formulated as

$$\begin{aligned} \max_{\tau} \quad & \hat{R}(\tau) = E_\tau R^m(\mu_s) \\ \text{s.t.} \quad & \sum_{\mu} \tau(\mu_s)\mu_s(\omega) = \mu_0(\omega), \forall \omega. \end{aligned} \tag{4.1}$$

Kamenica and Gentzkow (2011) establish the result that the maximum of $E_\tau R^m(\mu_s)$ is exactly the value of the concave closure of $R^m(\cdot)$ at prior μ_0. Therefore, we need to construct the concave closure of $R^m(\cdot)$. In this paper, there are only two states, G and B, it's without loss of generality to denote μ_s with $\mu_s(G)$. Thus, we are actually constructing the concave closure of $R^m(\mu_s(G))$ for $\mu_s(G) \in [0, 1]$.

Lemma 2. *$R^m(\mu(G))$ is concave in $\mu(G)$.*

Proof. See Appendix. □

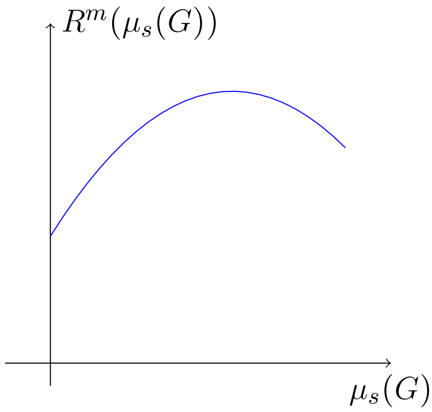


Figure 4.3: Expected revenue in posterior game: $v \geq v_0$

Lemma 2 tells us that $R^m(\mu_s(G))$ is concave in $\mu_s(G)$. Thus, its concave closure is exactly itself. By the result established in Kamenica and Gentzkow (2011), the maximum of the organizer's expected total revenue $E_\tau R^m(\mu_s)$ equals $R^m(\mu_0)$, which can be achieved by distribution of posteriors τ^* with $\tau^*(\mu_0) = 1$. Obviously, τ^* is induced by an uninformative signal.

Proposition 2. *If the two value types are sufficiently different, i.e., $v \geq v_0$, the optimal signal is uninformative.*

When $v \geq v_0$, for whatever belief about state, the unique equilibrium in the all-pay auction is monotone. The high value player always bids more than a low value player, and the allocation of the object is efficient. By disclosing information, the trading efficiency is unaffected, but the affiliation of players' values are changed. Although the organizer could benefit from higher affiliation, the indeterministic nature of signal realization fails to guarantee a higher affiliation. Thus, an informative signal isn't necessarily good for the organizer.

4.2 Sufficiently close types: $v \leq v_0$

In the scenario when the two possible types are sufficiently different, i.e., $v < v_0$, there exists an open interval inside $[0, 1]$ on which $\phi(\mu_s(G)) < 0$. In a posterior game induced by such a μ_s, the unique equilibrium is non-monotone. For μ_s with $\mu_s(G)$ outside that interval, we have $\phi(\mu_s(G)) \geq 0$, and the unique equilibrium in the induced posterior game is monotone. Therefore, the organizer's expected revenue from a posterior game induced by μ_s in this scenario is

$$R(\mu_s(G)) = \begin{cases} R^{nm}(\mu_s(G)) & \text{if } \phi(\mu_s(G)) < 0; \\ R^{m}(\mu_s(G)) & \text{if } \phi(\mu_(G)) \geq 0. \end{cases}$$

Still, following the well-established result in Kamenica and Gentzkow (2011), we need to construct the concave closure of revenue function $R(\mu_s(G))$ on its whole range, i.e., $[0, 1]$. To facilitate the construction of the concave closure, we first examine the continuity of $R(\mu_s(G))$, the result of which is shown in the following lemma.

Lemma 3. *For the μ_s such that $\phi(\mu_s(G)) = 0$, $R^{nm}(\mu_s) = R^m(\mu_s)$.*

Proof. See Appendix. □

From Lemma 3, we learn that $R(\mu_s(G))$ is continuous in $\mu_s(G)$. That is, at the switching point of monotone and non-monotone equilibrium, players' equilibrium strategies generate the same level of expected revenue for the organizer.

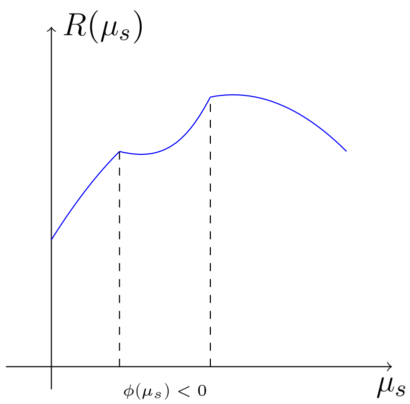


Figure 4.4: Expected revenue in posterior game: $v < v_0$

Now we have the continuity of revenue expression $R(\mu_s(G))$ and the concavity of expression $R^m(\mu_s(G))$ on $[0, 1]$ from Lemma 2, to construct the concave closure, we need to examine how $R^{nm}(\mu_s(G))$ changes on its domian,i.e., the interval on which $\phi(\mu_s(G)) < 0$. However, due to the complexity of revenue expression $R^{nm}(\mu_s(G))$, it's almost impossible to identify the shape of $R^{nm}(\mu_s(G))$ mathematically on that range. We can neither identify it's monotonicity nor concavity (or convexity) by riguous math. We only know the values $R^{nm}(\mu_s(G))$ takes at two endpoints of its domian, as a result of Lemma 3. Figure 4.4 depicts a shape that $R^{nm}(\mu_s)$ could possibly take.

Lemma 4. *For any μ_s such that $\phi(\mu_s) \leq 0$,*

$$R^{nm}(\mu_s(G)) \leq v_h + (v_h - v_l) \cdot \left[(\beta^2 - (1-\alpha)^2)\mu_s(G) - \beta^2\right].$$

The equality holds if and only if $\phi(\mu_s(G)) = 0$

Proof. See Appendix. □

If there is a linear function which dominates R^{nm} on the range and equals R^{nm} at the two endpoints, then it must be the concave closure of R^{nm}, since linear function is smallest concave function between two points. In Lemma 4, we find a linear function of $\mu_s(G)$, which dominants $R^{nm}(\mu_s(G))$ when $\phi(\mu_s(G)) \leq 0$ and equals $R^{nm}(\mu_s(G))$ when $\phi(\mu_s(G)) = 0$. By definition, the linear function is the concave closure of R^{nm} on the range.

Since revenue expression $R^m(\mu_s(G))$ is concave in $\mu_s(G)$ on the whole range, i.e., $[0, 1]$, it is also concave on the range where $\phi(\mu_s(G)) \leq 0$. By definition of concave closure, we have $R^m(\mu_s(G)) \geq v_h + (v_h - v_l) \cdot \left[(\beta^2 - (1-\alpha)^2)\mu_s(G) - \beta^2\right]$. Then the concave closure of $R(\mu_s)$ can be easily identified.

Lemma 5. *Define $\tilde{R} : [0, 1] \to [0, +\infty)$ as:*

$$\tilde{R}(\mu_s(G)) = \begin{cases} v_h + (v_h - v_l) \cdot \left[(\beta^2 - (1-\alpha)^2)\mu_s(G) - \beta^2\right] & \textit{if } \phi(\mu_s(G)) < 0; \\ R^m(\mu_s(G)) & \textit{if } \phi(\mu_s(G)) \geq 0. \end{cases}$$

$\tilde{R}$ is the concave closure of R.

In Lemma 5, we construct the concave closure of $R(\mu_s)$ for any μ_s. Concave closure $\tilde{R}$ coincides with R^m on the range where $\phi(\mu_s(G)) \geq 0$ and is below R^m on the remaining part of $[0, 1]$. By the result in Lemma 3 and Lemma 4, concave closure $\tilde{R}$ is continuous, which is depicted in Figure 4.5. By the well-established result in Kamenica and Gentzkow (2011), the optimal distribution of posterior can be directly identified from the graph of concave

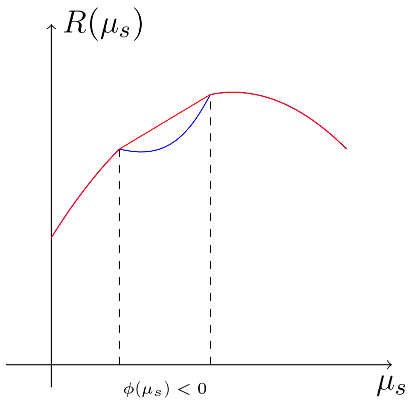


Figure 4.5: Concave closure $\tilde{R}$: $v < v_0$

closure. We see that if $\phi(\mu_0) \geq 0$, the maximum value of $E_\tau R^{\mu_s}$ is exactly $R(\mu_0)$, and the optimal distirbution of posteriors τ^* assigns probability one to the prior μ_0, which implies the optimal signal is uninformative. If $\phi(\mu_0) < 0$, the maximum of $E_\tau R^{\mu_s}$ is $\tilde{R}(\mu_0)$, which is greater than $R(\mu_0)$. That is, the uninformtiave signal isn't optimal in this case. The distribution of posterior induced by the optimal signal satisfies $\tau^*(\mu_1)\mu_1 + \tau^*(\mu_2)\mu_2 = \mu_0$, where $\phi(\mu_1(G)) = \phi(\mu_2(G)) = 0$.

Proposition 3. *When the two types are relatively close, i.e., $v < v_0$, if a monotone equilibrium is induced without information disclosure, then it's optimal for the organizer to adopt an uninformative signal. Otherwise, it's optimal for the organizer to adopt an informative signal which involves parital disclosure.*

Proposition 3 tells us that whether to disclose information about state depends on monotonicity of equilibrium in the original all-pay auction game. Specifically, if the original game generates a monotone equilibrium, the organizer couldn't do better by disclosing information. But if a non-monotone equilibrium arises in the original game, it's optimal for the organizer to disclose some information such that the generated equilibrium is monotone. From previous analysis, we learn that a non-monotone equilibrium arises when affiliation of players' values is large, and competition between players is so fierce that both players' rents are zero. From this perspective, it seems that a non-monotone equilibrium is better than a monotone equilibrium for the organizer. However, we should also notice that in a non-monotone equilibrium, a low value type could win the object against a high value type as the supports of their bidding strategies overlap with each other. It leads to a loss in efficiency.

Our result in Proposition 3 suggests that the gain from trading efficiency for the organizer is relatively larger than the his loss from smaller affiliation when $v < p_{\mu_0}(v_l|v_l)/p_{\mu_0}(v_l|v_h)$.

Corollary 2. *When $v < p_{\mu_0}(v_l|v_l)/p_{\mu_0}(v_l|v_h)$, the organizer can benefit from providing information.*

5 Conclusion

In this paper, we investigate the organizer's optimal public signal about value distribution in a two-player all-pay auction model with affiliated values. We restrict our analysis to two possible value distributions. We find that when the two private values are sufficiently close, i.e., $v < p_{\mu_0}(v_l|v_l)/p_{\mu_0}(v_l|v_h)$, it's optimal signal involves parital disclosure. Otherwise, no disclosure is optimal for the organizer.

The analysis in this paper can be extended to study the optimal private signal about value distribution in a similar auction setting. The difficulty in such a study lies in the characterization of equilibrium in a posterior game. With private persuasion, the two players may observe different signal realisation, which in effect doubles the palyers' types in the all-pay auction game. Chi, Murto, and Välimäki (2019) provide an analysis of all-pay auction game with affiliated private values and more than two players. Their work may help to work out the equilibrium in our all-pay auction with private signal realisations. We plan to study the private persusion in our future work, and examine whether the organizer can benefit from private persuasion or not.

Appendix A

Proof of Claim 1: Take the difference between the two conditional probabilities,

$$p_s(v_i|v_i) - p_s(v_i|v_j) = \frac{\sum_{\omega\in\Omega} p(v_i|\omega)p(v_i|\omega)\mu_s(\omega)}{\sum_{\omega\in\Omega} p(v_i|\omega)\mu_s(\omega)} - \frac{\sum_{\omega\in\Omega} p(v_i|\omega)p(v_j|\omega)\mu_s(\omega)}{\sum_{\omega\in\Omega} p(v_j|\omega)\mu_s(\omega)}.$$

Then $p_s(v_i|v_i) - p_s(v_i|v_j) \geq 0$ if and only if

$$\Big[\sum_{\omega\in\Omega} p(v_i|\omega)p(v_i|\omega)\mu_s(\omega)\Big]\Big[\sum_{\omega\in\Omega} p(v_j|\omega)\mu_s(\omega)\Big] - \Big[\sum_{\omega\in\Omega} p(v_i|\omega)p(v_j|\omega)\mu_s(\omega)\Big]\Big[\sum_{\omega\in\Omega} p(v_i|\omega)\mu_s(\omega)\Big] \geq 0.$$

The first part is extended as

$$\begin{aligned} &p(v_i|G)^2 p(v_j|G)\mu_s(G)^2 + p(v_i|G)^2\mu_s(G)p(v_j|B)\mu_s(B) \\ &+p(v_i|B)^2\mu_s(B)p(v_j|G)\mu_s(G) + p(v_i|B)^2 p(v_j|B)\mu_s(B)^2. \end{aligned}$$

The second part is extended as

$$\begin{aligned} &p(v_i|G)^2 p(v_j|G)\mu_s(G)^2 + p(v_i|G)p(v_j|G)\mu_s(G)p(v_i|B)\mu_s(B) \\ &+p(v_i|B)p(v_j|B)\mu_s(B)p(v_i|G)\mu_s(G) + p(v_i|B)^2 p(v_j|B)\mu_s(B)^2. \end{aligned}$$

Thus, the difference can be rewrite as

$$\mu_s(G)\mu_s(B)\Big(p(v_i|G) - p(v_i|B)\Big)\Big(p(v_i|G)p(v_j|B) - p(v_j|G)p(v_i|B)\Big).$$

Since there are just two possible values, we have $p(v_j|\omega) = 1 - p(v_i|\omega)$ for any ω. Then the above difference expression is

$$\mu_s(G)\mu_s(B)\Big(p(v_i|G) - p(v_i|B)\Big)^2 \geq 0.$$

Thus, we have $p_s(v_i|v_i) \geq p_s(v_i|v_j)$.

Proof of Corollary 1:

1) When $\phi(\mu_s(G)) \geq 0$, the equilibrium strategies for both value types are identified in Proposition 1. It's observed that the two value types employ a uniform bidding strategies on two connected intervals. Thus, it's easy to obtain that the expected bids from the two types are

$$E^{s,m}(x|v_l) = \frac{1}{2} v_l p_s(v_l|v_l);$$

$$E^{s,m}(x|v_h) = \frac{1}{2}v_h p_s(v_h|v_h) + v_l p_s(v_l|v_l).$$

The expected total bids is

$$\begin{aligned}
R^m(\mu_s) =& 2\sum_{\omega\in\{G,B\}} \mu_s(\omega) \underbrace{\left[p(v_h|\omega)E^{s,m}(x|v_h,s) + p(v_l|\omega)E^{s,m}(x|v_l,s)\right]}_{\text{revenue when state is } \omega \text{ and posterior is } \mu_s} \\
=& \sum_{\omega\in\{G,B\}} \mu_s(\omega)\Big[p(v_h|\omega)(v_h p_s(v_h|v_h) + v_l p_s(v_l|v_l)) + v_l p_s(v_l|v_l)\Big] \\
=& v_l p_s(v_l|v_l) + \Big(v_h p_s(v_h|v_h) + v_l p_s(v_l|v_l)\Big) \sum_{\omega\in\{G,B\}} \mu_s(\omega) p(v_h|\omega).
\end{aligned}$$

2) When $\phi(\mu_s(G)) < 0$, in equilibrium the low value type's strategy is a uniform distribution on $[0, \underline{x}(s)]$, and the high type's strategy is a piecewise-uniform randomization on $[0, v_h]$. For a low value type player, it's very direct and easy to obtain his expected bid, which is

$$E^{s,nm}(x|v_l) = \frac{1}{2}\underline{x}(s),$$

For a high value type player, its piecewise-uniform randomization strategy is identified in Proposition 1. Let

$$\bar{x}(s) = \frac{v_h v_l[p_s(v_h|v_h) - p_s(v_h|v_l)]}{v_l p_s(v_l|v_l) - v_h p_s(v_l|v_h)}.$$

The expected bid of a high value type player is

$$\begin{aligned}
E^{s,nm}(x|v_h) =& \int_0^{\underline{x}(s)} x d\frac{x}{\bar{x}(s)} + \int_{\underline{x}(s)}^{v_h} x d\frac{x - v_h p_s(v_l|v_h)}{v_h p_s(v_h|v_h)} \\
=& \frac{1}{2\bar{x}(s)} \cdot \underline{x}^2(s) + \frac{1}{v_h p_s(v_h|v_h)} \cdot \frac{1}{2}(v_h^2 - \underline{x}^2(s)) \\
=& \frac{1}{2} \cdot \frac{1}{\bar{x}(s)} \cdot \underline{x}^2(s) + \frac{1}{v_h p_s(v_h|v_h)} \cdot \frac{1}{2}(v_h - \underline{x}(s))(v_h + \underline{x}(s)).
\end{aligned}$$

Substitute the expression of $\underline{x}(s)$ into the the expression above, we have

$$E^{s,nm}(x|v_h) = \frac{1}{2\bar{x}(s)} \cdot \underline{x}^2(s) + \frac{1}{2} \cdot \frac{(v_h - v_l)}{v_h p_s(v_h|v_h) - v_l p_s(v_h|v_l)}(v_h + \underline{x}(s)).$$

On the other hand, by the definition of $\underline{x}(s)$ and $\bar{x}(s)$, we have

$$\underline{x}(s)/\bar{x}(s) = \frac{v_l p_s(v_l|v_l) - v_h p_s(v_l|v_h)}{v_h p_s(v_h|v_h) - v_l p_s(v_h|v_l)}.$$

Substitute it into the expression of $E^{s,nm}(x|v_h)$ we obtain

$$\begin{aligned}
2E^{s,nm}(x|v_h) =& \frac{v_l p_s(v_l|v_l) - v_h p_s(v_l|v_h)}{v_h p_s(v_h|v_h) - v_l p_s(v_h|v_l)} \cdot \underline{x}(s) + \frac{(v_h - v_l)}{v_h p_s(v_h|v_h) - v_l p_s(v_h|v_l)}(v_h + \underline{x}(s)) \\
=& \underline{x}(s) + \frac{v_h(v_h - v_l)}{v_h p_s(v_h|v_h) - v_l p_s(v_h|v_l)},
\end{aligned}$$

which implies that

$$E^{s,nm}(x|(v_h,)) = \frac{1}{2}\underline{x}(s) + \frac{1}{2}\frac{v_h(v_h - v_l)}{v_h p_s(v_h|v_h) - v_l p_s(v_h|v_l)}.$$

Thus, the expected total bids is

$$\begin{aligned} R^{s,nm}(\mu_s) =& 2\sum_{\omega\in\{G,B\}} \mu_s(\omega) \underbrace{\left[p(v_h|\omega)E^{s,nm}(x|v_h,s) + p(v_l|\omega)E^{s,nm}(x|v_l,s)\right]}_{\text{revenue when state is } \omega \text{ and posterior is } \mu_s} \\ =& \underline{x}(s) + \frac{v_h(v_h - v_l)}{v_h p_s(v_h|v_h) - v_l p_s(v_h|v_l)} \cdot \sum_{\omega\in\{G,B\}} \mu_s(\omega) p(v_h|\omega). \end{aligned}$$

Proof of Lemma 1:

1. **Step 1**: Rewrite function ϕ as

   $$\phi(x) = (v-1)(1-\alpha) + (\alpha+\beta-1)(1-x)\left[\frac{v(1-\beta)}{\alpha x + (1-\beta)(1-x)} - \frac{\beta}{(1-\alpha)x + \beta(1-x)}\right]$$

   The first order derivative is

   $$\phi'(x) = (\alpha+\beta-1)\cdot\left\{\frac{(1-\alpha)\beta}{[(1-\alpha)x + \beta(1-x)]^2} - \frac{v\alpha(1-\beta)}{[\alpha x + (1-\beta)(1-x)]^2}\right\}.$$

   Then $\phi'(x) \geq 0$ if and only if

   $$\frac{\sqrt{(1-\alpha)\beta}}{(1-\alpha)x + \beta(1-x)} \geq \frac{\sqrt{v\alpha(1-\beta)}}{\alpha x + (1-\beta)(1-x)},$$

   i.e.,

   $$(\alpha+\beta-1)(\sqrt{(1-\alpha)\beta} + \sqrt{v}\sqrt{\alpha(1-\beta)})x \geq \sqrt{\beta(1-\beta)}(\sqrt{v}\sqrt{\alpha\beta} - \sqrt{(1-\alpha)(1-\beta)}).$$

   Since $\alpha+\beta \geq 1$, we always have $\sqrt{v}\sqrt{\alpha\beta} - \sqrt{(1-\alpha)(1-\beta)} \geq 0$. That is, the left-hand side of the inequality above is always positive. Thus, there exists x_0 such that $\phi'(x_0) = 0$. And it's also unqiue.

   Step 2: $\phi'(x_0) = 0$ implies

   $$\frac{\sqrt{(1-\alpha)\beta}}{(1-\alpha)x_0 + \beta(1-x_0)} = \frac{\sqrt{v\alpha(1-\beta)}}{\alpha x_0 + (1-\beta)(1-x_0)}. \tag{A.1}$$

   $\phi(x_0) = 0$ implies

   $$v\cdot\frac{\alpha(1-\alpha)x_0 + \beta(1-\beta)(1-x_0)}{\alpha x_0 + (1-\beta)(1-x_0)} = \frac{(1-\alpha)^2 x_0 + \beta^2(1-x_0)}{(1-\alpha)x_0 + \beta(1-x_0)}. \tag{A.2}$$

Combine equation (A.1) and (A.2), we have

$$\sqrt{v}\cdot\frac{\alpha(1-\alpha)x_0+\beta(1-\beta)(1-x_0)}{(1-\alpha)^2x_0+\beta^2(1-x_0)}=\frac{\sqrt{\alpha(1-\beta)}}{\sqrt{(1-\alpha)\beta}}.$$

Rearrange it, we get

$$\begin{aligned}&\alpha^{1/2}(1-\alpha)^{3/2}\Big(\sqrt{v}\sqrt{\alpha\beta}-\sqrt{(1-\alpha)(1-\beta)}\Big)x_0\\=&\beta^{3/2}(1-\beta)^{1/2}\Big(\sqrt{\alpha\beta}-\sqrt{v}\sqrt{(1-\alpha)(1-\beta)}\Big)(1-x_0).\end{aligned}\tag{A.3}$$

Rearrange equation (A.1), we have

$$\sqrt{\alpha(1-\alpha)}\Big(\sqrt{\alpha\beta}-\sqrt{v}\sqrt{(1-\alpha)(1-\beta)}\Big)x_0=\sqrt{\beta(1-\beta)}\Big(\sqrt{v}\sqrt{\alpha\beta}-\sqrt{(1-\alpha)(1-\beta)}\Big)(1-x_0).\tag{A.4}$$

Combine equation (A.3) and (A.4), it follows that

$$\frac{(1-\alpha)\Big(\sqrt{v}\sqrt{\alpha\beta}-\sqrt{(1-\alpha)(1-\beta)}\Big)}{\Big(\sqrt{\alpha\beta}-\sqrt{v}\sqrt{(1-\alpha)(1-\beta)}\Big)}=\frac{\beta\Big(\sqrt{\alpha\beta}-\sqrt{v}\sqrt{(1-\alpha)(1-\beta)}\Big)}{\Big(\sqrt{v}\sqrt{\alpha\beta}-\sqrt{(1-\alpha)(1-\beta)}\Big)},$$

which is

$$\sqrt{v}\sqrt{(1-\alpha)\beta}=1-\sqrt{\alpha(1-\beta)}.$$

Thus we have

$$\begin{aligned}v_0=&\frac{1+\alpha(1-\beta)-2\sqrt{\alpha(1-\beta)}}{(1-\alpha)\beta}\\=&1+\frac{1-\beta+\alpha-2\sqrt{\alpha(1-\beta)}}{(1-\alpha)\beta}\geq 1.\end{aligned}$$

If $v\geq v_0$, then $\phi(x_0)\geq 0$. Since $\phi(x_0)$ is the minimum point of $\phi(x)$ on $[0,\infty)$, we have $\phi(\mu(G))\geq 0$ always holds.

2. If $v<v_0$, then the minimum point $\phi(x_0)<0$.

Claim 2. *If $v<v_0$, $\phi'(1)>0$ and $\phi(1)>0$.*

Proof. From the proof of Lemma 1,

$$\phi'(1)=(\alpha+\beta-1)\cdot(\frac{\beta}{1-\alpha}-\frac{v(1-\beta)}{\alpha}).$$

Thus, $\phi'(1) \geq 0$ if and only if $v \leq \frac{\alpha\beta}{(1-\alpha)(1-\beta)} = v_{00}$. Then we have

$$v_{00} - v_0 = \frac{\alpha+\beta-1}{(1-\alpha)(1-\beta)} - \frac{(\sqrt{\alpha}-\sqrt{1-\beta})^2}{(1-\alpha)\beta}.$$

Rearrange it, we get

$$v_{00} - v_0 = \frac{\alpha+\beta-1}{(1-\alpha)(1-\beta)} - \frac{(\alpha+\beta-1)(\sqrt{\alpha}-\sqrt{1-\beta})}{(1-\alpha)\beta(\sqrt{\alpha}+\sqrt{1-\beta})}$$

$$v_{00} - v_0 = \frac{\alpha+\beta-1}{1-\alpha}\left\{\frac{1}{1-\beta} - \frac{\sqrt{\alpha}-\sqrt{1-\beta}}{\beta(\sqrt{\alpha}+\sqrt{1-\beta})}\right\}$$

Since

$$\begin{aligned}\beta(\sqrt{\alpha}+\sqrt{1-\beta}) - (1-\beta)(\sqrt{\alpha}-\sqrt{1-\beta}) &\geq 2\beta\sqrt{1-\beta} - (1-\beta)(\sqrt{\alpha}-\sqrt{1-\beta}) \\ &\geq \sqrt{1-\beta}\left\{\beta - \sqrt{1-\beta}\sqrt{\alpha}+1\right\} > 0,\end{aligned}$$

we have $v_{00} > v_0$. Thus, when $v < v_0 < v_{00}$, we have $\phi'(1) \geq 0$. On the other hand, $\phi(1) = (v-1)(1-\alpha) > 0$. □

By the proof of Claim 2 and the shape of ϕ, we learn that there exist x_1 and x_2 in $(0,1)$ such that $\phi(x_1) = 0$ and $\phi(x_2) = 0$. And for any $\mu_s(G) \in (x_1, x_2)$, $\phi(\mu_s(G)) < 0$.

Proof of Lemma 2: Recall that

$$R^m(\mu_s) = (p_s(v_h|v_h)v_h + p_s(v_l|v_l)v_l)\Big[\mu_s(G)\alpha + (1-\mu_s(G))(1-\beta)\Big] + p_s(v_l|v_l)v_l.$$

Let $x = \mu_s(G)$, and

$$p_s(v_l|v_l) = \frac{(1-\alpha)^2 x + \beta^2(1-x)}{(1-\alpha)x + \beta(1-x)} = g(x);$$

$$p_s(v_h|v_h) = \frac{\alpha^2 x + (1-\beta)^2(1-x)}{\alpha x + (1-\beta)(1-x)} = h(x).$$

Then we have

$$\frac{R^m(x)}{v_l} = (h(x)\cdot v + g(x))\Big[x\alpha + (1-x)(1-\beta)\Big] + g(x).$$

The first order derivative is

$$\frac{dR^m(x)/v_l}{dx} = (h'(x)\cdot v + g'(x))\Big[x\alpha + (1-x)(1-\beta)\Big] + (h(x)\cdot v + g(x))(\alpha+\beta-1) + g'(x).$$

The second order derivative is

$$\frac{d^2 R^m(x)/v_l}{dx^2} = (h''(x) \cdot v + g''(x))\Big[x\alpha + (1-x)(1-\beta)\Big] + 2(h'(x) \cdot v + g'(x))(\alpha+\beta-1) + g''(x).$$

On the other hand,

$$h'(x) = \frac{(\alpha+\beta-1)\alpha(1-\beta)}{[\alpha x + (1-\beta)(1-x)]^2},$$
$$h''(x) = \frac{(\alpha+\beta-1)^2\alpha(1-\beta)(-2)}{[\alpha x + (1-\beta)(1-x)]^3},$$

it follows that

$$h''(x)\Big[x\alpha + (1-x)(1-\beta)\Big] + 2h'(x)(\alpha+\beta-1) = 0.$$

Thus, we have

$$\begin{aligned}\frac{d^2 R^m(x)/v_l}{dx^2} =& g''(x)\Big[x\alpha + (1-x)(1-\beta)\Big] + 2g'(x)(\alpha+\beta-1) + g''(x) \\ =& -g''(x)\Big[(1-\alpha)x + \beta(1-x)\Big] + 2g'(x)(\alpha+\beta-1) + 2g''(x).\end{aligned} \tag{A.5}$$

Since

$$g'(x) = \frac{-(\alpha+\beta-1)(1-\alpha)\beta}{[(1-\alpha)x + \beta(1-x)]^2},$$
$$g''(x) = \frac{-2(\alpha+\beta-1)^2(1-\alpha)\beta}{[(1-\alpha)x + \beta(1-x)]^3},$$

it follows that

$$-g''(x)[(1-\alpha)x + \beta(1-x)] + 2g'(x)(\alpha+\beta-1) = 0.$$

Then we have

$$\frac{d^2 R^m(x)/v_l}{dx^2} = 2g''(x) \le 0.$$

Therefore, $R^m(\mu)$ is concave.

Proof of Lemma 3: By Corollary 1, the organizer's expected revenue in a non-monotone equilibrium induced by μ_s is

$$R^{nm}(\mu_s) = \underline{x}(s) + \frac{v_h(v_h - v_l)}{v_h p_s(v_h|v_h) - v_l p_s(v_h|v_l)}\Big[\mu_s(G)\alpha + (1-\mu_s(G))(1-\beta)\Big].$$

Recall that

$$\underline{x}(s) = \frac{v_h v_l[p_s(v_h|v_h) - p_s(v_h|v_l)]}{v_h p_s(v_h|v_h) - v_l p_s(v_h|v_l)},$$

substitute it into the equation above, we obtain

$$R^{nm}(\mu_s) = \frac{v_h[p_s(v_h|v_h) - p_s(v_h|v_l)]}{vp_s(v_h|v_h) - p_s(v_h|v_l)} + \frac{v_h(v-1)\left[\mu_s(G)\alpha + (1-\mu_s(G))(1-\beta)\right]}{vp_s(v_h|v_h) - p_s(v_h|v_l)}.$$

Since

$$\begin{aligned}\phi(\mu_s(G)) =& vp_s(v_l|v_h) - p_s(v_l|v_l)\\ =& v - 1 - (vp_s(v_h|v_h) - p_s(v_h|v_l)),\end{aligned}$$

we can rewrite $R^{nm}(\mu_s)$ as

$$R^{nm}(\mu_s) = v_l\frac{v[p_s(v_h|v_h) - p_s(v_h|v_l)]}{v-1-\phi(\mu_s(G))} + \frac{v_h(v-1)\left[\mu_s(G)\alpha + (1-\mu_s(G))(1-\beta)\right]}{v-1-\phi(\mu_s(G))}.$$

On the other hand,

$$v[p_s(v_h|v_h) - p_s(v_h|v_l)] = vp_s(v_h|v_h) - p_s(v_h|v_l) - (v-1)p_s(v_h|v_l),$$

we have

$$\begin{aligned}R^{nm}(\mu_s) =& v_l\frac{v-1-\phi(\mu_s(G)) - (v-1)p_s(v_h|v_l)}{v-1-\phi(\mu_s(G))} + \frac{v_h(v-1)\left[\mu_s(G)\alpha + (1-\mu_s(G))(1-\beta)\right]}{v-1-\phi(\mu_s(G))}\\ =& v_l\frac{(v-1)p_s(v_l|v_l) - \phi(\mu_s(G))}{v-1-\phi(\mu_s(G))} + \frac{v_h(v-1)\left[\mu_s(G)\alpha + (1-\mu_s(G))(1-\beta)\right]}{v-1-\phi(\mu_s(G))}.\end{aligned}$$

Split v_h as following

$$\begin{aligned}v_h =& p_s(v_h|v_h)v_h + p_s(v_l|v_h)v_h\\ =& p_s(v_h|v_h)v_h + v_l(\phi(\mu_s(G)) + p_s(v_l|v_l)).\end{aligned}$$

Then we have

$$\begin{aligned}R^{nm}(\mu_s) =& v_l\frac{(v-1)p_s(v_l|v_l) - \phi(\mu_s(G))}{v-1-\phi(\mu_s(G))}\\ &+ \frac{(v-1)\left[p_s(v_h|v_h)v_h + v_l(\phi(\mu_s(G)) + p_s(v_l|v_l))\right]\left[\mu_s(G)\alpha + (1-\mu_s(G))(1-\beta)\right]}{v-1-\phi(\mu_s(G))}.\end{aligned}$$

Since $R^m(\mu_s) = (p_s(v_h|v_h)v_h + p_s(v_l|v_l)v_l)\left[\mu_s(G)\alpha + (1-\mu_s(G))(1-\beta)\right] + p_s(v_l|v_l)v_l$, it follows that

$$R^{nm}(\mu_s) = \frac{(v-1)R^m(\mu_s) + (v-1)v_l\phi(\mu_s(G))\left[\mu_s(G)\alpha + (1-\mu_s(G))(1-\beta)\right] - v_l\phi(\mu_s(G))}{v-1-\phi(x)},$$

which is

$$R^{nm}(\mu_s) = R^m(\mu_s) + \frac{\phi(\mu_s(G))}{v-1-\phi(\mu_s(G))} \cdot \Big\{R^m(\mu_s) + v_l\Big\{(v-1)\Big[\mu_s(G)\alpha + (1-\mu_s(G))(1-\beta)\Big] - 1\Big\}\Big\}.$$

If $\phi(\mu_s(G)) = 0$, then we have $R^{nm}(\mu_s) = R^m(\mu_s)$.

Proof of Lemma 4: In the proof of lemma 3, we've shown that

$$R^{nm}(\mu_s) = v_l \frac{v-1-\phi(\mu_s(G)) - (v-1)p_s(v_h|v_l)}{v-1-\phi(\mu_s(G))} + \frac{v_h(v-1)\Big[\mu_s(G)\alpha + (1-\mu_s(G))(1-\beta)\Big]}{v-1-\phi(\mu_s(G))},$$

which can be simplified as

$$\begin{aligned} R^{nm}(\mu_s) =& v_l - \frac{v_l(v-1)p_s(v_h|v_l)}{v-1-\phi(\mu_s(G))} + \frac{v_h(v-1)\Big[\mu_s(G)\alpha + (1-\mu_s(G))(1-\beta)\Big]}{v-1-\phi(\mu_s(G))} \\ =& v_l + \frac{v_l(v-1)}{v-1-\phi(\mu_s(G))} \cdot \Big\{v\Big[\mu_s(G)\alpha + (1-\mu_s(G))(1-\beta)\Big] - p_s(v_h|v_l)\Big\}. \end{aligned}$$

Split v as $v = v-1-\phi(\mu_s(G)) + (1+\phi(\mu_s(G)))$, then we have

$$\begin{aligned} R^{nm}(\mu_s) =& v_l + (v_h - v_l)\Big[\mu_s(G)\alpha + (1-\mu_s(G))(1-\beta)\Big] \\ &+ \frac{(v_h - v_l)}{v-1-\phi(\mu_s(G))} \cdot \Big\{(1+\phi(\mu_s(G)))\Big[\mu_s(G)\alpha + (1-\mu_s(G))(1-\beta)\Big] - p_s(v_h|v_l)\Big\}. \end{aligned}$$

Since

$$1 + \phi(x) = 1 + vp_s(v_l|v_h) - p_s(v_l|v_l) = vp_s(v_l|v_h) + p_s(v_h|v_l),$$

we have

$$\begin{aligned} R^{nm}(\mu_s) =& v_l + (v_h - v_l)\Big[\mu_s(G)\alpha + (1-\mu_s(G))(1-\beta)\Big] \\ &+ \frac{(v_h - v_l)}{v-1-\phi(\mu_s(G))} \cdot \Big\{(vp_s(v_l|v_h) + p_s(v_h|v_l))\Big[\mu_s(G)\alpha + (1-\mu_s(G))(1-\beta)\Big] - p_s(v_h|v_l)\Big\}, \end{aligned}$$

which is

$$\begin{aligned} R^{nm}(\mu_s) =& v_l + (v_h - v_l)\Big[\mu_s(G)\alpha + (1-\mu_s(G))(1-\beta)\Big] + \frac{(v_h - v_l)}{v-1-\phi(\mu_s(G))} \cdot \\ & \Big\{vp_s(v_l|v_h)\Big[\mu_s(G)\alpha + (1-\mu_s(G))(1-\beta)\Big] - p_s(v_h|v_l)\Big[(1-\alpha)\mu_s(G) + \beta(1-\mu_s(G))\Big]\Big\}. \end{aligned}$$

Recall that

$$p_s(v_l|v_h) = \frac{\alpha(1-\alpha)\mu_s(G) + (1-\beta)\beta(1-\mu_s(G))}{\alpha\mu_s(G) + (1-\beta)(1-\mu_s(G))},$$

$$p_s(v_h|v_l) = \frac{\alpha(1-\alpha)\mu_s(G) + (1-\beta)\beta(1-\mu_s(G))}{(1-\alpha)\mu_s(G) + \beta(1-\mu_s(G))},$$

so we have

$$R^{nm}(\mu_s) = v_l + (v_h - v_l)\Big[\mu_s(G)\alpha + (1-\mu_s(G))(1-\beta)\Big] + \frac{v_l(v-1)^2}{v-1-\phi(\mu_s(G))} \cdot \Big[\alpha(1-\alpha)\mu_s(G) + \beta(1-\beta)(1-\mu_s(G))\Big].$$

Since $\phi(\mu_s(G)) \leq 0$, we have

$$\begin{aligned} R^{nm}(\mu_s) \leq & v_l + (v_h - v_l)\Big[\mu_s(G)\alpha + (1-\mu_s(G))(1-\beta)\Big] \\ & + (v_h - v_l)\Big[\alpha(1-\alpha)\mu_s(G) + \beta(1-\beta)(1-\mu_s(G))\Big] \\ = & v_h - (v_h - v_l)\Big[\mu_s(G)(1-\alpha) + (1-\mu_s(G))\beta\Big] \\ & + (v_h - v_l)\Big[\alpha(1-\alpha)\mu_s(G) + \beta(1-\beta)(1-\mu_s(G))\Big] \\ = & v_h + (v_h - v_l) \cdot \Big[(\beta^2 - (1-\alpha)^2)\mu_s(G) - \beta^2\Big]. \end{aligned}$$

The equality holds when $\phi(\mu_s(G)) = 0$.

References

[1] Amann, E., & Leininger, W. (1996). Asymmetric all-pay auctions with incomplete information: the two-player case. Games and Economic Behavior, 14(1), 1-18.

[2] Barut, Y., & Kovenock, D. (1998). The symmetric multiple prize all-pay auction with complete information. European Journal of Political Economy, 14(4), 627-644.

[3] Baye, M. R., Kovenock, D., & De Vries, C. G. (1993). Rigging the lobbying process: an application of the all-pay auction. The american economic review, 83(1), 289-294.

[4] Baye, M. R., Kovenock, D., & De Vries, C. G. (1996). The all-pay auction with complete information. Economic Theory, 8(2), 291-305.

[5] Chen, Z. C. (2019). Information Disclosure in Contests: Private versus Public Signals. Available at SSRN 3326462.

[6] Chi, C. K., Murto, P., & Välimäki, J. (2019). All-pay auctions with affiliated binary signals. Journal of Economic Theory, 179, 99-130.

[7] Fu, Q., Jiao, Q., & Lu, J. (2011). On disclosure policy in contests with stochastic entry. Public Choice, 148(3-4), 419-434.

[8] Fu, Q., Jiao, Q., & Lu, J. (2014). Disclosure policy in a multi-prize all-pay auction with stochastic abilities. Economics Letters, 125(3), 376-380.

[9] Hillman, A. L., & Riley, J. G. (1989). Politically contestable rents and transfers. Economics & Politics, 1(1), 17-39.

[10] Liu, Z., & Chen, B. (2016). A symmetric two-player all-pay contest with correlated information. Economics Letters, 145, 6-10.

[11] Lu, J., Ma, H., & Wang, Z. (2018). Ranking Disclosure Policies In All-Pay Auctions. Economic Inquiry, 56(3), 1464-1485.

[12] Lu, J., & Parreiras, S. O. (2017). Monotone equilibrium of two-bidder all-pay auctions Redux. Games and Economic Behavior, 104, 78-91.

[13] Mathevet, L., Perego, J., & Taneva, I. (2017). On information design in games. Unpublished paper, Department of Economics, New York University.[1115].

[14] Morath, F., & Münster, J. (2008). Private versus complete information in auctions. Economics Letters, 101(3), 214-216.

[15] Kamenica, E., & Gentzkow, M. (2011). Bayesian persuasion. American Economic Review, 101(6), 2590-2615.

[16] Kamenica, E. (2018). Bayesian persuasion and information design. Annual Review of Economics, 11.

[17] Konrad, K. A., & Morath, F. (2018). To deter or to moderate? Alliance formation in contests with incomplete information. Economic Inquiry, 56(3), 1447-1463.

[18] Kovenock, D., Morath, F., & Münster, J. (2015). Information sharing in contests. Journal of Economics & Management Strategy, 24(3), 570-596.

[19] Krishna, V., & Morgan, J. (1997). An analysis of the war of attrition and the all-pay auction. journal of economic theory, 72(2), 343-362.

[20] Rentschler, L., & Turocy, T. L. (2016). Two-bidder all-pay auctions with interdependent valuations, including the highly competitive case. Journal of Economic Theory, 163, 435-466.

[21] Siegel, R. (2014). Asymmetric all-pay auctions with interdependent valuations. Journal of Economic Theory, 153, 684-702.

[22] Szech, N. (2011). Asymmetric all-pay auctions with two types. University of Bonn, Discussion paper, January.

[23] Zhang, J., & Zhou, J. (2016). Information disclosure in contests: A Bayesian persuasion approach. The Economic Journal, 126(597), 2197-2217.