Deposit Requirements in Auctions*

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Abstract

We examine optimal auction design when buyers may receive future outside offers. The winning bidder may choose to default upon observing her outside offer. Under the optimal mechanism, the bidder with the highest value wins if and only if her value is above a cutoff, and the winner never defaults. The optimal auction takes the form of a second-price auction with a reserve price and a deposit by the winning bidder. Under regularity conditions, both the optimal reserve price and the deposit increase when the distribution of outside offers worsens.

Keywords: Truthful direct mechanism; full compliance; outside offers; deposit requirements; reserve prices.
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It often takes time for the winning buyer and the seller to complete the transaction with high-value auction items after the auction concludes. During this time, the winner may receive an alternative outside offer and renege on the original transaction if the outside offer is better.[1] In such a situation, it is a common practice for the seller to demand a non-refundable deposit from the winning bidder before settling the final transaction. If the winner defaults on the transaction, the deposit is forfeited. Deposit requirements (normally capped at a certain amount) have been widely adopted by sellers in practice. For instance, sellers in eBay auto auctions are allowed to set a deposit that is no more than $2,000$ US dollars. After paying the deposit, the winners can make final payments within a week or ten days. In UK real estate auctions, a buyer is required to pay a deposit of $6,000$ UK pounds immediately after winning; the buyer then has to make the final payment within twenty eight days.

It is, therefore, worthwhile to examine the impact of a deposit requirement on bidding strategy and the seller's revenue in the auction. A deposit has a direct effect that compensates the seller by making post-auction default costly. However, there is an indirect effect, as bidders who account for a possible future default will adjust their bids in response to the deposit requirement. It is not clear whether the seller benefits from requiring a deposit, and if so, at what level the deposit should be set. In addition, a large body of literature following Myerson (1981) has shown that a reserve price helps screen out low-value bidders and improve seller revenue. We explore the different roles of, and the links between the reserve price and the deposit requirement.

Our investigation starts by examining a class of truthful direct mechanisms, which can be described as follows: A seller would like to maximize revenue by allocating a single object among some bidders. Each bidder's private value is independently drawn from a common distribution. In the first stage, bidders report their values, and at most, one winner is selected to enter the second stage. All bidders discover their random outside offers in the second stage. In the second stage, the chosen winner reports her outside price, and the mechanism decides whether the winner completes the transaction or takes the outside offer. Relying on the necessary condition for first-stage incentive compatibility, we establish an upper bound for seller revenue under this class of mechanisms. Moreover, the upper bound of revenue entails the following allocation rules: In the first stage, a bidder with the highest value is selected as the winner if and only if her private value is no less than a cutoff value, and in the second stage, the winner never defaults.

[1] A typical example is online auction marketplaces, i.e., eBay, where auctions are listed regularly and sequentially, and it is easy for buyers to conduct post-auction searches for better outside offers before the final transaction. In this case, sellers are faced with buyers' commitment issues. Resnick and Zeckhauser (2002) observe that the most common complaint by sellers in online auctions is that winning bidders do not follow through with the transactions. Dellarocas and Wood (2008) find that 81 percent of the negative feedback given to buyers in eBay auctions results from "bidders who backed out of their commitment to buy the items they won."

We next examine whether and how the upper bound of the seller's revenue can be achieved through a second-price auction with a winner-pay deposit and a reserve price. A winner-pay deposit increases default cost and therefore lowers the winner's incentive to exercise the outside offer. In the auction, the seller first announces the deposit and the reserve price. All bidders place their bids simultaneously, and the bidder with the highest bid wins (given that her bid is no less than the reserve price). The winning bidder then decides whether to pay the deposit before the random price of the outside offer is realized.[2] After the deposit is paid, the winning bidder can either complete the original transaction or take the outside offer and forfeit the deposit.[3] If the deposit is not paid, the winner only decides whether to take the outside offer. When the winner defaults on the original transaction after paying the deposit, she loses the deposit, and the seller keeps the object.

To better understand the impact of the deposit and the reserve price on equilibrium bidding strategies and seller revenue, we first study the optimal reserve price for each level of deposit and then examine the overall effect of a change in the deposit (with the optimal reserve price as a function of the deposit). We find that seller revenue strictly increases with the deposit until a specific cutoff value and then becomes flat. Maximal seller revenue, which equals the upper bound of revenue identified above, is achieved when the deposit is set sufficiently high such that the winner is deterred from defaulting. This result, in turn, indicates that a second-price auction with the identified deposit and reserve price is essentially the optimal design among the class of mechanisms considered.

The intuition of our result is as follows: bidders lower their bids when facing a higher deposit requirement in the auction. But setting a reserve price as the minimum bid excludes low-value bidders, which helps offset the negative effect of the deposit on seller revenue. Thus, it turns out that an auction with relatively low bids and full compliance is more profitable than one with high bids but partial compliance. Consequently, the seller raises the required deposit until the possibility of ex-post default decreases to zero.

To the best of our knowledge, our study is the first to examine the impact of deposit requirements on bidding strategies and seller revenue. Our paper is related to the literatures on auctions without buyer commitment and auctions with outside options, which we now review.

Auctions without buyer commitment. First, our paper is related to the growing literature on auctions without buyer commitment. Asker (2000) studies an auction model in which bidders face

[2] In the auction, the price of the outside offer can be private information for the winning bidder.

[3] In our model, we assume that if the final price is less than the deposit, the seller commits to pay back the difference when the winner chooses to complete the transaction.

uncertainty regarding their final valuations of the object, and this uncertainty can only be resolved after bidding has taken place. He shows that the inclusion of a withdrawal right (allowing the winner to default) raises the seller's expected revenue. Zheng (2001, 2009) considers the situation where bidders facing budget constraints can default on their bids. He shows that the default risk induced by financial constraints affects both equilibrium bidding strategies and seller revenues in auctions. Engelmann, Frank, Koch, and Valente (2015) study an auction model in which the seller can make a second-chance offer to the second-highest bidder if the auction winner fails to complete the transaction. Their analysis shows that the availability of such an offer reduces bidders' willingness to bid in the auction, thus lowering the seller's revenue, even when no default takes place.

Krähmer and Strausz (2015) investigate the effects of the withdrawal right on optimal sales contracts that involve only one buyer and one seller. In their contracting environment, the buyer, after having observed her private valuation, has the choice of either exercising her option as specified in the contract or withdrawing from it and choosing her outside option. Their findings show that the inclusion of default rights is equivalent to introducing ex-post participation constraints in the sequential screening model; even though sequential screening is still feasible with ex-post participation constraints, the seller no longer benefits from it. Instead, the optimal selling contract is static and coincides with the optimal posted price contract in the static screening model.[4]

Our study is closely related to that of Armstrong and Zhou (2016), who study optimal search deterrence in a one-seller-and-one-buyer setting. In their model, the buyer incurs a cost to search for an outside option. They primarily focus on the seller's choice between a buy-it-now discount offer and an exploding offer and study the optimal selling mechanism. Their analysis reveals that, at the optimum, the seller might charge a non-refundable deposit. Our paper is also linked to the literature on auction design with an optimal search.[5] One of the most important findings is from Crémer, Spiegel, and Zheng (2007, 2009), who consider the optimal auction design when the seller controls the set of participating bidders. They show that the optimal selling mechanism would feature fewer participants and more extended searches. Moreover, their papers assume the winner cannot recall the original transaction. On the contrary, our paper considers the case in which the winner can hold the object by paying the deposit and then compare it with the outside offer.

Although our study and the literature on auctions without buyer commitment share a common feature that the auction winner may renege on the original transaction, the focus of our paper differs. We provide a rationale for the seller's adoption of a deposit, which is commonly observed in real-world auctions, and characterize how the seller should set both the deposit and the reserve

[4]For example, see related studies by Ben-Shahar and Posner (2011) and Eidenmüller (2011).

[5]For example, see related studies from McAfee and McMillan (1988); Burguet (1996); Lee and Li (2020).

price in the auction.

Auctions with outside options. Our paper is also part of the literature on auctions with outside options. Cherry, Frykblom, Shogren, List, and Sullivan (2004) conduct a lab experiment to examine whether bidders consider the existence of outside options when formulating their bidding strategies in second-price auctions. Their results show that bidders reduce their bids whenever their resale values exceed the price of the outside option. Kirchkamp, Poen, and Reiss (2009) study the equilibrium bidding behavior of bidders in first-price and second-price auctions with outside options. They show that first-price auctions yield more revenue to sellers than second-price auctions, which may explain why first-price auctions are more common in practice.

Lauermann and Virág (2012) study how the presence of outside options influences whether an auctioneer prefers "opaque" or "transparent" auctions, which differ based on the information that bidders receive. They show that an auctioneer might choose opaque auctions to reduce the values of the bidders' outside options. Figueroa and Skreta (2007, 2009) examine revenue-maximizing auctions for multiple objects, where bidders' outside options depend on their private information and are endogenously chosen by the seller. They show that an optimal mechanism may or may not allocate the objects efficiently.

The existing models consider either only losing bidders having ex-post outside options or all bidders knowing their outside options before bidding. We instead consider a scenario where outside offers arrive after all bidders have submitted their bids, and they are available to all bidders, including the auction winner (i.e., the winner can choose to either complete the original transaction or default and take the outside offer). In addition, our analysis focuses on the role of a deposit requirement in such an environment.

The rest of the paper is organized as follows. In Section I., we present the model setup and analyze the upper bound on seller revenue. Section II. examines how a second-price auction with a deposit and a reserve price can achieve the revenue upper bound. Section III. discusses distributions of the outside offers and the robustness of our results when relaxing some assumptions. Section IV. concludes the paper. The proofs of our main results are in the Appendix, and all the other non-essential proofs and computational details are included in Appendices S1 and S2, which are for online publication.

I. The Model and Analysis of General Mechanism

There are N potential risk-neutral bidders, where $1 < N < \infty$, who compete for an indivisible object. The seller's reservation value of the object is normalized to be zero. Bidders' private values,

denoted by v_i, $i = 1, 2, ..., N$, are independent draws from a common atomless distribution $F(\cdot)$ with density $f(\cdot) > 0$ over the support of $[0, \bar{v}]$, where $\bar{v} > 0$. After the competition, each bidder i receives her own outside offer, which gives the same object (thus the same value v_i) but with a random price, denoted by p_i. Prices p_i, $i = 1, 2, ..., N$, are random draws from a common atomless distribution $\Phi(\cdot)$ with density $\varphi(\cdot) > 0$ over $[0, \bar{v}]$. Bidder i's v_i and p_i are independent and private information. We assume that F and Φ are common knowledge among the seller and bidders, and they are *regular* in the sense that the hazard rates $\frac{f(\cdot)}{1-F(\cdot)}$ and $\frac{\varphi(\cdot)}{1-\Phi(\cdot)}$ are increasing.

Instead of studying a specific auction format, we start by examining a class of direct mechanisms with two stages of bidder value reporting and arrival of future outside offers. As mentioned above, the purpose of doing this is to understand how to design the winner selection and the default deterrence, and more importantly, it allows us to explore the upper bound of seller revenue. In Section II., we will show that the upper bound is indeed achievable by a second-price auction with properly set reserve price and winner-pay deposit. In this sense, focusing on the second-price auction has no loss of generality.

The class of direct mechanisms we consider can be described as follows: In the *first* stage, all bidders are asked to report their values. We use $\mathbf{v}' = (v_1', v_2', ..., v_N')$ to denote the bidders' reports. Among those bidders, at most one winner is selected to participate in the second stage. The probability that bidder i is selected is denoted by $q_i^1(\mathbf{v}') \in [0, 1]$ with $\sum_i q_i^1(\mathbf{v}') \leq 1$ and bidder i's payment to the seller in the first stage is denoted by $m_i^1(\mathbf{v}') \in [0, +\infty)$.

In the *second* stage, the selected bidder observes the price p_i of her outside offer and makes a report, denoted by $p_i' \in [0, \bar{v}]$. The object is sold to the selected bidder if and only if $p_i' \geq \hat{p}_i(\mathbf{v}') \in [0, \bar{v}]$, and the associated payment to the seller in the second stage is denoted by $m_i^2(\mathbf{v}', p_i') \in [0, +\infty)$. If $p_i' < \hat{p}_i(\mathbf{v}')$, the current transaction is not completed and the selected bidder takes the outside offer by paying p_i. The probability of completing the transaction, denoted by $q_2^i(v_i', p_i'; \hat{p}_i)$, corresponding to $\hat{p}_i(v_i')$, can therefore be written as follows:[6]

$$q_i^2(v_i', p_i'; \hat{p}_i) \;=\; \begin{cases} 1 & \text{if } p_i' \;\geq\; \hat{p}_i(\mathbf{v}'); \\ 0 & \text{if } p_i' \;<\; \hat{p}_i(\mathbf{v}'). \end{cases} \tag{1}$$

We here look for the optimal truthful direct mechanism that maximizes the seller's expected revenue. In this design problem, the seller's choice variables are $\{q_i^1(\mathbf{v}'), m_i^1(\mathbf{v}'); \hat{p}_i(\mathbf{v}'), m_i^2(\mathbf{v}', p_i'); i = 1, 2, ...N\}$. Since we aim to establish an upper bound on seller revenue, in the following analysis,

[6] In fact, whether the object will be allocated to the selected bidder i in the second stage does not depend on other losing bidders' reports in the first stage, thereby implying that making the deposit a function of all the losing bids in the auction is unlikely to improve the seller's revenue.

we relax the design problem by assuming that p_i in the second stage is public information. This relaxation would weakly increase the seller's revenue and moreover, by doing so, we can ignore the individual rationality (IR) and incentive compatibility (IC) constraints in the second stage but focus on the design problem in the first stage.

I. A Implication of incentive compatibility in the first stage

Assuming that all other bidders tell the truth, let us now consider bidder i with value v_i and reporting v_i' in the first stage. In this case, the interim expected payoff of bidder i, denoted by π_i, is given by

$$\pi_i(v_i', v_i) \;=\; E_{\mathbf{v}_{-i},\, p_i}\left\{\begin{array}{c} q_i^1(v_i', \mathbf{v}_{-i})\Big[q_i^2(v_i', p_i; \hat{p}_i)v_i + (1 - q_i^2(v_i', p_i; \hat{p}_i))\max\{v_i - p_i, 0\} \;-\; m_i^2(v_i', \mathbf{v}_{-i}, p_i)\Big] \\ + (1 - q_i^1(v_i', \mathbf{v}_{-i}))\max\{v_i - p_i, 0\} - m_i^1(v_i', \mathbf{v}_{-i}) \end{array}\right\}. \tag{2}$$

where the first term in the curly brackets is the expected payoff from being selected to enter the second stage (with probability of $q_i^1(v_i', \mathbf{v}_{-i})$) and the second term in the curly brackets is the expected payoff from not being selected to enter but facing the outside offer (with probability of $1 - q_i^1(v_i', \mathbf{v}_{-i})$).

To guarantee that no one has any incentive to lie about her value, incentive compatibility (IC) requires that, for all i, for all v_i and v_i',

$$\pi_i(v_i, v_i) \geq \pi_i(v_i', v_i),$$

indicating that any untruthful reporting cannot be beneficial for the bidder in the mechanism.

When bidder i with private value v_i chooses not to make a report (or always report zero) in the first stage but faces the outside offer directly, her expected payoff from the outside offer is given by $\int_0^{v_i}(v_i - p_i)\varphi(p_i)dp_i$, which can be simplified as $\int_0^{v_i}\Phi(p_i)dp_i$. Individual rationality (IR) requires that a bidder should not be better off by not participating, that is, for all i and v_i,

$$\pi_i(v_i, v_i) \geq \int_0^{v_i}\Phi(p_i)dp_i.$$

In particular, a bidder with value 0 will get zero payoff from her outside offer, i.e., we must have $\pi(0, 0) \geq 0$ from IR.

We then have the following lemma (See Appendix for the proof).

Lemma 1. *If the mechanism is incentive compatible, then bidder i's expected payoff can be expressed as follows:*[7]

$$\pi_i(v_i, v_i) \;=\; \pi_i(0,0) + \int_0^{v_i} E_{\mathbf{v}_{-i},\, p_i} \left\{ \begin{array}{c} q_i^1(t, \mathbf{v}_{-i}) \Big[q_i^2(t, p_i; \hat{p}_i) + (1 - q_i^2(t, p_i; \hat{p}_i)) \mathbf{1}\{t - p_i \geq 0\} \Big] \\ + (1 - q_i^1(t, \mathbf{v}_{-i})) \mathbf{1}\{t - p_i \geq 0\} \end{array} \right\} dt. \tag{3}$$

I. B An upper bound on seller revenue and full deterrence

Define virtual value as follows:

$$\lambda(v_i, p_i) \;=\; \begin{cases} p_i & \text{if } p_i \;\leq\; v_i; \\ J(v_i) & \text{if } p_i \;>\; v_i. \end{cases} \tag{4}$$

where $J(v_i) \equiv v_i - \frac{1-F(v_i)}{f(v_i)}$. Let us denote the seller's revenue function by R and given Lemma 1, we can then establish the following lemma (See Appendix for the proof).

Lemma 2. *The seller's revenue function R is given by*

$$R \;=\; E_{\mathbf{v}} \sum_i E_{p_i \leq v_i} \left\{ \left[v_i - \frac{1 - F(v_i)}{f(v_i)} \right] - p_i \right\} \;+\; E_{\mathbf{v}} \sum_i q_i^1(v_i, \mathbf{v}_{-i}) E_{p_i} \Big\{ q_i^2(v_i, p_i; \hat{p}_i) \lambda(v_i, p_i) \Big\}. \tag{5}$$

Now we are ready to establish an upper bound for seller revenue and characterize features of the associated allocation rules. Let $\check{v}^{so}$ and r^{so} be uniquely determined by

$$\begin{aligned} \frac{1 - F(\check{v}^{so})}{f(\check{v}^{so})} (1 - \Phi(\check{v}^{so})) \;&=\; \int_0^{\check{v}^{so}} [1 - \Phi(p_i)] dp_i, \;\; \text{and} \\ r^{so} \;&=\; \int_0^{\check{v}^{so}} [1 - \Phi(p_i)] dp_i. \end{aligned} \tag{6}$$

Let us further use $v^{(1)} \equiv \max\{v_i, i = 1, 2, ..., N\}$ to denote the highest value among bidders and set the following allocation rules:

$$\hat{p}_i^{so}(v_i, \mathbf{v}_{-i}) \;=\; \begin{cases} 0 & \text{if } v_i \;\geq\; \check{v}^{so}; \\ \bar{v} & \text{if } v_i \;<\; \check{v}^{so}, \end{cases} \quad \text{and} \quad q_i^{so}(v_i, \mathbf{v}_{-i}) \;=\; \begin{cases} 1 & \text{if } v_i = v^{(1)} \text{ and } v_i \geq \check{v}^{so}; \\ 0 & \text{otherwise.} \end{cases} \tag{7}$$

In our analysis, $\check{v}^{so}$ is called the winning threshold, indicating that in the first stage, a bidder with

[7] This lemma can also be obtained by appealing to the version of the envelope theorem in Milgrom and Segal (2002).

the highest value is selected as the winner if and only if her value is no less than the threshold $\check{v}^{so}$.[8] We therefore have $q_i^1(\mathbf{v}') = 0$ and $m_i^1(\mathbf{v}') = 0$ if $v_i' < \check{v}^{so}$. We then have the following result (See Appendix for the proof).

Proposition 1. *An upper bound of seller revenue under the class of the mechanisms is given by*

$$R^* \;=\; N\big(1 - F(\check{v}^{so})\big)Q(\check{v}^{so})r^{so} \;+\; N\int_{\check{v}^{so}}^{\bar{v}}(1 - F(t))\big(t - \int_0^t \Phi(x)dx\big)dQ(t), \tag{8}$$

where $Q(\cdot) \equiv F^{N-1}(\cdot)$.

Our characterization above shows that to achieve the upper bound of revenue R^*, the seller needs to screen out bidders using threshold $\check{v}^{so}$ in the first stage and then induce full compliance, i.e., $\hat{p}^{so} = 0$, in the second stage. The winning bidder i's virtual value is p_i if p_i is lower than value v_i, and the virtual value is $J(v_i) = v_i - \frac{1-F(v_i)}{f(v_i)}$ when $p_i \geq v_i$. The seller sets a cutoff price $\hat{p}_i$ to maximize the expected virtual values, subject to the constraint that the winning bidder defaults if and only if the outside price is lower than threshold $\hat{p}_i$. Note that the virtual value function as a function of p_i starts from 0 and increases until $p_i = v_i$, and then it drops to a flat level of $J(v_i)$. Therefore, the seller must set $\hat{p}_i$ to either 0 or $\overline{v}$ depending on the level of $J(v_i)$. Since $J(v_i)$ increases with v_i, for $v_i \geq \check{v}^{so}$, the seller finds it optimal to set $\hat{p}_i$ to 0; and, for $v_i < \check{v}^{so}$, it is optimal to set $\hat{p}_i$ to $\overline{v}$. Note that, given that p_i is assumed to be known publicly, the identified maximum of seller revenue gives the upper bound the seller can achieve. In the next section, we will show that a second-price auction with properly designed deposits and reserve prices can still achieve the upper bound, even when p_i is private information.

II. Implementation by SPA with Deposit and Reserve Price

In this section, we consider a second-price auction game with deposit D and reserve price r.[9] Note that charging a deposit increases the default cost of the winning bidder and therefore, it affects the cutoff $\hat{p}_i$ of whether to take the outside offer or complete the original transaction. Specifically, the auction game includes three stages: In stage $t = 0$, the seller sets up the deposit and reserve price. In stage $t = 1$, all the bidders submit simultaneous bids. Only those bids which are no less than the reserve price r are valid. In stage $t = 2$, the winning bidder decides whether to pay the deposit. In stage $t = 3$, an outside offer with price p_i arrives (which is the winning bidder's private

[8] In Proof of Proposition 1, we will provide the derivation of the threshold $\check{v}^{so}$.

[9] See the formal setup and timing of the auction game and characterizations of bidder strategies in online Appendix S1.

information) and then the winner decides whether to default and take the outside offer or complete the current transaction, conditional on paying the deposit D.

Given the auction setup, the characterizations of bidder i's decisions on paying the deposit at $t = 2$ and taking the outside offer at $t = 3$ are straightforward. If the bidder does not win the auction, whether to pay the deposit becomes irrelevant and the only option she faces is the outside offer: taking the offer if $v_i \geq p_i$; otherwise, not to make the purchase. Conditional on winning, bidder i along the equilibrium path pays the deposit and faces two options between the original transaction and the outside offer:[10] the bidder completes the original transaction if the price for the outside offer is no less than the rest of the final payment; otherwise, the bidder takes the outside offer. Based on the strategies at $t = 2$ and $t = 3$, we employ backward induction to derive a bidder's equilibrium bidding strategy $b(v_i)$. Given D and r from the seller, define two cutoff values $\check{v} \in [0, \bar{v}]$ and $\hat{v} \in [0, \bar{v}]$ satisfying the following conditions:

$$r \;=\; \int_0^{\check{v}} [1 - \Phi(p_i)]dp_i, \;\; \text{and} \;\; D \;=\; \int_0^{\hat{v}} [1 - \Phi(p_i)]dp_i.$$

Note that if $b(\bar{v}) \leq r$, we then have $\check{v} \geq \bar{v}$, and no bidder participates. If $b(\bar{v}) \leq D$, we have $\hat{v} \geq \bar{v}$, and no bidder will bid more than the deposit D.

In Case (I) where $r \leq D$, we have $\hat{v} \geq \check{v}$. Bidder i's bidding strategy can be characterized as follows:

$$b(v_i) \;=\; \begin{cases} v_i \;-\; \displaystyle\int_{b(v_i)-D}^{v_i} \Phi(p_i)dp_i & \text{if } \; v_i \;>\; \hat{v}; \\ v_i \;-\; \displaystyle\int_0^{v_i} \Phi(p_i)dp_i & \text{if } \; v_i \in [\check{v}, \hat{v}]; \\ \varnothing & \text{if } \; v_i \;<\; \check{v}. \end{cases} \tag{9}$$

In Case (II) where $r \geq D$, we have $\hat{v} \leq \check{v}$; only the bidder with $v_i \in [\check{v}, \bar{v}]$ will submit a valid bid. The bidding strategy can be summarized as follows:

$$b(v_i) \;=\; \begin{cases} v_i \;-\; \displaystyle\int_{b(v_i)-D}^{v_i} \Phi(p_i)dp_i & \text{if } \; v_i \;\geq\; \check{v}; \\ \varnothing & \text{if } \; v_i \;<\; \check{v}. \end{cases} \tag{10}$$

Now let us examine the seller's optimal choices of reserve price r and deposit D at $t = 0$. We

[10] If the winning bidder chooses not to pay the deposit, this is an off-equilibrium behavior, and the seller will interpret the behavior as a default. Strategically, this off-equilibrium behavior is equivalent to the case where the bidder chooses not to bid in the auction but waits for the outside offer directly.

denote the seller's expected revenue by $R(r, D)$, and the expected revenue-maximizing problem with both instruments r and D can be written as follows:

$$\max_{(r,D)\in[0,\bar{v}]\times[0,\bar{v}]} R(r, D). \tag{11}$$

Depending on r and D, the bidders' equilibrium biding strategies are as characterized above. We thus rewrite (11) as follows:

$$\max_{D\in[0,\bar{v}]} \max\left\{ \max_{r:\, r\leq D} R(r, D);\ \max_{r:\, r\geq D} R(r, D)\right\},$$

where the sub-problem $\max_{r:\, r\leq D} R(r, D)$ corresponds to Case (I) where $r \leq D$ and the other sub-problem $\max_{r:\, r\geq D} R(r, D)$ corresponds to Case (II) where $r \geq D$. In each case, the seller's revenue equals the auction price if the winner does not default, the deposit if the winner does, and zero in the absence of any valid bids. For notation simplification, we write the seller's expected revenue in the former sub-problem as $\mathbb{E}_S^I[R(r, D)]$ and the latter as $\mathbb{E}_S^{II}[R(r, D)]$ in the following analysis.

In Case (I) where $r \leq D$, we write the bidding strategy in (9) as $b(v_i)$ and $\tilde{b}(v_i)$ for $v_i \in [\check{v}, \hat{v}]$ and $v_i \in (\hat{v}, \bar{v}]$, respectively. Then, the seller's expected revenue $\mathbb{E}_S^I[R(r, D)]$ is given by the following equation. According to Lemma 4 in online Appendix S1, $\check{v}$ is increasing in r but independent of D and $\hat{v}$ is increasing in D but independent of r in Case (I).

$$\begin{aligned}\mathbb{E}_S^I[R(r, D)] \;=\; & N\big(1 - F(\check{v})\big)Q(\check{v})r \;+\; N\int_{\check{v}}^{\hat{v}}\int_{\check{v}}^{v_i} b(x)dQ(x)dF(v_i) \;+\; N\int_{\hat{v}}^{\bar{v}}\int_{\check{v}}^{\hat{v}} b(x)dQ(x)dF(v_i) \\ & +\; N\int_{\hat{v}}^{\bar{v}}\int_{\hat{v}}^{v_i}\Big[\big(1 - \Phi(\tilde{b}(x) - D)\big)\tilde{b}(x) + \Phi(\tilde{b}(x) - D)D\Big]dQ(x)dF(v_i),\end{aligned} \tag{12}$$

where $Q(\cdot) \equiv F^{N-1}(\cdot)$. The first term occurs when only the highest value of all bidders is above $\check{v}$. The second term occurs when the highest and the second-highest values are between $\check{v}$ and $\hat{v}$. The third term occurs when the highest value is higher than $\hat{v}$ while the second-highest value is between $\check{v}$ and $\hat{v}$. The last term occurs when both the highest and the second-highest values are above $\hat{v}$.

In Case (II) where $r \geq D$, the bidding strategy $b(v_i)$ is given by (10), and the seller's expected revenue $\mathbb{E}_S^{II}[R(r, D)]$ is given by the following equation. According to Lemma 4 in online Appendix

S1, $\check{v}$ is increasing in both r and D in Case (II).

$$\begin{aligned}\mathbb{E}_S^{II}[R(r,D)] \;=\; & N\big(1-F(\check{v})\big)Q(\check{v})\Big[\big(1-\Phi(r-D)\big)r \;+\; \Phi(r-D)D\Big] \\ & +\; N\int_{\check{v}}^{\bar{v}}\int_{\check{v}}^{v_i}\Big[\big(1-\Phi(b(x)-D)\big)b(x)+\Phi(b(x)-D)D\Big]dQ(x)dF(v_i),\end{aligned} \tag{13}$$

where the first term occurs when only the highest value of all bidders is above $\check{v}$, while the second term occurs when both the highest and the second-highest values are above $\check{v}$.

Our analysis will be carried out using the following two steps to examine the seller's problem with the instruments (r, D). *Step one*: for each given deposit level, we pin down the optimal reserve price. For this purpose, we analyze the seller's revenue maximization problems separately in Case (I) where $r \leq D$ and Case (II) where $r \geq D$. Let us denote the overall maximized expected seller revenue by $R^*(D)$ and the seller's optimal choice regarding the reserve price by $r^*(D)$. Then, combining the solutions of the two sub-problems gives $R^*(D)$ and $r^*(D)$, that is,

$$\begin{aligned}R^*(D) \;&:=\; \max\Big\{\max_{r:\, r\leq D} R(r,D);\; \max_{r:\, r\geq D} R(r,D)\Big\}, \quad \text{and} \\ r^*(D) \;&:=\; \arg\max\Big\{\arg\max_{r:\, r\leq D} R(r,D);\; \arg\max_{r:\, r\geq D} R(r,D)\Big\}.\end{aligned}$$

The characterization of $r^*(D)$ helps us identify the connection between the deposit and the optimal reserve price, which shows us how to set the reserve price optimally for any given deposit. *Step two*: we relax the condition so the seller is free to choose any level of deposit and examine the optimal deposit that maximizes $R^*(D)$ with the optimal reserve price $r^*(D)$, that is, $\max_{D\in[0,\bar{v}]} R^*(D)$. By doing so, we obtain the optimal deposit and the associated reserve price.

Following the steps mentioned above, we first separately characterize the seller's optimal choice of the reserve price for a given deposit under Case (I) and Case (II). Before proceeding further, we introduce notation of D^{so} as follows:

$$D^{so} \;=\; \int_0^{\hat{v}^{so}} [1-\Phi(p_i)]dp_i, \tag{14}$$

where $\hat{v}^{so} = \bar{v}$. Recall that $\check{v}^{so}$ and r^{so} are defined in (6). We then have $b(\hat{v}^{so}) = D^{so}$ and $b(\check{v}^{so}) = r^{so}$. Clearly, D^{so} is strictly greater than r^{so}.

We then establish Lemmas 5 – 8 to help us characterize the seller's optimal choice of the reserve price r, given that D is set in the intervals of $[0, r^{so})$, $[r^{so}, D^{so}]$, and $(D^{so}, \bar{v}]$, respectively. Since

these lemmas are rather technical, we instead put all the technical proofs for the seller's choice on reserve price for any given deposit in online Appendix S1. Let $r^I(D)$ and $r^{II}(D)$ denote the optimal reserve price maximizing $\mathbb{E}_S^I[R(r,D)]$ under Case (I) where $r \leq D$ and maximizing $\mathbb{E}_S^{II}[R(r,D)]$ in Case II where $r \geq D$, respectively. After obtaining the solutions for the sub-problems separately, we then compare the maximized seller revenues $\mathbb{E}_S^I[R(r^I(D),D)]$ and $\mathbb{E}_S^{II}[R(r^{II}(D),D)]$ across Cases (I) and (II) for a given $D \in [0, D^{so}]$. Note that with $D = r$ at the 45-degree line, $\check{v} = \hat{v}$, and revenue functions in both cases are the same, i.e., $\mathbb{E}_S^I[R(r^I(D),D)] = \mathbb{E}_S^{II}[R(r^{II}(D),D)]$. By doing so, we can establish the following result (See Appendix for the proof).

Proposition 2. *The optimal reserve price $r^*(D)$ can be characterized as follows:*

$$r^*(D) \;=\; \begin{cases} r^I(D) & \text{if } \; D \in [r^{so}, D^{so}]; \\ r^{II}(D) & \text{if } \; D \in [0, r^{so}). \end{cases} \tag{15}$$

From Proposition 2, it is easy to see that $r^*(D) = r^{so} \leq D$ when $D \in [r^{so}, D^{so}]$ and $r^*(D) \geq D$ when $D \in [0, r^{so})$.[11] In practice, due to (a) the time and/or budget constraints faced by the buyers (bidders), especially for the items with high values, and/or (b) regulations from the market regulators, like those real examples mentioned in the introduction, often an upper limit on the deposit that the seller can charge in the auction is imposed. Our result demonstrates how to set the corresponding optimal reserve price for a given deposit level. Let $\bar{D} \in [0, D^{so})$ denote the upper limit on the deposit, and then the reserve price is given by $r^*(\bar{D})$, as shown in Proposition 2. In this case, the cutoff $\hat{v}$ is lower than $\bar{v}$, and the possibility of default from the winner is not fully deterred.

Proposition 2 further allows us to define the optimal revenue function $R^*(D)$ as follows:

$$R^*(D) \;=\; \begin{cases} \mathbb{E}_S^I[R(r^*(D),D)] & \text{if } \; D \in [r^{so}, D^{so}]; \\ \mathbb{E}_S^{II}[R(r^*(D),D)] & \text{if } \; D \in [0, r^{so}). \end{cases} \tag{16}$$

Using (16), we move to step two to examine the optimal deposit and the associated reserve price. Suppose that the seller is free to set the level of the deposit in the auction; the following result can then be established (see Appendix for the proof).

[11] To provide some insights on the seller's choices of reserve price r and deposit D, we present numerical example 1 in online Appendix S2. Assume for simplicity that there are only two bidders ($N = 2$) and that both bidder valuations and outside offers are uniformly distributed on the unit interval ($v_i \sim U[0,1]$ and $p_i \sim U[0,1]$). Our computations show $r^{so} = 0.33333$ and $D^{so} = 0.5$. Note that if no post-auction outside offer exists, i.e., $\Phi(\cdot) = 0$, the optimal reserve price is 0.5. Moreover, when the seller sets $r \geq 0.5$, no valid bids will be submitted, and the seller's revenue decreases to zero. All details regarding the computations are in online Appendix S2.

Proposition 3. *$\frac{dR^*(D)}{dD} \geq 0$ for any $D \in [0, D^{so}]$. In particular, $\frac{dR^*(D)}{dD} = 0$ when $D = D^{so}$. As a result, the combination of D^{so} and r^{so} defined in (14) and (6) maximizes the seller's overall expected revenue.*

Proposition 3 describes how the overall seller revenue changes with the deposit requirement. A higher deposit requirement induces a higher overall expected revenue to the seller. The maximum is achieved at $D = D^{so}$, and the associated optimal reserve price is given by r^{so}. Then, $R^*(D^{so})$ is given by the following equation:

$$\begin{aligned} R^*(D^{so}) &= \mathbb{E}_S^I[R(r^{so}, D^{so})] \\ &= N\big(1 - F(\check{v}^{so})\big)Q(\check{v}^{so})r^{so} + N\int_{\check{v}^{so}}^{\hat{v}^{so}}\int_{\check{v}^{so}}^{v_i} b(x)dQ(x)dF(v_i) + N\int_{\hat{v}^{so}}^{\bar{v}}\int_{\check{v}^{so}}^{\hat{v}} b(x)dQ(x)dF(v_i) \\ &\quad + N\int_{\hat{v}^{so}}^{\bar{v}}\int_{\hat{v}^{so}}^{v_i}\Big[\big(1 - \Phi(\tilde{b}(x) - D)\big)\tilde{b}(x) + \Phi(\tilde{b}(x) - D)D\Big]dQ(x)dF(v_i). \end{aligned}$$

Given that $\hat{v}^{so} = \bar{v}$ and $b(x) = \int_0^x (1 - \Phi(p_i))dp_i$, the third and fourth terms become zero and the second term can be re-written as $\int_{\check{v}^{so}}^{\bar{v}}(1 - F(x))b(x)dQ(x)$ by changing order of the integration. The equation above can then be simplified as follows

$$\begin{aligned} R^*(D^{so}) &= N\big(1 - F(\check{v}^{so})\big)Q(\check{v}^{so})r^{so} + N\int_{\check{v}^{so}}^{\bar{v}}(1 - F(x))b(x)dQ(x), \\ &= N\big(1 - F(\check{v}^{so})\big)Q(\check{v}^{so})r^{so} + N\int_{\check{v}^{so}}^{\bar{v}}(1 - F(t))\big(t - \int_0^t \Phi(x)dx\big)dQ(t), \end{aligned}$$

showing that $R^*(D^{so})$ is identical to R^* in Proposition 1. Note that charging any deposit $D > D^{so}$ generates the same expected overall revenue as $D = D^{so}$ to the seller. It is clear by (14) that, at the optimum, $D = D^{so}$ implies that the cutoff $\hat{v}^{so}$ is equal to $\bar{v}$; the default possibility for the winning bidder is fully prevented, thereby resulting in full compliance in the auction, i.e., $\hat{p}^{so} = 0$. The analysis here suggests that

Theorem 1. The second-price auction with D^{so} and r^{so} generates the upper bound of seller revenue R^*.

Setting a high deposit, on the one hand, makes winning become less attractive and in turn gives bidders incentives to lower their bids in the auction; on the other hand, the deposit requirement increases the winner's default cost, which reduces the likelihood that she will withdraw from the original transaction. Our analysis indicates that the latter positive effect always dominates the former negative effect; the seller's revenue is maximized by setting a sufficiently high deposit to fully deter the possibility of default. In this case, the reserve price also plays an important role

in excluding the possibility of selling the item to bidders with very low bids, thereby helping the seller offset the negative effect of the deposit requirement and increase revenue. Figure 4 in online Appendix S1 confirms that it is not optimal to use the deposit requirement alone; given any deposit requirement $D \in [0, D^{so}]$ in the auction, the overall seller revenue with a zero reserve price is strictly less than that of $r^*(D)$. However, if the reserve price is too high, it would (a) screen out bidders with reasonably high values and (b) induce a high ending price, which gives a higher incentive to the auction winner to default; both effects hurt the seller's expected revenue and thus limit the level of the reserve price the seller can set. As shown in Proposition 3, the optimal reserve price is set at r^{so}.

It is also of interest to compare the optimal reserve price r^{so} to that of Myerson (1981). We denote Myerson's optimal reserve price by r^m, which is given by $\frac{1-F(r^m)}{f(r^m)} = r^m$. From the characterization of r^{so} in (6), we have the following equation:

$$\begin{aligned} \frac{1-F(\check{v}^{so})}{f(\check{v}^{so})} &= \frac{\int_0^{\check{v}^{so}}[1-\Phi(p_i)]dp_i}{1-\Phi(\check{v}^{so})} \\ &\geq \frac{\check{v}^{so}(1-\Phi(\check{v}^{so}))}{1-\Phi(\check{v}^{so})} \\ &= \check{v}^{so}. \end{aligned}$$

This implies that $r^m \geq \check{v}^{so}$, as $\frac{1-F(v)}{f(v)}$ is decreasing in v. Furthermore, given that $r^{so} = \int_0^{\check{v}^{so}}[1 - \Phi(p_i)]dp_i \leq \check{v}^{so}$, we have $r^m \geq \check{v}^{so} \geq r^{so}$. Clearly, $r^m = \check{v}^{so} = r^{so}$ if and only if there exists no outside offer, i.e., $\Phi(\cdot) = 0$. Summarizing the comparison yields the following result.

Remark 1. $r^m \geq \check{v}^{so} \geq r^{so}$. In particular, $r^m = \check{v}^{so} = r^{so}$ only when no outside offer exists, i.e., $\Phi(\cdot) = 0$.

III. Discussion

We now discuss different distributions of the distribution of the outside offers and the robustness of our results when relaxing some important assumptions.[12]

III. A Distribution of the outside offer

In this section, we study the impact of the outside offer in the auction. When the price of the outside offer is equal to 0 with probability 1, bidders will not enter the auction but choose to wait

[12] We also analyze the cases of percentage deposit and deposit proportionally deducted from full payment. See the related discussion in online Appendix S1.

for the outside offer, regardless of r and D set by the seller. In this case, (14) implies that the optimal deposit D^{so} is 0. Another extreme case is when no outside offer exists for bidders after the auction, i.e., $\Phi(\cdot) = 0$. In this case, the bidding strategy and the seller revenue are the same as those in the standard second-price auction (with Myerson's reserve price). (14) states that the optimal deposit D^{so} in this case is equal to $\overline{v}$. It is clear in both cases that charging any deposit D will not affect the seller's revenue.

Next, we examine how different distributions of the outside offers affect the auction design. Given r and D, let $b(v_i, \Phi_1)$ and $b(v_i, \Phi_2)$ denote the equilibrium bidding functions corresponding to the distributions of the outside offer $\Phi_1(\cdot)$ and $\Phi_2(\cdot)$, respectively; and further denote the optimal reserve price, the optimal deposit, and the overall seller revenue by r_k^{so}, D_k^{so}, and $R^*(r_k^{so}, D_k^{so}, \Phi_k)$, respectively, corresponding to $\Phi_k(\cdot), k = 1, 2$. Our results are stated as follows (see Appendix for the proof).[13]

Proposition 4. *If the distribution of the outside offer worsens (in the sense of first-order stochastic dominance) from the bidders' perspective, i.e.,* $\Phi_1(\cdot) < \Phi_2(\cdot)$,

(i) *the equilibrium bid submitted by a bidder is higher for any given reserve price* r *and deposit* D*, that is,* $\check{v}_1 < \check{v}_2$*,* $\hat{v}_1 < \hat{v}_2$*, and* $b(v_i, \Phi_1) > b(v_i, \Phi_2)$*;*

(ii) *the seller sets a higher optimal deposit, that is,* $D_1^{so} > D_2^{so}$*;*

(iii) *the optimal seller revenue is higher, that is,* $R^*(r_1^{so}, D_1^{so}, \Phi_1) > R^*(r_2^{so}, D_2^{so}, \Phi_2)$.

Part (i) of Proposition 4 is intuitive; when the possibility for an attractive outside offer is smaller, the original auction becomes more attractive, which induces the bidders to submit higher bids. Interestingly, part (ii) shows that a worse distribution for the outside offer allows the seller to charge a higher deposit. This result can be explained as follows: A worse distribution induces bidders to bid more aggressively, thereby increasing the probability of post-auction default by the winner. As shown above, it is preferable for the seller to have full compliance in the auction, although it would lower bids from the bidders. Therefore, to deter the winner from defaulting, a higher deposit, combined with a higher reserve price (see Proposition 5), must be demanded in the auction. As a result, a worse distribution leads to a higher revenue to the seller, which gives part (iii).

[13] To further illustrate the impacts of the distribution of an outside offer, we present numerical example 2, assuming that bidder i's valuation still follows a uniform distribution, but the distribution of the outside offer takes the form of $\Phi(p_i) = p_i^{\alpha}$ over $[0, 1]$, where $\alpha \geq 0$. We then compute how a change in α affects the sellers' choices regarding the optimal reserve price r^{so}, the optimal deposit D^{so}, and the optimal expected revenue R^*. The computational results are consistent with the predictions in Proposition 4, and all the details are presented in online Appendix S2.

We then turn to the comparisons of the threshold $\check{v}^{so}$ and the optimal reserve price r^{so}. Define function $\tau(v_i)$ as $\tau(v_i) \equiv \frac{\int_0^{v_i}[1-\Phi(p_i)]dp_i}{1-\Phi(v_i)}$. Differentiating $\tau(v_i)$ with respect to v_i yields

$$\begin{aligned}\tau'(v_i) &= 1 + \frac{\int_0^{v_i}(1-\Phi(p_i))dp_i}{1-\Phi(v_i)} \cdot \frac{\varphi(v_i)}{1-\Phi(v_i)} \\ &= 1 + \tau(v_i) \cdot \frac{\varphi(v_i)}{1-\Phi(v_i)} \\ &> 0.\end{aligned}$$

$\tau(v_i)$ is thus increasing in v_i, and $\tau(0) = 0$. If $\Phi_1(\cdot)$ dominates $\Phi_2(\cdot)$ in terms of the hazard rate, i.e., $\frac{\varphi_1(\cdot)}{1-\Phi_1(\cdot)} < \frac{\varphi_2(\cdot)}{1-\Phi_2(\cdot)}$, then $\tau_1(\cdot) < \tau_2(\cdot)$.[14] This indicates that, in order to satisfy $\frac{1-F(\check{v}_1^{so})}{f(\check{v}_1^{so})} = \tau_1(\check{v}_1^{so})$ and $\frac{1-F(\check{v}_2^{so})}{f(\check{v}_2^{so})} = \tau_2(\check{v}_2^{so})$, we must have $\check{v}_1^{so} > \check{v}_2^{so}$ and $r_1^{so} = \int_0^{\check{v}_1^{so}}[1-\Phi_1(p_i)]dp_i > \int_0^{\check{v}_2^{so}}[1-\Phi_2(p_i)]dp_i = r_2^{so}$. This gives us the following result.

Proposition 5. *$\check{v}_1^{so} > \check{v}_2^{so}$ and $r_1^{so} > r_2^{so}$, if $\frac{\varphi_1(\cdot)}{1-\Phi_1(\cdot)} < \frac{\varphi_2(\cdot)}{1-\Phi_2(\cdot)}$. The seller sets a higher optimal reserve price when the distribution of the outside offer worsens (in the sense of hazard rate dominance) from the bidders' perspective.*

III. B Stochastic number of bidders

Rather than being fixed, the number of bidders may be stochastic. E.g., in internet auctions, an entering bidder does not know the number of rival bidders, but only the distribution of the number of potential bidders. One may then question whether the characterization of the optimal auction design with r and D still holds. First, regardless of whether the participation process is stochastic, conditional on winning, it is still optimal for bidder i to adopt the deposit strategy and the outside offer strategy characterized in (A4) and (A1) in online Appendix S1. Second, the equilibrium bidding strategy still follows the property of the second-price auction mechanism itself, that is, a bidder's bidding strategy does not depend on the number of entering bidders. Thus, if bidders' participation is stochastic, the combination of r^{so} and D^{so} is still optimal. Note that the optimal reserve price r^{so} and deposit D^{so} do not depend on the number of bidders.

III. C English ascending auction format

It is natural to examine whether our result still holds in the English ascending auction format instead of the simultaneous bidding format. We still solve the game by backward induction. Clearly,

[14] This also implies that $\Phi_1(\cdot)$ first-order stochastically dominates $\Phi_2(\cdot)$, that is, $\Phi_1(\cdot) \leq \Phi_2(\cdot)$; see more details in online Appendix B of Krishna (2002).

the optimality of the deposit strategy and the outside offer strategy in (A4) and (A1), respectively, will not change. The equilibrium bidding strategy we constructed is a weakly dominant strategy for bidders (see details in online Appendix S1); in the bidding stage of the English ascending auction, it cannot be optimal for the bidder to stay in after the current price exceeds the equilibrium bid or drop out before the current price reaches the equilibrium bid. Thus, the combination of r^{so} and D^{so} is still optimal if the auction takes the English ascending format.

III. D Deposit requirement before bidding

Our analysis in the main text focused on the winner-pay deposit. In practice, another type of deposit exists where the seller requires all potential bidders to pay a certain amount as a deposit before submitting bids. After the auction ends, the seller refunds the deposits to all the bidders, except the winner. Then, the winner can choose to either default and lose the deposit or complete the current transaction by paying the final auction price minus the deposit. Such a deposit requirement is commonly used in art or antiques auctions. For instance, Sotheby's requires such a deposit for items. We call it *the all-pay deposit*.

It is then interesting to examine whether these two types of deposit requirements would result in different equilibrium strategies and seller revenues. Across the two deposit requirements, a bidder's strategy regarding the outside offer is the same conditional on winning, and the only difference is the timing of paying the deposit. Although a bidder needs to pay the deposit before bidding with the all-pay deposit requirement, this does not affect the expected surplus of the bidder and the construction of the equilibrium bid since the deposit will be refunded conditional on losing the auction. Thus, we can conclude that, in the auction game, the winner-pay deposit is strategically equivalent to the all-pay deposit requirement. However, this equivalence relies on the seller's commitment to refund the deposits to the losing bidders being credible. This may not always be true in reality; the seller's historical reputation and commitment issue may break down the equivalence.[15]

IV. Conclusion

In this paper, we study the role of a deposit requirement in auctions with future outside offers for all bidders and possibly defaulting from the winner. To examine the design problem and identify

[15] In short, the ascending bidding procedure in Section III. C and the deposit requirement before bidding in Section III. D do not affect threshold $\check{v}^{so}$ and full compliance $\hat{p}^{so} = 0$ in the truthful direct mechanism. Therefore, it explains why they render the same optimal auction design.

the upper bound on seller revenue, we start by examining a truthful direct mechanism, which includes two stages of bidder value reporting and arrival of outside offers (public information). Our characterization shows that it is optimal to set up threshold $\check{v}^{so}$ in the first stage and full compliance $\hat{p}^{so} = 0$ in the second stage. We next examine a second-price auction with a deposit and a reserve price, showing that the upper bound of revenue can be achieved when a sufficiently high deposit is charged to deter the winner from default fully. At the same time, a lower optimal reserve price (which is uniquely determined by $\check{v}^{so}$) than that of Myerson (1981) is required. Our study provides a rationale for the widely adopted deposit requirements in auctions where winner default is a recognized concern.

The environment we consider in the current study rules out some possible situations.[16] One relevant situation is that a bidder's private valuation may be positively or negatively correlated with the price of her outside offer. A positive correlation means that a higher outside price is more likely for a higher value. In this case, the auction winner who tends to value the object higher than the losers is less likely to default, and one can expect that a deposit requirement would play a less significant role helping the seller. When the correlation is negative, the winner tends to have a better outside offer, which means that the deposit requirement can play a more effective role in enhancing the seller's expected revenue. It would also be interesting to consider and examine the role of a deposit requirement in a common value or an affiliated private value setting. The mechanism design literature typically assumes independent private information across players due to the full surplus extraction result of Crémer and McLean (1988). With affiliated private values, their insight still applies in principle. As a result, the seller's revenue under the optimal mechanism is unlikely to be achievable by a second-price auction with deposit and reserve price. Another possible extension is to allow correlation in bidders' outside offers, which could arise since some common outside offers might be accessible to all of them. Our analysis should still apply, as bidders' payoffs only depend on their own outside offers. Moreover, the bidders' outside offers might also be correlated to the sellers' reservation values since the same market conditions can affect all of them. Our insight should also be able to accommodate this feature. These extensions are left for future research.

References

Armstrong, M., and J. Zhou (2016): "Search deterrence," *The Review of Economic Studies*, 83, 26–57.

[16] We thank Robert Porter for raising this point.

Asker, J. (2000): "Bidding up, buying out and cooling-off: An examination of auctions with withdrawal rights," *Economic Theory*, 16(3), 585–611.

Ben-Shahar, O., and E. A. Posner (2011): "The right to withdraw in contract law," *The Journal of Legal Studies*, 40(1), 115–148.

Burguet, R. (1996): "Optimal repeated purchases when sellers are learning about costs," *Journal of Economic Theory*, 68(2), 440–455.

Cherry, T., P. Frykblom, J. Shogren, J. List, and M. Sullivan (2004): "Laboratory testbeds and non-market valuation: The case of bidding behavior in a second-price auction with an outside option," *Environmental and Resource Economics*, 29(3), 285–294.

Crémer, J., and R. P. McLean (1988): "Full extraction of the surplus in Bayesian and dominant strategy auctions," *Econometrica*, pp. 1247–1257.

Crémer, J., Y. Spiegel, and C. Z. Zheng (2007): "Optimal search auctions," *Journal of Economic Theory*, 134(1), 226–248.

Crémer, J., Y. Spiegel, and C. Z. Zheng (2009): "Auctions with costly information acquisition," *Economic Theory*, 38(1), 41–72.

Dellarocas, C., and C. A. Wood (2008): "The sound of silence in online feedback: Estimating trading risks in the presence of reporting bias," *Management Science*, 54(3), 460–476.

Eidenmüller, H. (2011): "Why withdrawal rights?," *European Review of Contract Law*, 7(1), 1–24.

Engelmann, D., J. Frank, A. K. Koch, and M. Valente (2015): "Does a buyer benefit from bad reputation? Theory and evidence on auctions with default," *Working paper*.

Figueroa, N., and V. Skreta (2007): "The role of outside options in auction design," *Working paper*.

——— (2009): "The role of optimal threats in auction design," *Journal of Economic Theory*, 144(2), 884–897.

Kirchkamp, O., E. Poen, and J. P. Reiss (2009): "Outside options: Another reason to choose the first-price auction," *European Economic Review*, 53(2), 153–169.

Krähmer, D., and R. Strausz (2015): "Optimal sales contracts with withdrawal rights," *The Review of Economic Studies*, 82(2), 762–790.

Krishna, V. (2002): *Auction Theory*. Elsevier, 1 edn.

Lauermann, S., and G. Virág (2012): "Auctions in markets: Common outside options and the continuation value effect," *American Economic Journal: Microeconomics*, 4(4), 107–130.

Lee, J., and D. Z. Li (2020): "Search for Bidders by a Deadline," *Available at SSRN 3133797*.

McAfee, R. P., and J. McMillan (1988): "Search mechanisms," *Journal of Economic Theory*, 44(1), 99–123.

Milgrom, P., and I. Segal (2002): "Envelope theorems for arbitrary choice sets," *Econometrica*, 70(2), 583–601.

Myerson, R. B. (1981): "Optimal auction design," *Mathematics of operations research*, 6(1), 58–73.

Resnick, P., and R. Zeckhauser (2002): "Trust among strangers in internet transactions: Empirical analysis of eBay's reputation system," in *The Economics of the Internet and E-commerce*, pp. 127–157. Emerald Group Publishing Limited.

Zheng, C. Z. (2001): "High bids and broke winners," *Journal of Economic Theory*, 100(1), 129–171.

——— (2009): "The default-prone US toxic asset auction plan," *The BE Journal of Economic Analysis & Policy*, 9(1).

Mathematical Appendix: Proofs

In the Appendix, we provide proofs of Lemmas 1 and 2, and Propositions 1, 2, 3, and 4.

Proof of Lemma 1

Given (2), we can construct the following equation

$$\pi_i(v_i', v_i') \;=\; \pi_i(v_i', v_i) \;+\; E_{\mathbf{v}_{-i},\, p_i}\left\{\begin{array}{c} q_i^1(v_i', \mathbf{v}_{-i})\Big[q_i^2(v_i', p_i; \hat{p}_i)(v_i' - v_i) \\ +\,(1 - q_i^2(v_i', p_i; \hat{p}_i))(\max\{v_i' - p_i, 0\} - \max\{v_i - p_i, 0\})\Big] \\ +\,(1 - q_i^1(v_i', \mathbf{v}_{-i}))(\max\{v_i' - p_i, 0\} - \max\{v_i - p_i, 0\}) \end{array}\right\}.$$

Therefore, for $v_i' < v_i$, $\pi_i(v_i', v_i) \leq \pi_i(v_i, v_i)$ and that yields

$$\pi_i(v_i', v_i') \;\leq\; \pi_i(v_i, v_i) \;+\; E_{\mathbf{v}_{-i},\, p_i}\left\{\begin{array}{c} q_i^1(v_i', \mathbf{v}_{-i})\Big[q_i^2(v_i', p_i; \hat{p}_i)(v_i' - v_i) \\ +\,(1 - q_i^2(v_i', p_i; \hat{p}_i))(\max\{v_i' - p_i, 0\} - \max\{v_i - p_i, 0\})\Big] \\ +\,(1 - q_i^1(v_i', \mathbf{v}_{-i}))(\max\{v_i' - p_i, 0\} - \max\{v_i - p_i, 0\}) \end{array}\right\},$$

which we can re-write as follows

$$\frac{\pi_i(v_i, v_i) \;-\; \pi_i(v_i', v_i')}{v_i - v_i'} \\ \geq\; E_{\mathbf{v}_{-i},\, p_i}\left\{\begin{array}{c} q_i^1(v_i', \mathbf{v}_{-i})\Big[q_i^2(v_i', p_i; \hat{p}_i) \;+\; (1 - q_i^2(v_i', p_i; \hat{p}_i))(\max\{v_i - p_i, 0\} - \max\{v_i' - p_i, 0\})/(v_i - v_i')\Big] \\ +\,(1 - q_i^1(v_i', \mathbf{v}_{-i}))(\max\{v_i - p_i, 0\} - \max\{v_i' - p_i, 0\})/(v_i - v_i') \end{array}\right\}. \tag{17}$$

Again, given (2), we can construct the following equation

$$\pi_i(v_i, v_i) \;=\; \pi_i(v_i, v_i') \;+\; E_{\mathbf{v}_{-i},\, p_i}\left\{ \begin{array}{c} q_i^1(v_i, \mathbf{v}_{-i})\Big[q_i^2(v_i, p_i; \hat{p}_i)(v_i - v_i') \\ +\, (1 - q_i^2(v_i, p_i; \hat{p}_i))(\max\{v_i - p_i, 0\} - \max\{v_i' - p_i, 0\})\Big] \\ +\, (1 - q_i^1(v_i, \mathbf{v}_{-i}))(\max\{v_i - p_i, 0\} - \max\{v_i' - p_i, 0\}) \end{array} \right\}.$$

Therefore, for $v_i' < v_i$, $\pi_i(v_i, v_i') \leq \pi_i(v_i', v_i')$ and that yields

$$\pi_i(v_i, v_i) \;\leq\; \pi_i(v_i', v_i') \;+\; E_{\mathbf{v}_{-i},\, p_i}\left\{ \begin{array}{c} q_i^1(v_i, \mathbf{v}_{-i})\Big[q_i^2(v_i, p_i; \hat{p}_i)(v_i - v_i') \\ +\, (1 - q_i^2(v_i, p_i; \hat{p}_i))(\max\{v_i - p_i, 0\} - \max\{v_i' - p_i, 0\})\Big] \\ +\, (1 - q_i^1(v_i, \mathbf{v}_{-i}))(\max\{v_i - p_i, 0\} - \max\{v_i' - p_i, 0\}) \end{array} \right\},$$

which we can re-write as follows

$$\begin{aligned} &\frac{\pi_i(v_i, v_i) \;-\; \pi_i(v_i', v_i')}{v_i - v_i'} \\ &\leq\; E_{\mathbf{v}_{-i},\, p_i}\left\{ \begin{array}{c} q_i^1(v_i, \mathbf{v}_{-i})\Big[q_i^2(v_i, p_i; \hat{p}_i) \;+\; (1 - q_i^2(v_i, p_i; \hat{p}_i))(\max\{v_i - p_i, 0\} - \max\{v_i' - p_i, 0\})/(v_i - v_i')\Big] \\ +\, (1 - q_i^1(v_i, \mathbf{v}_{-i}))(\max\{v_i - p_i, 0\} - \max\{v_i' - p_i, 0\})/(v_i - v_i') \end{array} \right\}. \end{aligned} \tag{18}$$

Combining (17) and (18) gives the left derivative of $\pi_i(v_i, v_i)$:

$$\frac{d\pi_i^-(v_i, v_i)}{dv_i} \;=\; E_{\mathbf{v}_{-i},\, p_i}\left\{ \begin{array}{c} q_i^1(v_i, \mathbf{v}_{-i})\big[q_i^2(v_i, p_i; \hat{p}_i) + (1 - q_i^2(v_i, p_i; \hat{p}_i))\mathbf{1}\{v_i - p_i \geq 0\}\big] \\ +\, (1 - q_i^1(v_i, \mathbf{v}_{-i}))\mathbf{1}\{v_i - p_i \geq 0\} \end{array} \right\}.$$

Then, let us consider the case where $v_i' > v_i$, we can construct the similar inequalities and obtain the right derivative of $\pi_i(v_i, v_i)$:

$$\frac{d\pi_i^+(v_i, v_i)}{dv_i} \;=\; E_{\mathbf{v}_{-i},\, p_i}\left\{ \begin{array}{c} q_i^1(v_i, \mathbf{v}_{-i})\big[q_i^2(v_i, p_i; \hat{p}_i) + (1 - q_i^2(v_i, p_i; \hat{p}_i))\mathbf{1}\{v_i - p_i \geq 0\}\big] \\ +\, (1 - q_i^1(v_i, \mathbf{v}_{-i}))\mathbf{1}\{v_i - p_i \geq 0\} \end{array} \right\}.$$

We can hence conclude that $\pi_i(v_i) = \pi_i(v_i, v_i)$ is differentiable everywhere and

$$\frac{d\pi_i(v_i, v_i)}{dv_i} \;=\; E_{\mathbf{v}_{-i},\, p_i}\left\{ \begin{array}{c} q_i^1(v_i, \mathbf{v}_{-i})\big[q_i^2(v_i, p_i; \hat{p}_i) + (1 - q_i^2(v_i, p_i; \hat{p}_i))\mathbf{1}\{v_i - p_i \geq 0\}\big] \\ +\, (1 - q_i^1(v_i, \mathbf{v}_{-i}))\mathbf{1}\{v_i - p_i \geq 0\} \end{array} \right\}.$$

Taking the integral of its derivative, we then have

$$\pi_i(v_i, v_i) \;=\; \pi_i(0, 0) \;+\; \int_0^{v_i} E_{\mathbf{v}_{-i},\, p_i}\left\{ \begin{array}{c} q_i^1(t, \mathbf{v}_{-i})\big[q_i^2(t, p_i; \hat{p}_i) + (1 - q_i^2(t, p_i; \hat{p}_i))\mathbf{1}\{t - p_i \geq 0\}\big] \\ +\, (1 - q_i^1(t, \mathbf{v}_{-i}))\mathbf{1}\{t - p_i \geq 0\} \end{array} \right\} dt.$$

We complete the proof. □

Proof of Lemma 2

Given Lemma 1 and $\pi_i(0,0)=0$, we can write the bidder i's expected payoff before learning type v_i as follows:

$$\begin{aligned}
&\int_0^{\bar{v}} \pi_i(v_i,v_i)f(v_i)dv_i \\
=\ & \int_0^{\bar{v}}\int_0^{v_i} E_{\mathbf{v}_{-i},\,p_i}\left\{\begin{array}{c} q_i^1(t,\mathbf{v}_{-i})\big[q_i^2(t,p_i;\hat{p}_i)+(1-q_i^2(t,p_i;\hat{p}_i))\mathbf{1}\{t-p_i\geq 0\}\big] \\ +(1-q_i^1(t,\mathbf{v}_{-i}))\mathbf{1}\{t-p_i\geq 0\} \end{array}\right\} f(v_i)dtdv_i \\
=\ & E_{\mathbf{v},\,\mathbf{p}}\left(\left\{\begin{array}{c} q_i^1(v_i,\mathbf{v}_{-i})\big[q_1^2 i(v_i,p_i;\hat{p}_i)+(1-q_i^2(v_i,p_i;\hat{p}_i))\mathbf{1}\{v_i-p_i\geq 0\}\big] \\ +(1-q_i^1(v_i,\mathbf{v}_{-i}))\mathbf{1}\{v_i-p_i\geq 0\} \end{array}\right\}\cdot[\frac{1-F(v_i)}{f(v_i)}]\right).
\end{aligned}$$

Now let us look at the total expected surplus of the seller and buyers, denoted by TE, which is given by

$$TE\ =\ E_{\mathbf{v},\,\mathbf{p}}\sum_i\left[\begin{array}{c} q_i^1(v_i,\mathbf{v}_{-i})\big[q_i^2(v_i,p_i;\hat{p}_i)v_i+(1-q_i^2(v_i,p_i;\hat{p}_i))\max\{v_i-p_i,0\}\big] \\ +(1-q_i^1(v_i,\mathbf{v}_{-i}))\max\{v_i-p_i,0\}. \end{array}\right]$$

Note that the total surplus means the sum of payoffs of the seller and the bidders in our paper. Therefore, the expected surplus for the "third party" who offers the outside offer is not counted in. The seller's revenue R is the difference between the total expected surplus and the buyers' expected payoffs, that is,

$$\begin{aligned}
R\ =\ & TE-\sum_i\int_0^{\bar{v}}\pi_i(v_i,v_i)f(v_i)dv_i \\
=\ & E_{\mathbf{v}}\sum_i E_{p_i}\left[\begin{array}{c} q_i^1(v_i,\mathbf{v}_{-i})\big[q_i^2(v_i,p_i;\hat{p}_i)v_i+(1-q_i^2(v_i,p_i;\hat{p}_i))\max\{v_i-p_i,0\}\big] \\ +(1-q_i^1(v_i,\mathbf{v}_{-i}))\max\{v_i-p_i,0\} \end{array}\right] \\
& -E_{\mathbf{v}}\sum_i E_{p_i}\left(\left\{\begin{array}{c} q_i^1(v_i,\mathbf{v}_{-i})\big[q_i^2(v_i,p_i;\hat{p}_i)+(1-q_i^2(v_i,p_i;\hat{p}_i))\mathbf{1}\{v_i-p_i\geq 0\}\big] \\ +(1-q_i^1(v_i,\mathbf{v}_{-i}))\mathbf{1}\{v_i-p_i\geq 0\} \end{array}\right\}\cdot[\frac{1-F(v_i)}{f(v_i)}]\right) \\
=\ & E_{\mathbf{v}}\sum_i E_{p_i\leq v_i}\left\{q_i^1(v_i,\mathbf{v}_{-i})q_i^2(v_i,p_i;\hat{p}_i)p_i+[v_i-\frac{1-F(v_i)}{f(v_i)}]-p_i\right\} \\
& +E_{\mathbf{v}}\sum_i E_{p_i>v_i}\left\{q_i^1(v_i,\mathbf{v}_{-i})q_i^2(v_i,p_i;\hat{p}_i)[v_i-\frac{1-F(v_i)}{f(v_i)}]\right\}.
\end{aligned}$$

Let us further define virtual value as follows:

$$\lambda(v_i,p_i)\ =\ \left\{\begin{array}{ll} p_i & \text{if } p_i\ \leq\ v_i; \\ J(v_i) & \text{if } p_i\ >\ v_i. \end{array}\right.$$

where $J(v_i) \equiv v_i - \frac{1-F(v_i)}{f(v_i)}$. The seller's revenue function R can be re-written as follows:

$$R \;=\; E_{\mathbf{v}} \sum_i E_{p_i \le v_i} \left\{ \left[v_i - \frac{1-F(v_i)}{f(v_i)} \right] - p_i \right\} \;+\; E_{\mathbf{v}} \sum_i q_i^1(v_i, \mathbf{v}_{-i}) E_{p_i} \left\{ q_i^2(v_i, p_i; \hat{p}_i) \lambda(v_i, p_i) \right\}.$$

We complete the proof. □

Proof of Proposition 1

From (5), we can now identify the cutoff $\hat{p}_i^{so}(v_i, \mathbf{v}_{-i})$ such that the corresponding $q_i^2(v_i, p_i; \hat{p}_i)$ maximizes $E_{p_i}\left\{q_i^2(v_i, p_i; \hat{p}_i)\lambda(v_i, p_i)\right\} \;=\; E_{p_i \ge \hat{p}_i}\left\{\lambda(v_i, p_i)\right\}$, that is,

$$E_{p_i \ge \hat{p}_i}\left\{\lambda(v_i, p_i)\right\} \;=\; \begin{cases} \displaystyle\int_{\hat{p}_i}^{\bar{v}} J(v_i) d\Phi(p_i) & \text{if } \hat{p}_i \ge v_i; \\ \displaystyle\int_{\hat{p}_i}^{v_i} p_i d\Phi(p_i) + \int_{v_i}^{\bar{v}} J(v_i) d\Phi(p_i) & \text{if } \hat{p}_i < v_i. \end{cases}$$

Differentiating $E_{p_i \ge \hat{p}_i}\left\{\lambda(v_i, p_i)\right\}$ with respect to $\hat{p}_i$ yields

$$\frac{\partial E_{p_i \ge \hat{p}_i}\left\{\lambda(v_i, p_i)\right\}}{\partial \hat{p}_i} \;=\; \begin{cases} -J(v_i)\varphi(\hat{p}_i) & \text{if } \hat{p}_i \ge v_i; \\ -\hat{p}_i \varphi(\hat{p}_i) & \text{if } \hat{p}_i < v_i. \end{cases}$$

Clearly, if $J(v_i) \ge 0$, then $\frac{\partial E_{p_i \ge \hat{p}_i}\{\lambda(v_i, p_i)\}}{\partial \hat{p}_i} \le 0, \forall \hat{p}_i$. Therefore, for $v_i < v_i^M$ where $J(v^M) = 0$, we should have

$$\hat{p}_i^{so}(v_i, \mathbf{v}_{-i}) = 0, \text{ if } v_i \ge v^M.$$

If $J(v_i) < 0$, then $\frac{\partial E_{p_i \ge \hat{p}_i}\{\lambda(v_i, p_i)\}}{\partial \hat{p}_i} \le 0, \forall \hat{p}_i < v_i$; and $\frac{\partial E_{p_i \ge \hat{p}_i}\{\lambda(v_i, p_i)\}}{\partial \hat{p}_i} > 0, \forall \hat{p}_i > v_i$. Therefore, $\hat{p}_i^{so}(v_i, \mathbf{v}_{-i})$ is either 0 or $\bar{v}$. We thus compare $\varsigma(v_i) = \int_0^{v_i} p_i d\Phi(p_i) + \int_{v_i}^{\bar{v}} J(v_i) d\Phi(p_i)$ and 0 to pin down the optimal $\hat{p}_i^{so}(v_i, \mathbf{v}_{-i})$:

$$\hat{p}_i^{so}(v_i, \mathbf{v}_{-i}) \;=\; \begin{cases} 0 & \text{if } \varsigma(v_i) \ge 0; \\ \bar{v} & \text{if } \varsigma(v_i) < 0. \end{cases}$$

To this end, we study the property of $\varsigma(v_i)$ for $v_i < v^M$:

$$\varsigma'(v_i) = \frac{1-F(v_i)}{f(v_i)} \Phi'(v_i) > 0.$$

Furthermore, $\varsigma(0) = \int_0^{\bar{v}} J(0) d\Phi(p_i) < 0$, and $\varsigma(v^M) = \int_0^{v^M} p_i d\Phi(p_i) > 0$. Thus, there must exist a unique cutoff $\check{v}^{so}$ such that $\varsigma(\check{v}^{so}) = 0$, i.e., $\varsigma(\check{v}^{so}) = \int_0^{\check{v}^{so}} p_i d\Phi(p_i) + \int_{\check{v}^{so}}^{\bar{v}} J(\check{v}^{so}) d\Phi(p_i) = 0$. This indicates that $\hat{p}_i^{so}(v_i, \mathbf{v}_{-i}) = 0$ if $v_i \in [\check{v}^{so}, v^M)$ and $\hat{p}_i^{so}(v_i, \mathbf{v}_{-i}) = \bar{v}$ if $v_i < \check{v}^{so}$. Furthermore, simplifying $\int_0^{\check{v}^{so}} p_i d\Phi(p_i) = -\int_{\check{v}^{so}}^{\bar{v}} J(\check{v}^{so}) d\Phi(p_i)$ gives

$$\frac{\int_0^{\check{v}^{so}} [1 - \Phi(p_i)] dp_i}{1 - \Phi(\check{v}^{so})} \;=\; \frac{1 - F(\check{v}^{so})}{f(\check{v}^{so})}. \tag{19}$$

Summarizing the discussion above yields

$$\hat{p}_i^{so}(v_i, \mathbf{v}_{-i}) \;=\; \begin{cases} 0 & \text{if } v_i \;\geq\; \check{v}^{so}; \\ \bar{v} & \text{if } v_i \;<\; \check{v}^{so}. \end{cases} \tag{20}$$

Under (20), the selected bidder i with $v_i \geq \check{v}^{so}$ will not choose to take the outside offer. We further have

$$R \;=\; E_{\mathbf{v}} \sum_i E_{p_i \leq v_i} \left\{ \left[v_i - \frac{1-F(v_i)}{f(v_i)} \right] - p_i \right\} \;+\; E_{\mathbf{v}} \sum_i q_i^1(v_i, \mathbf{v}_{-i}) E_{p_i} \left\{ q_i^2(v_i, p_i; \hat{p}_i^{so}(v_i, \mathbf{v}_{-i}))\lambda(v_i, p_i) \right\}.$$

It is clear that $E_{p_i}\left\{q_i^2(v_i, p_i; \hat{p}_i^{so}(v_i, \mathbf{v}_{-i}))\lambda(v_i, p_i)\right\} = 0$ for $v_i \leq \check{v}^{so}$, since $q_i^2(v_i, p_i; \hat{p}_i^{so}(v_i, \mathbf{v}_{-i})) = 0$ for $v_i \leq \check{v}^{so}$. When $v_i > \check{v}^{so}$, $q_i^2(v_i, p_i; \hat{p}_i^{so}(v_i, \mathbf{v}_{-i})) = 1$ and thus $E_{p_i}\left\{q_i^2(v_i, p_i; \hat{p}_i^{so}(v_i, \mathbf{v}_{-i}))\lambda(v_i, p_i)\right\} = E_{p_i}\lambda(v_i, p_i)$. Note that $\lambda(v_i, p_i)$ strictly increases with v_i. To maximize R, it is clear to set

$$q_i^{so}(v_i, \mathbf{v}_{-i}) \;=\; \begin{cases} 1 & \text{if } v_i = v^{(1)} \text{ and } v_i \geq \check{v}^{so}; \\ 0 & \text{otherwise,} \end{cases} \tag{21}$$

where $v^{(1)} = \max\{v_i, i = 1, 2, ..., N\}$. In other words, only the bidder with the highest value is invited to enter the second stage, provided the bidder's value is no less than $\check{v}^{so}$. This is equivalent to setting the threshold at $\check{v}^{so}$ in the first stage.

In the mechanism, the total expected payment (the seller's revenue) R can be written as follows:

$$\begin{aligned} R \;&=\; E_{\mathbf{v}} \sum_i E_{p_i \leq v_i} \left\{ \left[v_i - \frac{1-F(v_i)}{f(v_i)} \right] - p_i \right\} \;+\; E_{\mathbf{v}} \sum_i q_i^1(v_i, \mathbf{v}_{-i}) E_{p_i} \left\{ q_i^2(v_i, p_i; \hat{p}_i^{so}(v_i))\lambda(v_i, p_i) \right\} \\ &=\; N \int_0^{\bar{v}} \left(\big[v_i - \frac{1-F(v_i)}{f(v_i)} \big] \Phi(v_i) - \int_0^{v_i} p_i \varphi(p_i) dp_i \right) dF(v_i) \\ &\quad +\; N \int_0^{\bar{v}} \left(\int_0^{v_i} p_i \varphi(p_i) dp_i + \int_{v_i}^{\bar{v}} \big[v_i - \frac{1-F(v_i)}{f(v_i)} \big] \varphi(p_i) dp_i \right) Q(v_i) f(v_i) dv_i \\ &=\; N \int_0^{\bar{v}} \left(v_i Q(v_i) + (1 - Q(v_i)) \int_0^{v_i} \Phi(p_i) dp_i - \frac{1-F(v_i)}{f(v_i)} \big(Q(v_i) + (1 - Q(v_i))\Phi(v_i) \right) f(v_i) dv_i \\ &=\; N \int_0^{\bar{v}} (1 - F(t)) \big(t - \int_0^t \Phi(x) dx \big) dQ(t). \end{aligned}$$

where $Q(v_i) = F^{N-1}(v_i)$. When bidder i is selected, v_i should be no less than $\check{v}^{so}$. Let us further define $r^{so} \;=\; \int_0^{\check{v}^{so}} [1 - \Phi(p_i)] dp_i$, which gives the payment bidder i with value $\check{v}^{so}$ will pay conditional on being

selected. Then, the expected payment from bidder i can be written as follows:

$$\int_{\check{v}^{so}}^{\bar{v}} \Big(Q(\check{v}^{so}) r^{so} + \int_{\check{v}^{so}}^{v_i} \big(t - \int_0^t \Phi(x)dx\big) Q'(t)dt \Big) f(v_i)dv_i$$
$$= \big(1 - F(\check{v}^{so})\big) Q(\check{v}^{so}) r^{so} + \int_{\check{v}^{so}}^{\bar{v}} (1 - F(t)) \big(t - \int_0^t \Phi(x)dx\big) dQ(t).$$

With N bidders in the mechanism, we then have

$$R^* \;=\; N\big(1 - F(\check{v}^{so})\big) Q(\check{v}^{so}) r^{so} + N \int_{\check{v}^{so}}^{\bar{v}} (1 - F(t)) \big(t - \int_0^t \Phi(x)dx\big) dQ(t). \tag{22}$$

We complete the proof. □

Proof of Proposition 2

Given the fact that $\mathbb{E}_S^I[R(r, D)] = \mathbb{E}_S^{II}[R(r, D)]$ when $r = D$ (implying that $\check{v} = \hat{v}$), Lemma 7 indicates that for any $D \in [r^{so}, D^{so}]$, $\mathbb{E}_S^I[R(r^I(D) = r^{so}, D)] > \mathbb{E}_S^I[R(r^I(D) = D, D)] = \mathbb{E}_S^{II}[R(r^{II}(D) = D, D)]$. Therefore, $r^*(D) = r^{so}$ for any $D \in [r^{so}, D^{so}]$; this proves part (i). Regarding part (ii), Lemma 8 states that for any $D \in [0, r^{so})$, $\mathbb{E}_S^I[R(r^I(D) = D, D)] = \mathbb{E}_S^{II}[R(r^{II}(D) = D, D)] < \mathbb{E}_S^{II}[R(r^{II}(D) = \tilde{r}^{II}(D), D)]$. Hence, $r^*(D) = \tilde{r}^{II}(D)$ for any $D \in [0, r^{so})$. Note that in this case, $r^*(D)$ is not necessary to be monotone in D when $D \in [0, r^{so})$. □

Proof of Proposition 3

For $D \in [r^{so}, D^{so}]$, we have $R^*(D) = \mathbb{E}_S^I[R(r^*(D), D)]$ where $r^*(D) = r^{so}$. We have the following two steps to establish the result:

Step (i). Differentiating $\mathbb{E}_S^I[R(r^*(D), D)]$ with respect to $\hat{v}$, and evaluating it at $\hat{v} = \hat{v}(D)$ give the following equation:

$$\frac{\partial}{\partial \hat{v}} \frac{\mathbb{E}_S^I[R(r^*(D), D)]}{N}\Big|_{\hat{v}=\hat{v}(D)}$$
$$= \Bigg[\big(1 - F(\hat{v}(D))\big) b(\hat{v}(D)) \;-\; \big(1 - F(\hat{v}(D))\big) \Big[\big(1 - \Phi(\tilde{b}(\hat{v}(D)) - D)\big) \tilde{b}(\hat{v}(D)) + \Phi(\tilde{b}(\hat{v}(D)) - D) D \Big] \Bigg] q(\hat{v}(D)), \tag{23}$$

where $q(\cdot) = Q'(\cdot)$. Since $\tilde{b}(\hat{v}(D)) - D = b(\hat{v}(D)) - D = 0$, (23) can be re-written as follows:

$$\frac{\partial}{\partial \hat{v}} \frac{\mathbb{E}_S^I[R(r^*(D), D)]}{N}\Big|_{\hat{v}=\hat{v}(D)} \;=\; \Big[\big(1 - F(\hat{v}(D))\big) D - \big(1 - F(\hat{v}(D))\big) D \Big] q(\hat{v}(D)) \;=\; 0. \tag{24}$$

This indicates that $\hat{v}(D)$ has no impact on $\mathbb{E}_S^I[R(r^*(D), D)]$.

Step (ii). Step (i) indicates that $\frac{\partial}{\partial \hat{v}} \frac{\mathbb{E}_S^I[R(r^*(D),D)]}{N}|_{\hat{v}=\hat{v}(D)} = 0$, and we then have that the impact of D through $r^*(D)$ is zero by envelope theorem, as $r^*(D) = r^{so}$ is an interior optimum for the given D. We thus have the following equation

$$\begin{aligned}
\frac{d}{dD}\frac{\mathbb{E}_S^I[R(r^*(D),D)]}{N} &= \frac{\partial}{\partial D}\frac{\mathbb{E}_S^I[R(r,D)]}{N}|_{r=r^*(D),\hat{v}=\hat{v}(D)} \\
&= \int_{\hat{v}(D)}^{\bar{v}} \big(1-F(x)\big)\Bigg[-\varphi(\tilde{b}(x)-D)\big(\Phi(\tilde{b}(x)-D)(\frac{\partial \tilde{b}(x)}{\partial D}-1)-1\big)\tilde{b}(x) \\
&\quad + \big(1-\Phi(\tilde{b}(x)-D)\big)\Phi(\tilde{b}(x)-D)(\frac{\partial \tilde{b}(x)}{\partial D}-1) \\
&\quad + \varphi(\tilde{b}(x)-D)\big(\Phi(\tilde{b}(x)-D)(\frac{\partial \tilde{b}(x)}{\partial D}-1)-1\big)D + \Phi(\tilde{b}(x)-D)\Bigg]dQ(x),
\end{aligned} \tag{25}$$

Note that $\frac{\partial}{\partial D}\frac{\mathbb{E}_S^I[R(r,D)]}{N}$ denotes the partial derivative of $\frac{\mathbb{E}_S^I[R(r,D)]}{N}$ with respect to D while fixing r and $\hat{v}$. Since $\frac{\partial \tilde{b}(x)}{\partial D} = \frac{-\Phi(\tilde{b}(x)-D)}{1-\Phi(\tilde{b}(x)-D)}$, we have

$$\frac{\partial \tilde{b}(x)}{\partial D} - 1 = \frac{-\Phi(\tilde{b}(x)-D)}{1-\Phi(\tilde{b}(x)-D)} - \frac{1-\Phi(\tilde{b}(x)-D)}{1-\Phi(\tilde{b}(x)-D)} = \frac{-1}{1-\Phi(\tilde{b}(x)-D)}. \tag{26}$$

Plugging (26) into (25) shows

$$\begin{aligned}
&\frac{d}{dD}\frac{\mathbb{E}_S^I[R(r^*(D),D)]}{N} \\
&= \int_{\hat{v}(D)}^{\bar{v}} (1-F(x))\Bigg[-\varphi(\tilde{b}(x)-D)\big(\Phi(\tilde{b}(x)-D)(\frac{\partial \tilde{b}(x)}{\partial D}-1)-1\big)\tilde{b}(x) \\
&\quad + \varphi(\tilde{b}(x)-D)\big(\Phi(\tilde{b}(x)-D)(\frac{\partial \tilde{b}(x)}{\partial D}-1)-1\big)D\Bigg]dQ(x) \\
&= \int_{\hat{v}(D)}^{\bar{v}} (1-F(x))\Bigg[-\varphi(\tilde{b}(x)-D)\big(\frac{-1}{1-\Phi(\tilde{b}(x)-D)}\big)\tilde{b}(x) + \varphi(\tilde{b}(x)-D)\big(\frac{-1}{1-\Phi(\tilde{b}(x)-D)}\big)D\Bigg]dQ(x) \\
&= \int_{\hat{v}(D)}^{\bar{v}} (1-F(x))\big(\frac{\varphi(\tilde{b}(x)-D)(\tilde{b}(x)-D)}{1-\Phi(\tilde{b}(x)-D)}\big)dQ(x),
\end{aligned} \tag{27}$$

Clearly, $\frac{d}{dD}\frac{\mathbb{E}_S^I[R(r^*(D),D)]}{N} \geq 0$ when D is in the interval of $[r^{so}, D^{so}]$. The equality holds if and only if $\hat{v}(D) = \bar{v}$, which implies that the seller charges $D = D^{so}$.

For any $D \in [0, r^{so})$, we have $R^*(D) = \mathbb{E}_S^{II}[R(r^*(D),D)]$ where $r^*(D) = \tilde{r}^{II}(D) > D$. The impact of r on seller revenue is solely through its impact on $\check{v}(r,D)$. At optimal $r^*(D)$, the marginal effect is zero, which

means $\frac{\partial}{\partial \check{v}} \frac{\mathbb{E}_S^{II}[R(r^*(D),D)]}{N}|_{\check{v}=\check{v}(r^*(D),D)} = 0$. This result, together with envelop theorem, gives

$$\begin{aligned}
\frac{d}{dD}\frac{\mathbb{E}_S^{II}[R(r^*(D),D)]}{N} &= \frac{\partial}{\partial D}\frac{\mathbb{E}_S^{II}[R(r,D)]}{N}|_{r=r^*(D),\check{v}=\check{v}(r^*(D),D)} \\
&= \big(1-F(\check{v}(r^*(D),D))\big)Q(\check{v}(r^*(D),D))\big(\frac{\varphi(b(\check{v}(r^*(D),D))-D)(b(\check{v}(r^*(D),D))-D)}{1-\Phi(b(\check{v}(r^*(D),D))-D)}\big) \\
&\quad + \int_{\check{v}(r^*(D),D)}^{\bar{v}}(1-F(x))\big(\frac{\varphi(b(x)-D)(b(x)-D)}{1-\Phi(b(x)-D)}\big)dQ(x),
\end{aligned} \tag{28}$$

Note that $\frac{\partial}{\partial D}\frac{\mathbb{E}_S^{II}[R(r,D)]}{N}$ denotes the partial derivative of $\frac{\mathbb{E}_S^{II}[R(r,D)]}{N}$ with respect to D while fixing r and $\check{v}$. When the seller charges any D in the interval of $[0, r^{so})$, we have $\check{v}(r^*(D),D) < \bar{v}$ which indicates that $\frac{d}{dD}\frac{\mathbb{E}_S^{II}[R(r^*(D),D)]}{N} > 0$.

Summarizing the discussion above, we can conclude that charging $D = D^{so}$ is optimal. Any (D, r^{so}) with $D > D^{so}$ also maximizes seller's revenue, since seller revenue does not change in D by Lemma 5. □

Proof of Proposition 4

We separately prove parts (i), (ii), and (iii) as follow:

Part (i). Case (I) where $r - D \leq 0$. First, given that $\Phi_1(\cdot)$ first-order stochastically dominates $\Phi_2(\cdot)$, i.e. $\Phi_1(\cdot) < \Phi_2(\cdot)$, which means that $\Phi_1(\cdot)$ gives a worse outside offer to the buyers, we can easily establish the following facts: (a) $\int_0^{\check{v}_2}[1-\Phi_2(p_i)]dp_i = r = \int_0^{\check{v}_1}[1-\Phi_1(p_i)]dp_i$ implies $\check{v}_1 < \check{v}_2$; (b) $\int_0^{\hat{v}_2}[1-\Phi_2(p_i)]dp_i = D = \int_0^{\hat{v}_1}[1-\Phi_1(p_i)]dp_i$ gives $\hat{v}_1 < \hat{v}_2$.

Second, we compare equilibrium bidding strategies across $\Phi_1(\cdot)$ and $\Phi_2(\cdot)$, given $\Phi_1(\cdot) < \Phi_2(\cdot)$. For convenience, we write $b(v_i, \Phi_k)$ and $\tilde{b}(v_i, \Phi_k), k = 1, 2$ for $v_i \in [\check{v}_k, \hat{v}_k]$ and $v_i > \hat{v}_k$, respectively. We then have the following:

1. For $v_i \in [\check{v}_1, \check{v}_2)$, $b(v_i, \Phi_1) = \int_0^{v_i}[1-\Phi_1(p_i)]dp_i \geq r$ under $\Phi_1(.)$, but the bidder under $\Phi_2(.)$ does not submit a valid bid, equivalently, $b(v_i, \Phi_2) = 0$. Thus, $b(v_i, \Phi_1) > b(v_i, \Phi_2)$.

2. For $v_i = \check{v}_2$, given that the equilibrium bidding strategy is monotone and increasing, the following inequality must be true: $b(v_i, \Phi_1) > b(\check{v}_1, \Phi_1) = b(v_i, \Phi_2) = r$. Thus, $b(v_i, \Phi_1) > b(v_i, \Phi_2)$.

3. For $v_i \in (\check{v}_2, \hat{v}_1]$, we have $b(v_i, \Phi_1) = \int_0^{v_i}[1-\Phi_1(p_i)]dp_i > \int_0^{v_i}[1-\Phi_2(p_i)]dp_i = b(v_i, \Phi_2)$. Thus, $b(v_i, \Phi_1) > b(v_i, \Phi_2)$.

4. For $v_i \in (\hat{v}_1, \hat{v}_2)$, given that the equilibrium bidding strategy is monotone and increasing, we have $\tilde{b}(v_i, \Phi_1) > b(v_i, \Phi_1) = \int_0^{v_i}[1-\Phi_1(p_i)]dp_i > \int_0^{v_i}[1-\Phi_2(p_i)]dp_i = b(v_i, \Phi_2)$. Thus, $\tilde{b}(v_i, \Phi_1) > b(v_i, \Phi_2)$.

5. For $v_i = \hat{v}_2$, given that the equilibrium bidding strategy is monotone and increasing, the following inequality must be true: $\tilde{b}(v_i, \Phi_1) > b(\hat{v}_1, \Phi_1) = b(v_i, \Phi_2) = D$. Thus, $\tilde{b}(v_i, \Phi_1) > b(v_i, \Phi_2)$.

6. For $v_i \in (\hat{v}_2, \bar{v}]$, recall that we can re-write $\tilde{b}(v_i, \Phi)$ as $\int_{\tilde{b}(v_i,\Phi)-D}^{v_i}[1-\Phi(p_i)]dp_i = D$, therefore, we have $\int_{\tilde{b}(v_i,\Phi_1)-D}^{v_i}[1-\Phi_1(p_i)]dp_i = D = \int_{\tilde{b}(v_i,\Phi_2)-D}^{v_i}[1-\Phi_2(p_i)]dp_i$, which immediately indicates $\tilde{b}(v_i, \Phi_1) > \tilde{b}(v_i, \Phi_2)$.

Case (II) where $r - D \geq 0$. Recall that $\int_{r-D}^{\check{v}}[1-\Phi(p_i)]dp_i = D$. First, given that $\Phi_1(\cdot)$ first-order stochastically dominates $\Phi_2(\cdot)$, i.e. $\Phi_1(\cdot) < \Phi_2(\cdot)$, we can easily establish the following fact: $\int_{r-D}^{\check{v}_1}[1-\Phi_1(p_i)]dp_i = D = \int_{r-D}^{\check{v}_2}[1-\Phi_2(p_i)]dp_i$ implies $\check{v}_1 < \check{v}_2$.

Second, we compare equilibrium bidding strategies across $\Phi_1(\cdot)$ and $\Phi_2(\cdot)$, given $\Phi_1(\cdot) < \Phi_2(\cdot)$.

- For $v_i \in [\check{v}_1, \check{v}_2)$, $b(v_i, \Phi_1) \geq r$ under $\Phi_1(.)$, but the bidder under $\Phi_2(.)$ does not submit a valid bid, equivalently, $b(v_i, \Phi_2) = 0$. Thus, $b(v_i, \Phi_1) > b(v_i, \Phi_2)$.

- For $v_i = \check{v}_2$, given that the equilibrium bidding strategy is monotone and increasing, the following inequality must be true: $b(v_i, \Phi_1) > b(\check{v}_1, \Phi_1) = b(v_i, \Phi_2) = r$. Thus, $b(v_i, \Phi_1) > b(v_i, \Phi_2)$.

- For $v_i \in (\check{v}_2, \bar{v}]$, recall that we can re-write $b(v_i, \Phi)$ as $\int_{b(v_i,\Phi)-D}^{v_i}[1-\Phi(p_i)]dp_i = D$, therefore, we have $\int_{b(v_i,\Phi_1)-D}^{v_i}[1-\Phi_1(p_i)]dp_i = D = \int_{b(v_i,\Phi_2)-D}^{v_i}[1-\Phi_2(p_i)]dp_i$, which immediately indicates $b(v_i, \Phi_1) > b(v_i, \Phi_2)$.

Therefore, we can conclude that given reserve price r and deposit D, the equilibrium bid submitted by a bidder is higher when the distribution of the outside offer becomes worse in the sense of first-order stochastic dominance.

Part (ii). We examine how the optimal deposit D^{so} changes across different distributions of the outside offers. Recall that $\hat{v}_1 = \hat{v}_2 = \overline{v}$ corresponds to bids D_1^{so} and D_2^{so}. It is obvious that if $\Phi_1(\cdot) < \Phi_2(\cdot)$, then $D_1^{so} = \int_0^{\bar{v}}[1-\Phi_1(p_i)]dp_i > \int_0^{\bar{v}}[1-\Phi_2(p_i)]dp_i = D_2^{so}$.

Part (iii). Given that $\Phi_1(\cdot)$ first-order stochastically dominates $\Phi_2(\cdot)$, i.e., $\Phi_1(\cdot) < \Phi_2(\cdot)$, let us denote the optimal reserve price under $\Phi_i(\cdot)$ by $r_i^{so}, i = 1, 2$. Note that any sufficiently high D is optimal and fully deters the winner's default, and the optimal revenues under $\Phi_i(\cdot), i = 1, 2$ do not depend on D when it is optimally set. Taking such a D, under (r_2^{so}, D) the winner's default is fully deterred even with $\Phi_1(\cdot)$. Since $\Phi_1(\cdot) < \Phi_2(\cdot)$, under (r_2^{so}, D), we then have $\check{v}_1 < \check{v}_2$ and $b(v_i, \Phi_1) > b(v_i, \Phi_2)$ by Proposition 4 and its proof.

Under (r_2^{so}, D), seller revenue with $\Phi_1(\cdot)$ is given by

$$R(r_2^{so}, D, \Phi_1) \;=\; N\big(1-F(\check{v}_1)\big)Q(\check{v}_1)r_2^{so} + N\int_{\check{v}_1}^{\bar{v}}\big(1-F(x)\big)b(x,\Phi_1)dQ(x),$$

and the optimal seller revenue under $\Phi_2(\cdot)$ is given by

$$R^*(r_2^{so}, D, \Phi_2) \;=\; N\big(1 - F(\check{v}_2)\big)Q(\check{v}_2)r_2^{so} + N\int_{\check{v}_2}^{\bar{v}} \big(1 - F(x)\big)b(x, \Phi_2)dQ(x).$$

Define $R(v_1) = N(1-F(v_1))Q(v_1)r_2^{so} + N\int_{v_1}^{\bar{v}} \big(1-F(x)\big)b(x, \Phi_1)dQ(x)$. Differentiating $R(v_1)$ with respect to v_1 yields

$$\frac{dR(v_1)}{dv_1} \;=\; -Nf(v_1)Q(v_1)r_2^{so} + N\big(1 - F(v_1)\big)Q'(v_1)\big(r_2^{so} - b(v_1, \Phi_1)\big) \;<\; 0,\ \forall v_1 > \check{v}_1.$$

Let $R^*(r_1^{so}, D, \Phi_1)$ denote the optimal seller revenue under Φ_1. Thus, it is clear that $R^*(r_1^{so}, D, \Phi_1) \geq R(r_2^{so}, D, \Phi_1) = R(\check{v}_1) > R(\check{v}_2) \geq R^*(r_2^{so}, D, \Phi_2)$. $\square$