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Optimal prize allocation in contests: The role of negative prizes ☆

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Abstract

In this paper, we analyze the role of negative prizes in contest design with a fixed budget, risk-neutral contestants, and independent private abilities. The effort-maximizing prize allocation rule features a threshold. When the highest effort is above the threshold, all contestants with lower efforts receive negative prizes. These negative prizes are used to augment the prize to the contestant with the highest effort, which better incentivizes contestants with higher abilities. When no contestant's effort exceeds the threshold, all contestants equally split the initial budget (or a portion of it) to ensure their participation. We find that allowing

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negative prizes can increase the expected total effort dramatically. In particular, if no bound is imposed on negative prizes, the expected total effort can be arbitrarily close to the highest possible effort inducible when all contestants have the maximum ability with certainty. The above contest is shown to be the optimal mechanism for a more general class of mechanisms.

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1. Introduction

Many activities and events, such as promotions within organizations, school admissions, political elections, R&D races, and sports, can be viewed as contests. As a well-established institution, contests essentially incentivize contestants to exert costly and irreversible effort through awarding prizes to winners according to their relative rankings. It is often the case that contestants have private information about their own abilities, valuations, competence, etc. A contest designer desires an optimal prize architecture in order to deal with such private information to achieve her objective. In this paper, the objective is to maximize the total effort.[1]

To illustrate the problem we analyze in this paper, imagine that a contest designer has a million dollars to distribute among some ex ante symmetric, but privately informed, risk-neutral workers to incentivize them to exert effort in an all-pay auction. In this environment, Moldovanu and Sela (2001) show that when restricting to fixed nonnegative prizes, the optimal prize structure to maximize the expected total effort is to award the entire prize to the worker with the highest effort. Their result can be understood from the mechanism design intuition: The designer should rank workers according to their "virtual abilities," and the one with the highest virtual ability—who happens to be the one with the highest ability—should be awarded the entire prize (under some regularity assumption), assuming symmetry in workers' ability distributions.

One interesting question is to what extent the insight from Moldovanu and Sela (2001) would generalize if we allow negative prizes to contestants. The intuition to maximize the reward to the highest ability worker in the optimal mechanism is that a dollar can induce more effort from a high-ability worker than from a low-ability worker, since exerting effort is more costly for a lower ability worker. Therefore, the contest designer would like to not only award her entire prize budget to the worker with the highest ability, but also transfer money from those lower ability workers to the highest ability worker if she is allowed to do so. If such monetary cross-type transfers do happen, a lower ability worker may end up with a negative prize ex post. The challenge is then to provide enough incentives for a low-ability worker to participate in the contest. One potential solution is to award every worker if all are of low abilities. As a result, although a low-ability worker may obtain a negative prize with some probability, he obtains a positive prize with the rest of the probability and his interim participation constraint can be satisfied. In Moldovanu and Sela (2001), monetary cross-type transfers involving negative prizes are not allowed, which implies that the total effort elicited from their model is likely to be lower if negative prizes are allowed.

[1] This objective function has been used in many papers in the contest literature; for example, Moldovanu and Sela (2001).

The above discussion suggests that the expected total effort can be further improved if we introduce an effort threshold in the multi-prize all-pay auction of Moldovanu and Sela (2001). In this paper, we consider the optimal prize allocation rule in the following setup. The contest designer has a fixed prize budget V that she can allocate to contestants in an all-pay auction. The prize a contestant can obtain depends only on his relative ranking and whether the highest effort is higher than a threshold. The designer is allowed to use both positive and negative prizes, and an upper bound (K) is imposed on the magnitude of negative prizes due to limited liability.

We would like to emphasize that in our paper, negative prizes are only money, similar to Fullerton and McAfee (1999), and the money collected can be used by the contest designer to strengthen the incentives provided to contestants.[2] Moldovanu et al. (2012) allow negative prizes in a different way. They assume that a negative prize, which can be viewed as a physical or mental/psychological punishment—i.e., a kind of "stick"—can be either costly or costless for the organizer to implement. In addition, Moldovanu et al. (2012) mainly study the case in which contestants cannot quit the contest—i.e., the participation constraint can be ignored in their analysis[3]—while we focus on the case in which the interim participation constraint of every type of contestants must hold.

We fully characterize the optimal prize structure in this environment. When K is smaller than a certain value, i.e., when contestants cannot be penalized too severely, the ability corresponding to the optimal threshold effort is exactly the cut-off type in Myerson (1981). When K is larger than that value, however, the ability corresponding to the threshold effort is strictly higher than the Myerson cut-off. Furthermore, when the highest effort is lower than the threshold, all contestants share equally either a portion (in the former case) or the entirety (in the latter case) of the original prize budget. In either case of K, when the highest effort is above the threshold, the contestant with the highest effort will receive an enhanced prize, which is equal to the original prize budget plus the extra money collected through the negative prizes (i.e., $-K$) imposed on the rest of the contestants. We show that the expected total effort is increasing in K, and when K goes to infinity, the expected total effort goes to the highest possible effort inducible when all contestants have the maximum ability with certainty.

One distinct feature of the optimal prize structure is that when all contestants' efforts are below the threshold, they equally share the prize budget (or a portion of it). The purpose of this is to ensure the participation of low-ability contestants, which renders transfers from low-ability contestants to high-ability ones (through negative prizes) feasible. Some real-life contests that share this feature include the performance-based salary adjustments (or supplements) adopted by many departments at universities around the world. If no faculty members perform significantly well, the salary adjustment (or supplement) will be equally divided among all faculty members instead of being canceled. The possible reason for this division, as will be explained by our analysis, is to ensure their participation in the system so that they will not complain or sue the department to obtain a fair share ex ante.

The above example also illustrates the use of negative prizes in the optimal contest. Every year, the total budget V for salary increases for a department is usually fixed, while inflation

2 Fullerton and McAfee (1999) study the optimal shortlisting in a procurement environment, and allow the sponsor to use entry fees to screen firms and offset procurement costs. Recently, Ghosh and McAfee (2012) find that free entry is dominated by taxing entry (which is a form of entry fees) in crowdsourcing tournaments, a conclusion echoed by our paper, though in a different environment.

3 Moldovanu et al. (2012) also consider the situation in which punishments are costless. In this case, it is always optimal to punish, even when contestants can quit.

adjustment can be viewed as the bound K for every faculty member. When some faculty member's performance stands out, the inflation adjustment for those with subpar performance could be denied in order to disproportionately increase the pay adjustment for the outstanding member. In this case, those with subpar performance literally suffer from a negative prize. This feature of negative prizes is also found in many real-life contests. In the FCC-organized contest to set the standard for high-definition television, any firm can enter the contest by paying a $200,000 entry fee (cf. Taylor, 1995). Thus, a firm that enters the contest but does not win gets a negative prize. Similarly, professional tournaments, such as golf, sailing, chess, and horse racing, often charge significant fees for membership, registration, nomination, and/or starting, which constitute a significant portion of the prizes. As a result, participants who lose the tournament receive a negative prize. As another example, to compete in certain poker tournaments, a player must pay a "buy-in," which is an upfront payment that goes toward the winning prize pool. In this case, the buy-in becomes a negative prize if the player loses the game.

In their pioneering work on contest design, Moldovanu and Sela (2001) focus on the contest structure of all-pay auctions and allow the designer to distribute the prize budget according to a contestant's ranking. They find that with linear utility functions or concave cost of effort functions, it is optimal to award the entire prize to the best contestant. Meanwhile, with convex cost of effort functions, such a single-prize rule may not be optimal. Minor (2012) reexamines this single-prize principle in cases in which contestants have convex costs of effort and the contest designer has concave benefit of effort. Olszewski and Siegel (2017) consider large contests, in which the number of players goes to infinity, with complete or incomplete information, and ex ante asymmetric contestants. They show that splitting the awards is optimal when contestants are risk averse or have convex cost functions. Moldovanu and Sela (2006) generalize their investigation to a two-stage all-pay auction framework. Meanwhile, Moldovanu et al. (2007) analyze the environment in which contestants care about their relative status. None of the above papers allows negative prize, which is the main feature of our model.

Although we focus on all-pay auctions, as in the above strand of literature, due to the added analytical complexity introduced by negative prizes, the effort threshold, and the contingent sequences of prizes, we adopt a mechanism design approach to identify the optimal prize structure. Such an approach enables us to establish the optimality of our optimal prize structure in the all-pay auction among a wider range of mechanisms that allocate the budget contingent on the contestants' reported type profile. They include some widely adopted incentive mechanisms, such as piece rate and bonus schemes. Technically, our paper is mostly related to Polishchuk and Tonis (2013), Chawla et al. (forthcoming), and Kirkegaard (2012). The main difference is that they focus on nonnegative prizes. As we show in this paper, the possibility of negative prizes not only introduces new technical challenges, but also results in very different characteristics of the optimal mechanism. For example, the optimal mechanism in Polishchuk and Tonis (2013) can be derived from pointwise maximization. However, the possibility of negative prizes invalidates such an approach, and we need to adopt continuous linear programming in the analysis. Chawla et al. (forthcoming) assume that the designer cares about the maximum effort (instead of the total effort) and derive the corresponding optimal symmetric mechanism. They show that the optimal symmetric contest is still winner-take-all, but it requires a strong regularity condition. Kirkegaard (2012) adopts a similar approach to study the optimal favoritism in contests with asymmetric players. Our paper differentiates from these studies by providing a first investigation of the role of negative prizes. Our analysis sheds light on the necessity of negative prizes in the optimal contest and the leverage of prizes on contestants with different virtual abilities.

The rest of the paper is organized as follows. In Section 2, we present a model of negative prizes. In Section 3, we derive the optimal direct mechanism and the optimal prize allocation using the mechanism design approach. In Section 4, we characterize the optimal contest rule that implements the optimal direct mechanism and examine its properties. Section 5 concludes. All technical proofs are relegated to the Appendix.

2. The model

A risk-neutral contest designer has a total prize budget of $V > 0$ to elicit effort from $N \geq 2$ risk-neutral contestants in a contest.[4] Each contestant has an ability for the contest. The cost for contestant i with ability t_i to exert effort $e_i \geq 0$ is given by $c(e_i, t_i) = e_i/t_i$. This ability or type t_i,[5] is the private information of contestant i, and it follows an independent and identical distribution with cumulative distribution function $F(\cdot)$ and probability density function $f(\cdot)$, which is strictly positive on support $[a, b]$ with $a > 0$. The parameter b is referred to as the maximum ability. Similar to Myerson (1981), we define $J(t_i) = t_i - \frac{1-F(t_i)}{f(t_i)}$ as the virtual ability function. We make the standard regularity assumption that $J(t_i)$ is strictly increasing.

The payoff of a contestant in the contest is equal to the prize he receives minus his cost of effort. The contest designer uses the prize budget V to induce effort from contestants. At the same time, if there is money left in the budget, she values that money as well. To simplify notation, assume that there is a linear relationship between effort and money for the contest designer. Let $1/t_0$ be the marginal benefit of effort for the mechanism designer. Assume that t_0 is common knowledge. Note that the cost of 1 unit of effort for the maximum ability (b) contestant is $1/b$, which needs to be less than $1/t_0$; otherwise, it is obviously optimal for the designer not to spend any of the prize budget. Therefore, we assume that $t_0 < b$.

As mentioned in the introduction, an important feature of our contests is whether the highest effort among the contestants is above a certain threshold. To this end, assume that there is an effort threshold $\hat{e} \geq 0$, and there are two prize allocation vectors $\mathbf{v}_h = (v_{h1}, v_{h2}, \ldots, v_{hN})$ and $\mathbf{v}_l = (v_{l1}, v_{l2}, \ldots, v_{lN})$, where $\mathbf{v}_h$ ($\mathbf{v}_l$) is the prize allocation vector when the highest effort is above (below) the threshold $\hat{e}$. More specifically, when the highest effort is above or equal to $\hat{e}$, the contestant with the i-th highest effort wins prize v_{hi}; when the highest effort is below $\hat{e}$, the contestant with the i-th highest effort wins prize v_{li}, $i = 1, \ldots, N$. Ties are randomly broken. These two vectors satisfy the budget constraint: $\sum_{i=1}^{N} v_{pi} \leq V$ for $p \in \{h, l\}$.

2.1. Positive and negative prizes

One distinct feature of the prize structure in this paper is that negative prizes are allowed. This implies that the contest designer can, in principle, collect negative prizes from contestants and then “top up” the final prize pool. We assume that contestants have limited liabilities (or endowments) so that there is a bound ($K > 0$) for the negative prize, i.e., $v_{pi} \geq -K, \forall p \in \{h, l\}$

[4] This budget V does not need to be all in cash; it could be the value to the contestants of nonmonetary and sometimes indivisible rewards, such as honors, recognition, and gifts. We will illustrate, in the discussions following Proposition 3 in Section 4, how our results can be modified to accommodate nondivisible prizes.

[5] In Moldovanu and Sela (2001), a contestant's ability is defined as $c_i = \frac{1}{t_i}$, which is mathematically equivalent.

and $i \in \{1, \ldots, N\}$.[6] Note that monotonicity constraints, such as $v_{p1} \geq v_{p2} \geq \ldots \geq v_{pN}$ (i.e., a higher rank is paid a higher prize), are not imposed on our prize allocation vectors.

Given the above notation, our contest rule can be expressed as $(\hat{e}, \mathbf{v}_h, \mathbf{v}_l)$, which consists of the choice of three instruments: the effort threshold and the two prize allocation vectors, which correspond to whether the highest effort is above the threshold. Note that each prize allocation vector depends only on the relative rankings of contestants' efforts. The contest designer's objective is to choose a contest rule to maximize the expected total effort of the contestants plus the effort equivalent of any money left in the prize budget.

A direct approach to determining the optimal contest rule is given in three steps: 1) fix an effort threshold $\hat{e}$, then characterize the equilibrium bidding strategies for any prize allocation vector pairs $(\mathbf{v}_h, \mathbf{v}_l)$; 2) among all of these pairs, find the one that yields the highest total effort; and 3) vary across all possible effort thresholds to pin down the optimal threshold, and hence the optimal contest rule. Clearly, every step in this multistep direct procedure is calculation-intensive, and thus this approach is not very practical.

We adopt an indirect approach to get around the technical difficulties of the above direct approach. For any given contest rule $(\hat{e}, \mathbf{v}_h, \mathbf{v}_l)$, assume that it induces a (pure-strategy or mixed-strategy) Bayesian Nash equilibrium. According to the revelation principle, any equilibrium outcome (including prize allocation and effort provision) of the contest can be reproduced by a direct mechanism. Our approach is to first solve for the optimal direct mechanism, and then show that the optimal direct mechanism can be implemented by a particular contest rule $(\hat{e}, \mathbf{v}_h, \mathbf{v}_l)$. This implies that the contest rule that implements the optimal direct mechanism must be the optimal contest rule.

3. The optimal direct mechanism

In this section, we adopt a mechanism design approach and identify the optimal direct mechanism. A direct mechanism is formally defined as below. Let $\tilde{t}_i \in [a, b]$ be contestant i's reported ability. Given the profile of reports $\tilde{\mathbf{t}} = (\tilde{t}_1, \cdots, \tilde{t}_N)$, the contest designer gives a prize of $v_i(\tilde{\mathbf{t}})$ to contestant i and demands an effort of $e_i(\tilde{\mathbf{t}})$ from him.[7] A direct mechanism can thus be denoted by $(\mathbf{v}(\tilde{\mathbf{t}}), \mathbf{e}(\tilde{\mathbf{t}}))$, where $\mathbf{v}(\tilde{\mathbf{t}}) = (v_1(\tilde{\mathbf{t}}), \cdots, v_N(\tilde{\mathbf{t}}))$ and $\mathbf{e}(\tilde{\mathbf{t}}) = (e_1(\tilde{\mathbf{t}}), \cdots, e_N(\tilde{\mathbf{t}}))$.

Define the expected prize of contestant i with report $\tilde{t}_i$ as

$$V_i(\tilde{t}_i) = \int_{\mathbf{t}_{-i}} v_i(\tilde{t}_i, \mathbf{t}_{-i})\mathbf{f}_{-i}(\mathbf{t}_{-i})d\mathbf{t}_{-i}, \tag{1}$$

where $\mathbf{t}_{-i} = (t_1, \cdots, t_{i-1}, t_{i+1}, \cdots, t_N)$ and $\mathbf{f}_{-i}(\mathbf{t}_{-i})$ denotes the density of $\mathbf{t}_{-i}$.

Given that other contestants truthfully report their abilities, contestant i's expected payoff when reporting $\tilde{t}_i$ is

$$u_i(\tilde{t}_i, t_i) = \int_{\mathbf{t}_{-i}} v_i(\tilde{t}_i, \mathbf{t}_{-i})\mathbf{f}_{-i}(\mathbf{t}_{-i})d\mathbf{t}_{-i} - \frac{\int_{\mathbf{t}_{-i}} e_i(\tilde{t}_i, \mathbf{t}_{-i})\mathbf{f}_{-i}(\mathbf{t}_{-i})d\mathbf{t}_{-i}}{t_i}$$

[6] The case for $K = 0$ is trivial. As we shall see later, its analysis is equivalent to the optimal auction design, and Myerson's result applies. We will deal with this degenerate case by letting K go to zero in our analysis. The case of no limited liability corresponds to $K \to +\infty$, which will be discussed in Section 4.

[7] As a contestant's payoff is linear in effort and prize, it is without loss of generality to focus on deterministic mechanisms. In fact, $v_i(\tilde{\mathbf{t}})$ and $e_i(\tilde{\mathbf{t}})$ can be interpreted as the expected prize and the expected effort.

$$= V_i(\tilde{t}_i) - \frac{\int_{\mathbf{t}_{-i}} e_i(\tilde{t}_i, \mathbf{t}_{-i})\mathbf{f}_{-i}(\mathbf{t}_{-i})d\mathbf{t}_{-i}}{t_i}.$$

The contest designer's objective can be expressed as:

$$\max_{\{v_i(\cdot), e_i(\cdot), \forall i\}} R = \int_{\mathbf{t}} \left[\sum_i e_i(\mathbf{t}) + t_0(V - \sum_i v_i(\mathbf{t})) \right] \mathbf{f}(\mathbf{t})d\mathbf{t} \tag{2}$$

subject to the following feasibility constraints:

$$u_i(t_i, t_i) \geq u_i(\tilde{t}_i, t_i), \forall \tilde{t}_i, t_i, \forall i, \tag{3}$$

$$u_i(t_i, t_i) \geq 0, \forall t_i, \forall i, \tag{4}$$

$$\sum_i v_i(\mathbf{t}) \leq V, \forall \mathbf{t}, \tag{5}$$

$$v_i(\mathbf{t}) \geq -K, \forall \mathbf{t}, \forall i, \tag{6}$$

$$e_i(\mathbf{t}) \geq 0, \forall \mathbf{t}, \forall i. \tag{7}$$

The feasibility constraints consist of five parts: (3) is the incentive compatibility constraint, (4) is the participation constraint, (5) is the designer's budget constraint, (6) is the lower bound imposed on prizes, and (7) is the nonnegative effort constraint.

Define $\tilde{u}_i(\tilde{t}_i, t_i) = t_i \cdot u_i(\tilde{t}_i, t_i)$. Then

$$\tilde{u}_i(\tilde{t}_i, t_i) = t_i V_i(\tilde{t}_i) - \int_{\mathbf{t}_{-i}} e_i(\tilde{t}_i, \mathbf{t}_{-i})\mathbf{f}_{-i}(\mathbf{t}_{-i})d\mathbf{t}_{-i}.$$

Constraints (3) and (4) can be rewritten in terms of $\tilde{u}_i(\cdot, \cdot)$. From (3) and the envelope theorem, we have

$$\frac{d\tilde{u}_i(t_i, t_i)}{dt_i} = \left.\frac{\partial \tilde{u}_i(\tilde{t}_i, t_i)}{\partial t_i}\right|_{\tilde{t}_i = t_i} = V_i(t_i),$$

which leads to

$$\tilde{u}_i(t_i, t_i) - \tilde{u}_i(a, a) = \int_a^{t_i} V_i(s)ds.$$

Standard derivations, such as those in Myerson (1981), lead to the following lemma. The proof is omitted here.

Lemma 1. *Mechanism* $(\mathbf{v}(\cdot), \mathbf{e}(\cdot))$ *is feasible if and only if the following conditions hold, together with (5), (6), and (7):*

$$\int_{\mathbf{t}_{-i}} e_i(t_i, \mathbf{t}_{-i})\mathbf{f}_{-i}(\mathbf{t}_{-i})d\mathbf{t}_{-i} = t_i V_i(t_i) - \int_a^{t_i} V_i(s)ds - a \cdot u_i(a, a), \forall t_i, \forall i, \tag{8}$$

$$V_i(t_i') \geq V_i(t_i),\ \forall t_i' > t_i, \forall i, \tag{9}$$

$$u_i(a, a) \geq 0, \forall i.$$

Note that in the optimal mechanism, we must have $u_i(a,a)=0$—i.e., the lowest ability contestant must earn zero informational rent. If $u_i(a,a)>0$, the contest designer can simply decrease the informational rent for every ability and yield a higher level of expected total effort. Given (1) and (8), we can replace effort $\mathbf{e}(\cdot)$ with the prize function $\mathbf{v}(\cdot)$ and rewrite the contest designer's objective function as

$$\max \int_{\mathbf{t}} \sum_i [J(t_i)-t_0]\, v_i(\mathbf{t})\mathbf{f}(\mathbf{t})d\mathbf{t} + t_0 V. \tag{10}$$

Therefore, the contest designer's optimization problem can be restated as maximizing (10), subject to (5), (6), (7), (8), and (9). We denote this optimization problem as Problem (P) and the resulting mechanism as the optimal direct mechanism.

Define $\hat{t}(K) = F^{-1}((\frac{NK}{V+NK})^{\frac{1}{N-1}})$, $t^*(K) = \max\{J^{-1}(t_0), \hat{t}(K)\}$ and $\Lambda(K) = \frac{K}{F^{N-1}(t^*(K))} - K$. Note that $\Lambda(K) \in (0, \frac{V}{N}]$. Let $S^*(K)=\{j : t_j > t^*(K)\}$—i.e., the set of contestants with abilities higher than the cut-off—and let $t^{(1)}$ denote the first-order statistics of $\mathbf{t}$. We have the following proposition.

Proposition 1. *The following mechanism defines an optimal direct mechanism for Problem (P): The prize allocation function is given by*

$$v_i^*(\mathbf{t}; K) = \begin{cases} \Lambda(K), & \text{if } S^*(K)=\emptyset, \\ -K, & \text{if } S^*(K)\neq\emptyset \text{ and } t_i < t^{(1)}, \\ V+(N-1)K, & \text{if } S^*(K)\neq\emptyset \text{ and } t_i = t^{(1)}. \end{cases}$$

The effort function is given by

$$e_i^*(\mathbf{t}; K) = \varepsilon(t_i; K) \equiv \begin{cases} 0, & \text{if } t_i \le t^*(K), \\ t_i V_i^*(t_i; K) - \int_{t^*(K)}^{t_i} V_i^*(s; K)ds, & \text{if } t_i > t^*(K), \end{cases} \tag{11}$$

where $V_i^*(t_i; K) = \int_{\mathbf{t}_{-i}} v_i^*(\mathbf{t}; K)\mathbf{f}_{-i}(\mathbf{t}_{-i})d\mathbf{t}_{-i}$ *is the expected prize function.*

Unlike in Myerson (1981), the standard pointwise maximization technique is not applicable in characterizing the optimal mechanism in Problem (P). To obtain the results in Proposition 1, we rely on a continuous linear programming approach to establish key conditions for characterizing the optimal mechanism. Furthermore, a two-step, sequential relaxation procedure is necessary in our analysis. The first step of this procedure is to partially relax the nonnegative effort constraint (7) in such a way that the optimization problem would depend solely on the prize structure. We also relax the ex post budget constraint (5) to an ex ante budget constraint. The second step is to rewrite the relaxed optimization problem by adding its first-order conditions (found using the continuous linear programming method), and then drop the constraint of lower bound on prizes (6). We then show that the relaxed problem has an optimal solution that satisfies the feasibility conditions in the original optimization Problem (P), and thus it must also be an optimal solution for Problem (P). Details are relegated to the proof of Proposition 1 in the Appendix. Note that we cannot drop the lower bound on prizes before adding the appropriate first-order conditions, and these first-order conditions are obtained after relaxing the original optimization Problem (P).

In the optimal direct mechanism characterized by Proposition 1, there is a cut-off ability $t^*(K)$. If none of the contestants has an ability higher than this cut-off, then every contestant gets a prize $\Lambda(K)$, which is no bigger than V/N. But if at least one of the contestants has an ability higher than the cut-off, then every contestant (except the highest ability contestant) will be punished by a negative prize $-K$. This highest ability contestant gets the prize budget V plus the extra money generated by the negative prizes from other contestants. The interim incentive to participate for the lower ability contestants is maintained by the positive prize $\Lambda(K)$ when no contestant has an ability above the cut-off.

In the equilibrium of this mechanism, those contestants with abilities below the cut-off exert zero effort. Meanwhile, contestants with abilities higher than the cut-off exert a large amount of effort. Note that the above effort function $e_i^*(\mathbf{t}; K)$ depends only on contestant i's type t_i, and is symmetric among all contestants. Furthermore, it is strictly increasing in t_i for $t_i > t^*(K)$.

The intuitions behind this result can be illustrated as follows. In the optimal direct mechanism, the contest designer ranks contestants according to their abilities. The marginal benefit (in terms of effort generated) of giving one extra dollar to the contestant with the highest ability is higher than the marginal cost (in terms of effort lost) of charging one dollar from the lower ability contestants (i.e., negative prize), since the marginal cost of effort is lower for a contestant with a higher ability. With negative prizes being allowed, the contest designer thus desires to make prize transfers across contestants with different abilities such that the highest ability contestant will exert more effort and the lower ability contestants will exert less.[8] Moreover, since the marginal benefit and marginal cost of these transfers are both constant, the contest designer is willing to perform these transfers as long as it is feasible—i.e., until the negative prizes hit their bound K. Meanwhile, the contest designer is constrained by the participation constraints of the lower ability contestants and must compensate them for the negative prizes charged. The optimal way to achieve this balancing is to reward contestants when they are all of low abilities, since the contest designer cannot extract much effort from them due to the high cost of exerting effort. This is why the optimal contest features a cut-off $t^*(K)$. As a result, when all are below the cut-off, they equally share the entire prize (or a portion of it); in all other cases, only the best contestant obtains the entire prize plus all negative prizes collected from the rest of the contestants, whose negative prizes are equal to the bound K.

From the perspective of economic implications, there are a few distinctive features associated with the prize allocation rule in this optimal direct mechanism. First, a maximal negative prize K is imposed on all contestants except the one with the highest ability, unless all of them have abilities lower than the cut-off. This illustrates the important role of negative prizes in the optimal contest rule. Second, when all of the contestants have abilities lower than the cut-off, they are all treated equally, regardless of their ability rankings. In this case, they all obtain an equal positive prize. Together these two features ensure that the maximum incentive to exert effort is granted to the highest ability contestant, while lower ability contestants are still willing to participate (and provide the necessary cross-ability subsidies to the highest ability contestant). Third, similar to the cut-off type in Myerson (1981), a cut-off value $t^*(K)$ is determined in our analysis, but in a somewhat different way. The cut-off value in our optimal direct mechanism is always weakly higher than the one in Myerson (1981), and strictly higher if K is high enough. When K is low—and thus the negative prizes are not allowed to be large—our cut-off value coincides with the one in Myerson (1981). In this case, only a portion of the original prize budget is awarded to contestants when every contestant's ability is below the threshold. This feature in the prize

8 If negative prizes are not allowed, similar to the auction design literature such as that of Myerson (1981), it is easy to show that the optimal prize structure is to allocate the entire prize to the highest ability contestant, provided that his ability is higher than $J^{-1}(t_0)$.

structure is necessary to maintain the incentive constraints for those contestants with abilities above the cut-off. This partial award scheme in our optimal contest design resembles the no-sale situation in an optimal auction, when all bidders' virtual values are below the seller's valuation, even when some of their true valuations are higher than the seller's valuation.

The following corollary specifies the exact condition for the prize budget V to always be completely awarded to contestants in the optimal direct mechanism.

Corollary 1. *The prize budget V is always completely spent regardless of the contestants' type profile if and only if $K > 0$ and $J(\hat{t}(K)) \geq t_0$.*

When $K = 0$, or when $K > 0$ but $J(\hat{t}(K)) < t_0$, the budget constraint is not binding in the case in which every t_i is lower than $t^*(K)$. Note that this result may hold even when the contest designer does not derive any benefit from any unspent prize budget, i.e., when $t_0 = 0$.

4. The optimal contest

We have pinned down the optimal direct mechanism in the previous section. We know that the equilibrium of any contest rule $(\hat{e}, \mathbf{v}_h, \mathbf{v}_l)$ can be reproduced by a direct mechanism. As such, if we are able to uncover a contest rule $(\hat{e}, \mathbf{v}_h, \mathbf{v}_l)$ that implements the optimal direct mechanism, then such a contest rule must be an optimal contest rule. Let $\hat{e}^*(K) = \lim_{t_i \to t_+^*} \varepsilon(t_i; K)$, where $\varepsilon(t_i; K)$ is given by (11), which is the expected effort of type t_i in the optimal direct mechanism. Define prize allocation vectors $\mathbf{v}_h^*(K) = (V + (N-1)K, -K, -K, \dots, -K)$ and $\mathbf{v}_l^*(K) = (\Lambda(K), \Lambda(K), \dots, \Lambda(K))$. Contest rule $(\hat{e}^*(K), \mathbf{v}_h^*(K), \mathbf{v}_l^*(K))$ is an all-pay auction with an effort threshold $\hat{e}^*(K)$ and prize sequences $\mathbf{v}_h^*(K)$ and $\mathbf{v}_l^*(K)$. We then have the following proposition.

Proposition 2. *Contest rule $(\hat{e}^*(K), \mathbf{v}_h^*(K), \mathbf{v}_l^*(K))$ implements the optimal direct mechanism $(\mathbf{v}^*(\cdot; K), \mathbf{e}^*(\cdot; K))$ in Proposition 1, and is thus the optimal contest rule.*

In this contest, every type of contestants will participate; for abilities above $t^*(K)$, the optimal bid (in terms of effort) is given by the above expected effort function $\varepsilon(\cdot; K)$; and contestants with abilities below $t^*(K)$ bid zero. The intuition behind Proposition 2 is as follows. Recall that a contest rule consists of three instruments: an effort threshold $\hat{e}$ and two prize allocation vectors, $\mathbf{v}_h$ and $\mathbf{v}_l$. One additional dollar can extract more effort from a higher ability contestant, because such contestant's marginal cost is lower. Since a contestant's marginal cost is constant, ideally the designer should collect money (i.e., charge a negative prize) from low-ability contestants in order to top up the prize pool to better incentivize the most able contestant to exert more effort. As long as there is still room for such a cross-type transfer, the designer should continue doing this. Pushing this to the extreme, all contestants (except the highest ability one) should receive a negative prize $-K$, hitting the lower bound, and a grand prize of $V + (N-1)K$ should be awarded to the highest ability contestant. This explains why $\mathbf{v}_h^*(K) = (V + (N-1)K, -K, -K, \dots, -K)$. Of course, doing so would violate the participation constraints for low-ability contestants. Therefore, the low-ability contestants must somehow be compensated. The most "efficient" and "powerful" way to accomplish this is to let all of the contestants share the total prize (or a portion of it) equally when no one's effort is above the threshold. This leads to the second vector $\mathbf{v}_l^*(K) = (\Lambda(K), \Lambda(K), \dots, \Lambda(K))$. In this way, the designer optimally balances the two conflicting demands—the cross-type transfers to incentivize high type contestants to exert more effort, and satisfaction of the participation constraints of low-ability contestants.

Moldovanu et al. (2012) model negative prizes in contests in a different way. They consider a negative prize as a physical or mental/psychological punishment, which can be either costly or costless for the organizer to implement. Contestants may or may not be allowed to quit the contests. Moldovanu et al. (2012) show that when punishments are costless, it is always optimal to punish contestants even when they can quit; indeed, in equilibrium these punishments generate partial participation. The negative prizes in our setup are only money, as in Fullerton and McAfee (1999), and the money collected can be used by the contest designer to strengthen the incentives provided to other contestants. We focus on the case in which the interim participation constraint of every type of contestants must hold, and we fully characterize the optimal prize structure. From Section 3, it is clear that in our model the optimal prize structure that allows negative prizes induces full participation.[9] Full participation allows the designer more leeway to collect money to maximize the incentive for the most capable contestant. Since negative prizes in our model generate an additional budget for the contest organizer, one can expect that our optimal design would dominate the optimal design in settings in which such additional resources cannot be generated.

The optimal contest rule $(\hat{e}^*(K), \mathbf{v}_h^*(K), \mathbf{v}_l^*(K))$ has the following equivalent interpretation:

Proposition 3. *The optimal contest rule is equivalent to a modified all-pay auction with entry fee K and minimum bid $\hat{e}^*(K)$. Every participant pays the entry fee, whether he bids or not. The highest bidder wins V plus the entry fees collected from all of the participants. When no one bids, each participant receives $\Lambda(K)$ $(\leq V/N)$ plus the paid entry fee K. In equilibrium, every type of contestants participates.*

It is noteworthy that the optimal contest in Proposition 3 does not require that the prize V be divisible. If the prize is indivisible, winning a portion of V with probability one is equivalent to winning V with smaller probability due to the linearity of utility functions. As such, the optimal contest rule remains optimal, even when the prize is not divisible.

The main message of this paper is that negative prizes can significantly improve contest design. Therefore, it is important to know how the contest designer's payoff is affected. From (10) and Proposition 1, the expected total effort equivalent induced by the optimal contest rule $(\hat{e}^*(K), \mathbf{v}_h^*(K), \mathbf{v}_l^*(K))$ is given by

$$R(K) = N \int_{t^*(K)}^{b} [J(t_i) - t_0][(V + NK)F^{N-1}(t_i) - K]dF(t_i) + t_0 V.$$

The following proposition characterizes how the level of negative prize (i.e., K) affects the designer's payoff at the optimum.

Proposition 4. *(i) When $K = 0$, the optimal contest resembles the Myerson optimal auction. In particular, when $J(a) \geq t_0$, the optimal contest can be implemented by a standard all-pay auction with a single prize V for the winner (cf. Moldovanu and Sela, 2001).*

[9] Note that in our setup, any mechanism that induces partial participation can be duplicated by a mechanism that induces full participation by setting zero prize and zero effort for nonparticipating types.

(ii) The expected total effort equivalent $R(K)$ increases in K.

(iii) $\lim_{K\to+\infty} R(K) = bV$. *In the limit, the contestant with maximum ability b obtains a positive but finite informational rent* $\frac{V}{Nbf(b)}$*; all other types of contestants obtain zero informational rent, and full rent extraction is achieved.*

The first two results are not difficult to understand. When $K = 0$, no negative prizes are allowed in the contest. In this case, the constraints in contest design Problem (P) are more restrictive than those in the Myerson optimal auction design setup, as negative effort is not allowed in contests, but negative payments are allowed in auctions. Since the Myerson optimal mechanism does not involve negative monetary payments, the contest designer can do equally well. In particular, when $K = 0$ and $J(a) \geq t_0$, the effort threshold $\hat{e} = 0$ so that the winner-take-all prize allocation rule, as in Moldovanu and Sela (2001), prevails. An optimal mechanism for a smaller K must also be feasible when K becomes larger; the expected total effort equivalent $R(K)$ elicited from Problem (P) must be increasing in K. These two results imply that whenever negative prizes are allowed, it is optimal to use them.

To understand the third result, note that bV is the maximum effort level when all contestants are of maximum ability b with certainty. This can be achieved by asking each of the N contestants to exert effort $e = bV/N$ and awarding each of them a prize of V/N. We will show in the following proposition that this is the highest amount of total effort inducible, given budget V in our original optimization Problem (P) for any ability distribution with support bounded above by b. We call this the **utmost total effort**.

Proposition 5. *The total effort equivalent elicitable in Problem (P) is bounded by bV given any contestants' type distribution $F(\cdot)$ with a support bounded by b.*

A formal proof is provided in the Appendix. Fundamentally, Proposition 5 holds because of the participation constraints of the contestants. Note that we allow any type distribution, which can be degenerated in such a situation when all contestants are of maximum ability b with certainty.

Obviously, by Proposition 5, $R(K)$ is bounded above by bV. The question is why the contest designer can still elicit almost bV as K goes to infinity when the abilities of the contestants are distributed randomly (and therefore are almost surely to be less than the maximum b). Because of the lower cost of effort, higher ability contestants are willing to expend more effort for a given prize than lower ability ones. Therefore, there is some leeway to move effort from the lower ability contestants to the higher ability ones, as this will save money for the contest designer.

To accomplish this maneuver, first suppose that effort can be negative. Then a lower ability contestant is willing to accept a negative prize for exerting a negative effort to maintain his participation constraint. The contest designer can use the additional budget as a result of this negative prize to incentivize the higher ability contestant to exert more effort. Due to the ability difference, the increase in the effort of the higher ability contestant must dominate the decrease in the effort of the lower ability contestant. Therefore, the total effort must increase. As we let the prizes of lower ability contestants go to negative infinity, the additional budget created to incentivize higher ability ones goes to positive infinity. By doing this, an infinite amount of total effort can be generated.

Now efforts must be nonnegative. The optimal mechanism characterized by Propositions 1 to 3 must generate a total effort below bV due to the participation constraints of contestants. Surprisingly, Proposition 4(*iii*) shows that when K goes to infinity, $R(K)$ must get arbitrarily close to the bound bV. We will use the following two-contestant, two-type example to illustrate this result.

Example 1. Suppose that there are two contestants, i and j, whose private abilities take two possible values, a_1 and b_1, with probabilities q and $1-q$, respectively. Here $b_1 > a_1 > 0$ and $q \in (0,1)$. $\forall k \in \{i,j\}$, let $v_k(t_i,t_j)$ and $e_k(t_i,t_j)$ be contestant k's prize and effort when the type profile is (t_i,t_j), respectively. Consider the following mechanism:

$$v_k(a_1,t_{-k}) = \begin{cases} \frac{1}{2}V, & \text{if } t_{-k}=a_1, \\ -\tilde{K} & \text{if } t_{-k}=b_1, \end{cases}$$

$$v_k(b_1,t_{-k}) = \begin{cases} V+\tilde{K}, & \text{if } t_{-k}=a_1, \\ \frac{1}{2}V, & \text{if } t_{-k}=b_1, \end{cases}$$

and

$$e_k(a_1,t_{-k}) = 0, \ \forall t_{-k};$$

$$e_k(b_1,t_{-k}) = \begin{cases} e^*, & \text{if } t_{-k}=a_1, \\ \frac{1}{2}e^*, & \text{if } t_{-k}=b_1, \end{cases}$$

where $k \in \{i,j\}$ and $-k$ denotes the opponent of contestant k. Here $\tilde{K} = \frac{qV}{2(1-q)} > 0$ and $e^* = \frac{b_1V}{1-q^2} > 0$.

The following is what happens in the mechanism. When both contestants are weak (type a_1), both get prize $V/2$ without exerting effort. When one is weak and the other is strong (type b_1), then the weak one is asked to pay $\tilde{K}$ and exert zero effort, while the strong one gets prize $V+\tilde{K}$ and is asked to exert effort e^*. When both contestants are strong, then each contestant obtains prize $V/2$ and exerts effort $e^*/2$. Note that when $q \to 1$—i.e., when the probability of strong type b_1 approaches zero—we have $\tilde{K} \to \infty$ and $e^* \to \infty$, i.e., both negative prize $\tilde{K}$ and effort level e^* go to infinity.[10]

One can easily verify that in the above mechanism: 1) there is no incentive for each type of contestants to misreport, given that his opponent reports truthfully; 2) both type a_1 and type b_1 contestants' expected payoffs are exactly zero; 3) and the ex ante expected total effort is the utmost effort level b_1V. The proof of these properties is relegated to the Appendix.

The link between the above discrete-type example and our continuous-type setup can be illustrated as follows. Given any cut-off $t^* \in (a,b)$, let $q = F(t^*)$. We treat all types below the cut-off as type $a_1 = a$ and treat all types above the cut-off (inclusive) as type $b_1 = t^*$. Then we go back to the discrete-type example and restrict the message space to $\{a_1,b_1\}$ and propose the above mechanism in the example using $q = F(t^*)$. It is straightforward to verify that all types below t^* would report a_1, and all types above t^* would report b_1, in equilibrium. Thus the total expected effort induced would be t^*V, which goes to bV as t^* goes to b.

Given that the total expected effort goes to bV, it is not surprising that contestants obtain zero expected payoff. In equilibrium, a contestant with an ability below the cut-off $t^*(K)$ wins a positive (resp., negative) prize $\Lambda(K)$ (resp., $-K$) when the best contestant has ability lower (resp., higher) than this cut-off, and exerts no effort. A contestant with an ability higher than the

10 One can generalize the above mechanism to allow for N players.

cut-off wins a positive (resp., negative) prize $V + (N-1)K$ (resp., $-K$) if he is (resp., is not) the best contestant, and exerts positive effort. A higher K strengthens the incentive for a higher ability contestant to exert more effort. But if a lower ability contestant is sometimes asked to pay a higher negative prize, the probability of the event that he will end up only paying the fee must be decreased, implying that the cut-off in ability must move higher. When K goes to infinity, the cut-off goes to the upper bound b. Therefore, all contestants—except the one with upper bound ability b—obtain no informational rent. Since the probability of being type b is zero, the expected surplus of a contestant goes to zero.

Proposition 4 demonstrates that allowing negative prizes can dramatically boost the performance of the optimally designed contest. In particular, when K goes to infinity, the total expected effort equivalent can go arbitrarily close to the utmost total effort bV. One interesting and meaningful question is how this level of effort equivalent compares to that of Moldovanu and Sela (2001), who show that a winner-take-all allocation rule is optimal in their all-pay auction setting with linear cost function. For the purpose of comparison, we assume that $t_0 = 0$, so that the organizer always exhausts her budget in Moldovanu and Sela (2001). Note that the utmost total effort bV does not depend on t_0 in our setup.

Similar to the objective function in (10), the level of effort equivalent in the Moldovanu and Sela (2001) setup is:

$$R^{MS} = \sum_{i=1}^{N} \int_a^b J(t_i) V_i(t_i) f(t_i) dt_i,$$

where $V_i(t_i) = V F^{N-1}(t_i)$ since the equilibrium bidding strategy is monotone and symmetric and the highest type always wins. We thus have

$$R^{MS} = N \int_a^b J(t) V F^{N-1}(t) f(t) dt = V \int_a^b J(t) dF^N(t) = V \left[b - \int_a^b F^N(t) dJ(t) \right].$$

Therefore, we have

$$bV - R^{MS} = V \int_a^b F^N(t) dJ(t) > 0,$$

as $J(t)$ is increasing.

We use the following example to further quantify this comparison.

Example 2. Suppose $N = 2$, $t_0 = 0$, $V = 1$, and $F(t_i) = t_i - 1$ on $[1, 2]$. Recall that at the optimum, the expected total effort equivalent is given by

$$R(K) = N \int_{t^*(K)}^{b} [J(t_i) - t_0] \Big[(V + NK) F^{N-1}(t_i) - K \Big] dF(t_i) + t_0 V,$$

where $t^*(K) = \max\{J^{-1}(t_0), F^{-1}((\frac{NK}{V+NK})^{\frac{1}{N-1}})\}$. In this example, we have

$$R(K) = 2 \int_{t^*(K)}^{2} (2t_i - 2)[(1 + 2K)(t_i - 1) - K]dt_i,$$

where $t^*(K) = \frac{2K}{1+2K} + 1$.

In Moldovanu and Sela (2001), $K = 0$, which gives $t^*(0) = 1$ and

$$R(0) = 2 \int_{1}^{2} (2t_i - 2)(t_i - 1)dt_i = \frac{4}{3}.$$

When $K > 0$, we have[11]

$$\begin{aligned} R(K) &= 2 \int_{\frac{2K}{1+2K}+1}^{2} (2t_i - 2)[(1 + 2K)(t_i - 1) - K]dt_i \\ &= 4 \int_{\frac{2K}{1+2K}}^{1} t_i[(1 + 2K)t_i - K]dt_i \\ &= 2 - \frac{2}{3(2K + 1)} \left(\frac{3}{2} - \frac{1}{2(2K + 1)}\right). \end{aligned}$$

One can easily verify that $y(\frac{3}{2} - \frac{1}{2}y)$ increases with y when $y \leq 1$, which means that $R(K)$ increases with K for $K \geq 0$. Clearly, $\lim_{K\to\infty} R(K) = 2$, which is 1.5 times $R(0)$ in Moldovanu and Sela's (2001) setup, in which no negative prizes are allowed.

Note that the result in Proposition 4(*iii*) is related to the literature on full surplus extraction in auctions. Crémer and McLean (1988) and McAfee and Reny (1992) establish that the full surplus can be extracted from bidders when their types are correlated. Heifetz and Neeman (2006) and Chen and Xiong (2013) further examine the generality and robustness of the full surplus extraction result. However, when the bidders' types are independently distributed and when the interim participation constraints must be satisfied, it is believed that full surplus extraction cannot be achieved. In this paper, we show that if negative prizes are allowed, then the upper bound total effort can almost be achieved in the contest environment when contestants' types are distributed independently and when the interim participation constraints must be satisfied. The obtainability of this upper bound total effort is, in some sense, a stronger result than the full surplus extraction in auctions. The utmost total effort is the level of total effort achievable when all contestants have maximum ability b with certainty. The full surplus in auctions is much lower than what the seller can receive when all bidders have the maximum valuation with certainty. This shows that an optimal contest problem can be very different from an optimal auction problem.[12]

[11] Details are available from the authors upon request.

[12] While we obtain the utmost total effort under the condition of independent private abilities, we still assume risk neutrality, unlimited liability, no collusion among contestants, and no competing designers. Therefore, the critiques of full surplus extraction by Robert (1991), Laffont and Martimort (2000), Che and Kim (2006), and Peters (2001) still apply.

5. Concluding remarks

In this paper, we examine the role of negative prizes in contest design in an environment with multiple contestants who are endowed with private ability information. Under a regularity condition on the virtual ability function, we adopt an indirect mechanism design approach and completely characterize the optimal contest rule within the class of mechanisms characterized by an effort threshold and two contingent prize sequences. This approach implies that the optimal contest rule obtained is also optimal among a more general class of mechanisms. For example, define a contest rule as a mapping from the effort provision profile to the prize allocation rule—specifically, a contest rule is a mapping $\mathbf{g}: \mathbb{R}_+^N \to \mathbb{R}^N$, such that $\mathbf{g}(\mathbf{e}) = (g_1(\mathbf{e}), \ldots, g_N(\mathbf{e}))$ satisfies $\sum_{i=1}^N g_i(\mathbf{e}) \le V$ and $g_j(\mathbf{e}) \ge -K$, for any $\mathbf{e} \in \mathbb{R}_+^N$, and any $j = 1, \ldots, N$. Clearly, such a contest rule encompasses the contest rule in our main analysis as a special case. Nevertheless, the revelation principle still applies to this environment, and our analysis implies that the optimal contest rule characterized by Propositions 2 and 3 is still optimal, even among this more general class of mechanisms.

To some extent, the negative prizes in our model must be in the form of advance payments, such as entry fees. If advance payments are not allowed, and participation constraints are ex post, then negative prizes are infeasible. In that case, every contestant of every type must earn a nonnegative payoff in every profile, as prizes cannot be negative, and it follows that the optimal contest coincides with Myerson's (1981) optimal auction.

In the analysis, we focus on deriving the optimal mechanism that maximizes the total expected effort from all players. Alternative objective functions for the contest designer can be examined. For example, the contest designer may value only the highest effort among contestants, such as in innovation contests. In this case, we can modify the optimal mechanism in Proposition 1 so that only the highest ability player can exert a positive effort. It follows immediately that this modified mechanism would be the optimal mechanism when the contest designer values only the highest effort from the contestants.

Our study provides a few directions for future research. First, in the analysis, we consider only symmetric contestants. It would be worthwhile to investigate how to generalize the analysis to accommodate asymmetric contestants. Second, in the optimal contest with bounded negative prizes, we assume a common and fixed bound for the negative prizes. In some situations, this bound could be heterogenous among contestants. Furthermore, this bound could even be a contestant's private information. It would be of interest to investigate how the optimal leveraging on different virtual abilities should be arranged. Third, our analysis focuses on an environment with pure adverse selection. Extending the analysis to a setting of adverse selection and moral hazard may yield additional insights.

Appendix A

This appendix contains the proofs of Proposition 1, Lemmas A.1 and A.2, Corollary 1, Propositions 4 and 5, and Example 1.

Proof of Proposition 1. We need a two-step procedure to characterize the optimal mechanism in this proposition.

Our first step is to relax Problem (P) by "partially" dropping the nonnegative effort constraint (7) in such a way that the optimization problem would solely depend on the prize structure, but the optimization problem is not "overly" relaxed. In this step, we also relax the ex post budget constraint (5) to an ex ante budget constraint (A.2). This relaxed problem is labeled as Problem (P-Relax).

Although Problem (P-Relax) depends only on the prize structure, a standard pointwise maximization technique in Myerson (1981) is not applicable. We adopt a continuous linear programming approach to establish the necessary conditions for optimization Problem (P-Relax). In our second step, after adding these necessary conditions to Problem (P-Relax) to obtain an equivalent optimization problem, we further relax this equivalent problem by dropping constraint (6) on the lower bound for prizes. We fully characterize the optimal solution for this newly relaxed problem. This optimal solution can be supported by a feasible direct mechanism in Problem (P), and therefore this direct mechanism must also be an optimal mechanism for Problem (P).

In the analysis, we need to relax Problem (P) sequentially. We first obtain Problem (P-Relax) by relaxing the budget constraint (5) and the nonnegative effort constraint (7). The constraint for the lower bound of prizes (6) can be removed only after we add the necessary conditions characterized in Lemma A.1 to Problem (P-Relax). Dropping constraint (6) at the beginning would lead to unbounded negative prizes in the solution. Note that constraint (6) is essential to establish the key result in Lemma A.1.

Following the above road map, in Problem (P), first notice that

$$V_i(t_i) = \int_{\mathbf{t}_{-i}} v_i(t_i, \mathbf{t}_{-i})\mathbf{f}_{-i}(\mathbf{t}_{-i})d\mathbf{t}_{-i} \geq 0, \ \forall t_i, \forall i.$$

This is true because the monotonicity condition (9) holds and we must have $V_i(a) \geq 0, \forall i$. To see the latter, from (8), evaluating at $t_i = a$, we obtain $aV_i(a) = \int_{\mathbf{t}_{-i}} e_i(a, \mathbf{t}_{-i})\mathbf{f}_{-i}(\mathbf{t}_{-i})d\mathbf{t}_{-i}$, which is nonnegative from the nonnegative effort constraint (7). We then obtain that, in Problem (P), any feasible $V_i(t_i)$ is nonnegative and increasing. Thus, we define $\hat{t}_i = \sup\{t_i | V_i(t_i) = 0\}$. Without loss of generality, assume that $V_i(t_i)$ is left-continuous at $\hat{t}_i$. Therefore, $V_i(t_i) = 0$ for $t_i \leq \hat{t}_i$ and $V_i(t_i) > 0$ for $t_i > \hat{t}_i$.

Now we obtain the following relaxed optimization problem of (P), denoted as (P-Relax):

$$\max_{\{v_i(\mathbf{t}), \hat{t}_i, \forall i\}} \int_{\mathbf{t}} \sum_i [J(t_i) - t_0]\, v_i(\mathbf{t})\mathbf{f}(\mathbf{t})d\mathbf{t} + t_0 V \tag{A.1}$$

subject to

$$\int_{\mathbf{t}} \sum_i v_i(\mathbf{t})\mathbf{f}(\mathbf{t})d\mathbf{t} \leq V, \tag{A.2}$$

$$V_i(t_i) = \int_{\mathbf{t}_{-i}} v_i(t_i, \mathbf{t}_{-i})\mathbf{f}_{-i}(\mathbf{t}_{-i})d\mathbf{t}_{-i} = 0, \ \forall t_i \leq \hat{t}_i, \forall i, \tag{A.3}$$

$$V_i(t_i) = \int_{\mathbf{t}_{-i}} v_i(t_i, \mathbf{t}_{-i})\mathbf{f}_{-i}(\mathbf{t}_{-i})d\mathbf{t}_{-i} > 0, \ \forall t_i > \hat{t}_i, \forall i, \tag{A.4}$$

$$v_i(\mathbf{t}) \geq -K, \ \forall \mathbf{t}, \forall i, \tag{A.5}$$

$$a \leq \hat{t}_i \leq b, \forall i. \tag{A.6}$$

This is a relaxed problem of Problem (P), since the objective functions are the same in both problems, and the constraints in Problem (P-Relax) are less restrictive than those in Problem (P). To see this, constraint (A.2) follows from (5) by integrating over $\mathbf{t}$. Constraints (A.3) and (A.4) directly follow the definition of $\hat{t}_i$. Constraint (A.5) is the same as (6) in (P). Constraint (A.6) allows for all possible threshold values of $\hat{t}_i$.

Although Problem (P-Relax) depends only on the prize structure, the pointwise maximization technique is not applicable here, since a contestant's expected prize can be negative—i.e., (A.3) and (A.4) are violated. For example, for a contestant with the lowest type a, if we apply pointwise maximization, this contestant will obtain $-K$ with probability one so that his expected prize is $-K$, which would violate constraint (A.3).

To proceed, we construct the Lagrangian by introducing multipliers λ for constraint (A.2), $\mu_i(t_i)$ for constraints (A.3) and (A.4), and $\xi_i(\mathbf{t})$ for constraint (A.5):

$$\begin{aligned}
L = & \int_{\mathbf{t}} \sum_i [J(t_i) - t_0]\, v_i(\mathbf{t})\mathbf{f}(\mathbf{t})d\mathbf{t} + t_0 V + \lambda \int_{\mathbf{t}} \left[V - \sum_i v_i(\mathbf{t}) \right] \mathbf{f}(\mathbf{t})d\mathbf{t} \\
& + \sum_i \int_{t_i} \mu_i(t_i) \left(\int_{\mathbf{t}_{-i}} v_i(t_i, \mathbf{t}_{-i})\mathbf{f}_{-i}(\mathbf{t}_{-i})d\mathbf{t}_{-i} \right) f(t_i)dt_i \\
& + \sum_i \int_{\mathbf{t}} \xi_i(\mathbf{t})\, [v_i(\mathbf{t}) + K]\, \mathbf{f}(\mathbf{t})d\mathbf{t}.
\end{aligned}$$

The Kuhn–Tucker conditions for the optimization are:

$$\begin{aligned}
& \varsigma_i(\mathbf{t}) = [J(t_i) - t_0] - \lambda + \mu_i(t_i) + \xi_i(\mathbf{t}) = 0, \forall \mathbf{t}, \forall i, \\
& \lambda \geq 0,\ V - \int_{\mathbf{t}} \sum_i v_i(\mathbf{t})\mathbf{f}(\mathbf{t})d\mathbf{t} \geq 0,\ \text{ and } \lambda \left[V - \int_{\mathbf{t}} \sum_i v_i(\mathbf{t})\mathbf{f}(\mathbf{t})d\mathbf{t} \right] = 0, \forall \mathbf{t}, \\
& \mu_i(t_i) \geq 0,\ \int_{\mathbf{t}_{-i}} v_i(\mathbf{t})\mathbf{f}_{-i}(\mathbf{t}_{-i})d\mathbf{t}_{-i} \geq 0,\ \text{ and } \mu_i(t_i) \int_{\mathbf{t}_{-i}} v_i(\mathbf{t})\mathbf{f}_{-i}(\mathbf{t}_{-i})d\mathbf{t}_{-i} = 0, \forall t_i, \forall i, \\
& \xi_i(\mathbf{t}) \geq 0,\ v_i(\mathbf{t}) + K \geq 0,\ \text{ and } \xi_i(\mathbf{t})\, [v_i(\mathbf{t}) + K] = 0, \forall \mathbf{t}, \forall i.
\end{aligned}$$

These Kuhn–Tucker conditions lead to some important necessary conditions for the optimal solutions $\{v_i^{\circledast}(\mathbf{t}), \hat{t}_i^{\circledast}, \forall i\}$ for Problem (P-Relax), as specified in the following lemma. (Its proof is immediately after the proof of Proposition 1.)

Lemma A.1. *(i)* $\hat{t}_i^{\circledast} \geq F^{-1}((\frac{K}{V+NK})^{\frac{1}{N-1}})$*;*
(ii) For $t_i > \hat{t}_i^{\circledast}$*, we must have* $0 < V_i^{\circledast}(t_i) \leq (V + NK)F^{N-1}(t_i) - K$.

Lemma A.1 provides a set of necessary conditions for the optimal solution of (P-Relax). If we add these necessary conditions to the constraints in (P-Relax), we obtain a revised optimization Problem (P-Relax-Equivalent). The solutions to these two problems are the same, because the optimal solution of (P-Relax) must satisfy all of the constraints (i.e., the original feasibility constraints and the additional necessary conditions) in Problem (P-Relax-Equivalent). Thus the solution to Problem (P-Relax-Equivalent) cannot be worse than Problem (P-Relax). Meanwhile, Problem (P-Relax-Equivalent) is more restrictive, and therefore its solution cannot be better than Problem (P-Relax).

The equivalent Problem (P-Relax-Equivalent) can be rewritten as follows:

$$\max_{\{v_i(\mathbf{t}),\ \hat{t}_i, \forall i\}} \sum_{i=1}^{N} \int_a^b [J(t_i) - t_0]\, V_i(t_i) f(t_i) dt_i + t_0 V \tag{A.7}$$

subject to

$$\sum_{i=1}^{N} \int_a^b V_i(t_i) f(t_i) dt_i \le V, \tag{A.8}$$

$$V_i(t_i) = 0,\ \text{if } t_i \le \hat{t}_i, \forall i, \tag{A.9}$$

$$v_i(\mathbf{t}) \ge -K,\ \forall \mathbf{t}, \forall i, \tag{A.10}$$

$$F^{-1}((\frac{K}{V+NK})^{\frac{1}{N-1}}) \le \hat{t}_i \le b, \forall i, \tag{A.11}$$

$$0 < V_i(t_i) \le (V+NK)F^{N-1}(t_i) - K,\ \text{if } t_i > \hat{t}_i,\ \forall i. \tag{A.12}$$

Note that (A.8) simply rewrites (A.2).

Note that the optimal revenue from Problem (P) cannot be higher than that from Problem (P-Relax-Equivalent). If we can establish an upper bound of expected total effort for Problem (P-Relax-Equivalent) that is achievable by a feasible mechanism in the original Problem (P), then this mechanism must be an optimal direct mechanism in Problem (P).

Let us establish an upper bound of expected total effort for Problem (P-Relax-Equivalent). We first ignore constraint (A.10). As a result, the entire optimization problem now depends only on the expected prizes $V_i(\cdot)$ instead of the detailed prize structure $v_i(\cdot)$:

$$\max_{\{V_i(\cdot),\ \hat{t}_i, \forall i\}} \sum_{i=1}^{N} \int_a^b [J(t_i) - t_0]\, V_i(t_i) f(t_i) dt_i + t_0 V \tag{A.13}$$

subject to

$$\sum_{i=1}^{N} \int_a^b V_i(t_i) f(t_i) dt_i \le V, \tag{A.14}$$

$$0 < V_i(t_i) \le (V+NK)F^{N-1}(t_i) - K,\ \text{if } t_i > \hat{t}_i,\ \forall i, \tag{A.15}$$

$$V_i(t_i) = 0,\ \text{if } t_i \le \hat{t}_i, \forall i, \tag{A.16}$$

$$F^{-1}((\frac{K}{V+NK})^{\frac{1}{N-1}}) \le \hat{t}_i \le b, \forall i. \tag{A.17}$$

Now the choice variables are merely $\{V_i(\cdot), \hat{t}_i, \forall i\}$. The optimal solution is characterized in the following lemma. (Its proof is immediately after the proof of Lemma A.1.) Recall the following definitions immediately before Proposition 1: $\hat{t}(K) = F^{-1}((\frac{NK}{V+NK})^{\frac{1}{N-1}})$, $t^*(K) = \max\{J^{-1}(t_0), \hat{t}(K)\}$.

Lemma A.2. *$\forall i$, let* $\bar{V}_i(t_i) = \begin{cases} (V+NK)F^{N-1}(t_i) - K, & \text{if } t_i > t^*(K), \\ 0, & \text{if } t_i \le t^*(K). \end{cases}$ *Then $\{\bar{V}_i(t_i), \forall i\}$ is an optimal solution and yields an expected total effort* $\bar{R} = \sum_{i=1}^{N} \int_a^b [J(t_i) - t_0]\, \bar{V}_i(t_i) f(t_i) dt_i$.

Since (P-Relax-Equivalent) is a relaxed problem of (P), $\bar{R}$ must also be an upper bound of expected total effort for Problem (P). Suppose we find a set of prizes $\{v_i^*(\mathbf{t}), i = 1, 2, ..., N\}$ such that it generates $\bar{V}_i(t_i)$ and satisfies the constraints in the original Problem (P). Moreover, suppose we can find effort functions $\{e_i^*(\mathbf{t}), i = 1, 2, ..., N\}$ to support prizes $\{v_i^*(\mathbf{t}), i = 1, 2, ..., N\}$ and generate $u_i(a, a) = 0, \forall i$. Then prizes $\{v_i^*(\mathbf{t}), i = 1, 2, ..., N\}$ and effort functions $\{e_i^*(\mathbf{t}), i = 1, 2, ..., N\}$ would constitute an optimal solution to Problem (P). These supporting effort functions $\{e_i^*(\mathbf{t}), i = 1, 2, ..., N\}$ can be constructed based on (8) together with $u_i(a, a) = 0$.

It is easy to verify that the prize allocation function proposed in Proposition 1 generates

$$V_i^*(t_i) = \begin{cases} 0, & \text{if } t_i \le t^*(K); \\ (V + NK)F^{N-1}(t_i) - K > 0, & \text{if } t_i > t^*(K), \end{cases}$$

which is the optimal $\bar{V}_i(t_i)$ of Lemma A.2. Note that $V_i^*(t_i)$ is discontinuous at $t_i = t^*(K)$.

As we noted earlier, the supporting effort functions $\{e_i^*(\mathbf{t}), i = 1, 2, ..., N\}$ can be constructed using (8) with $u_i(a, a) = 0$, which gives us $e_i^*(\mathbf{t}) = \varepsilon(t_i; K)$. One can verify that $\varepsilon(t_i; K)$ is strictly positive and increasing in t_i for $t_i > t^*(K)$. Note also that because $V_i^*(t_i)$ is discontinuous at $t_i = t^*(K)$, $\varepsilon(t_i; K)$ is also discontinuous at $t_i = t^*(K)$.

We can easily verify that all constraints in Problem (P) are satisfied in mechanism $(\mathbf{v}^*(\cdot; K), \mathbf{e}^*(\cdot; K))$. We thus have established that $(\mathbf{v}^*(\cdot; K), \mathbf{e}^*(\cdot; K))$ is indeed an optimal solution for Problem (P). $\square$

Proof of Lemma A.1. We claim $v_i^{\circledast}(t_i, \mathbf{t}_{-i}) = -K$ for $t_i > \hat{t}_i^{\circledast}$ if there exists some contestant $j \ne i$ such that $t_j > t_i$. Suppose not—then $v_i^{\circledast}(t_i, \mathbf{t}_{-i}) > -K$, which means $\xi_i(\mathbf{t}) = 0$. In addition, we have $\mu_i(t_i) = 0$ from the fact $V_i^{\circledast}(t_i) > 0$. Thus $[J(t_i) - t_0] - \lambda = 0$. Note $[J(t_j) - t_0] - \lambda + \mu_j(t_j) + \xi_j(t_i, t_j, \mathbf{t}_{-ij}) = 0$. Thus $J(t_j) - t_0 = \lambda - \mu_j(t_j) - \xi_j(t_i, t_j, \mathbf{t}_{-ij}) \le \lambda = J(t_i) - t_0$, which contradicts the assumption that $J(\cdot)$ is a strictly increasing function.

When t_i is the highest among all contestants, contestant i can at most collect $V + (N-1)K$; when $t_i > \hat{t}_i^{\circledast}$ is not the highest, $v_i^{\circledast}(t_i, \mathbf{t}_{-i}) = -K$. For contestant i, when $t_i > \hat{t}_i^{\circledast}$, we must have

$$\begin{aligned} 0 < V_i^{\circledast}(t_i) &\le [V + (N-1)K]F^{N-1}(t_i) - K \cdot (1 - F^{N-1}(t_i)) \\ &= (V + NK)F^{N-1}(t_i) - K. \end{aligned} \tag{A.18}$$

(A.18) implies that $\hat{t}_i^{\circledast} \ge \tilde{t}_0 = F^{-1}((\frac{K}{V+NK})^{\frac{1}{N-1}})$. $\square$

Proof of Lemma A.2. It is straightforward to verify that $\bar{V}_i(t_i)$ satisfies all conditions in optimization Problem (P-Relax-Equivalent-Relax). To simplify notation, we hereafter use $\hat{t}$ and t^* to denote $\hat{t}(K)$ and $t^*(K)$, respectively. Define $t^M = J^{-1}(t_0)$. We next consider two cases to show the optimality of $\bar{V}_i(t_i)$. Case 1: $t^* = \hat{t}$, i.e., $J(\hat{t}) \ge t_0$. Case 2: $t^* = t^M$, i.e., $J(\hat{t}) \le t_0$.

First, we consider Case 1, where $J(\hat{t}) \ge t_0$, i.e., $t^* = \hat{t} \ge t^M$. We shall show that for any functions $V_i(t_i)$ satisfying (A.14) to (A.17), we have

$$\begin{aligned} \sum_{i=1}^{N} \int_a^b [J(t_i) - t_0]\, V_i(t_i) f(t_i) dt_i &\le \sum_{i=1}^{N} \int_a^b [J(t_i) - t_0]\, \bar{V}_i(t_i) f(t_i) dt_i \\ &= \sum_{i=1}^{N} \int_{t^*}^b [J(t_i) - t_0]\, \bar{V}_i(t_i) f(t_i) dt_i. \end{aligned}$$

This is equivalent to

$$\sum_{i=1}^{N}\int_{a}^{t^*}[J(t_i)-t_0]\,V_i(t_i)f(t_i)dt_i \le \sum_{i=1}^{N}\int_{t^*}^{b}[J(t_i)-t_0]\,(\bar V_i(t_i)-V_i(t_i))f(t_i)dt_i. \tag{A.19}$$

Note that $V_i(t_i)\ge 0$. In addition, when $t_i > t^* = \hat t$, $\bar V_i(t_i) = (V+NK)F^{N-1}(t_i)-K$. So $\bar V_i(t_i)-V_i(t_i)\ge 0$ for $t_i > t^* = \hat t$.

Since $J(\cdot)$ is strictly increasing and $J(t^*)\ge t_0$, we have

$$\sum_{i=1}^{N}\int_{a}^{t^*}[J(t_i)-t_0]\,V_i(t_i)f(t_i)dt_i \le [J(t^*)-t_0]\sum_{i=1}^{N}\int_{a}^{t^*}V_i(t_i)f(t_i)dt_i,$$

and

$$[J(t^*)-t_0]\sum_{i=1}^{N}\int_{t^*}^{b}(\bar V_i(t_i)-V_i(t_i))f(t_i)dt_i \le \sum_{i=1}^{N}\int_{t^*}^{b}[J(t_i)-t_0]\,(\bar V_i(t_i)-V_i(t_i))f(t_i)dt_i.$$

Thus, in order for (A.19) to hold, we only need to show that

$$\sum_{i=1}^{N}\int_{a}^{t^*}V_i(t_i)f(t_i)dt_i \le \sum_{i=1}^{N}\int_{t^*}^{b}(\bar V_i(t_i)-V_i(t_i))f(t_i)dt_i,$$

which is equivalent to

$$\sum_{i=1}^{N}\int_{a}^{b}V_i(t_i)f(t_i)dt_i \le \sum_{i=1}^{N}\int_{t^*}^{b}\bar V_i(t_i)f(t_i)dt_i. \tag{A.20}$$

According to constraint (A.14), the LHS of (A.20) must be bounded by V. For the RHS of (A.20), we have

$$\begin{aligned}\sum_{i=1}^{N}\int_{t^*}^{b}\bar V_i(t_i)f(t_i)dt_i &= N\int_{t^*}^{b}[(V+NK)F^{N-1}(t_i)-K]f(t_i)dt_i\\ &= (V+NK)F^{N}(t_i)|_{t_i=t^*}^{b} - NK(1-F(t^*))\\ &= (V+NK)(1-F^{N}(t^*)) - NK + NKF(t^*)\\ &= V-(V+NK)F^{N-1}(t^*)F(t^*)+NKF(t^*)\\ &= V-(V+NK)\cdot\frac{NK}{V+NK}\cdot F(t^*)+NKF(t^*)\\ &= V.\end{aligned}$$

Hence (A.20) holds.

Second, we consider Case 2, in which $J(\hat t)\le t_0$; i.e., $\hat t \le t^* = t^M$. We will show that for any $V_i(t_i)$ that satisfies (A.14) to (A.17), we have

$$\sum_{i=1}^{N}\int_{a}^{b}[J(t_i)-t_0]\,V_i(t_i)f(t_i)dt_i \le \sum_{i=1}^{N}\int_{a}^{b}[J(t_i)-t_0]\,\bar V_i(t_i)f(t_i)dt_i,$$

which is equivalent to

$$\sum_{i=1}^{N}\int_{\hat{t}_i}^{b}\left[J(t_i)-t_0\right]V_i(t_i)f(t_i)dt_i \le \sum_{i=1}^{N}\int_{t^M}^{b}\left[J(t_i)-t_0\right]\bar{V}_i(t_i)f(t_i)dt_i.$$

Consider contestant i. Suppose that $\hat{t}_i \ge t^M$. Note that when $t_i > t^M$, we have $V_i(t_i) \le (V+NK)F^{N-1}(t_i) - K = \bar{V}_i(t_i)$ and $[J(t_i)-t_0] > 0$. Therefore,

$$\begin{aligned}\int_{\hat{t}_i}^{b}\left[J(t_i)-t_0\right]V_i(t_i)f(t_i)dt_i &= \int_{t^M}^{b}\left[J(t_i)-t_0\right]V_i(t_i)f(t_i)dt_i \\ &\le \int_{t^M}^{b}\left[J(t_i)-t_0\right]\bar{V}_i(t_i)f(t_i)dt_i.\end{aligned}$$

Now suppose that $\hat{t}_i < t^M$. Note that $[J(t_i)-t_0] < 0$ when $t_i < t^M$. We have

$$\begin{aligned}&\int_{\hat{t}_i}^{b}\left[J(t_i)-t_0\right]V_i(t_i)f(t_i)dt_i \\ &\quad = \int_{\hat{t}_i}^{t^M}\left[J(t_i)-t_0\right]V_i(t_i)f(t_i)dt_i + \int_{t^M}^{b}\left[J(t_i)-t_0\right]V_i(t_i)f(t_i)dt_i \\ &\quad \le \int_{t^M}^{b}\left[J(t_i)-t_0\right]V_i(t_i)f(t_i)dt_i \le \int_{t^M}^{b}\left[J(t_i)-t_0\right]\bar{V}_i(t_i)f(t_i)dt_i.\end{aligned}$$

The last inequality holds, because when $t_i > t^M$, $V_i(t_i) \le (V+NK)F^{N-1}(t_i) - K = \bar{V}_i(t_i)$ and $[J(t_i)-t_0] > 0$.

To conclude, either when $\hat{t}_i \ge t^M$ or $\hat{t}_i < t^M$, we always have

$$\int_{\hat{t}_i}^{b}\left[J(t_i)-t_0\right]V_i(t_i)f(t_i)dt_i \le \int_{t^M}^{b}\left[J(t_i)-t_0\right]\bar{V}_i(t_i)f(t_i)dt_i.$$

Thus,

$$\sum_{i=1}^{N}\int_{a}^{b}\left[J(t_i)-t_0\right]V_i(t_i)f(t_i)dt_i \le \sum_{i=1}^{N}\int_{a}^{b}\left[J(t_i)-t_0\right]\bar{V}_i(t_i)f(t_i)dt_i.$$

This completes the proof for Lemma A.2. □

Proof of Corollary 1. Recall that $\hat{t}(K) = F^{-1}((\frac{NK}{V+NK})^{\frac{1}{N-1}})$, $t^*(K) = \max\{J^{-1}(t_0), \hat{t}(K)\}$ and $\Lambda(K) = \frac{K}{F^{N-1}(t^*(K))} - K$. The budget is always fully spent if and only if $\Lambda(K) = V/N$, which, by definition, is equivalent to $t^*(K) = \hat{t}(K)$, i.e., $\hat{t}(K) \geq J^{-1}(t_0)$. Therefore, the budget is always fully spent when and only when $J(\hat{t}(K)) \geq t_0$. $\square$

Proof of Proposition 4. Parts (*i*) and (*ii*) are proved in the main text immediately after Proposition 4. Here is the proof for part (*iii*). One can verify that the expected prize of type t_i under the contest rule $(\hat{e}^*(K), \mathbf{v}_h^*(K), \mathbf{v}_l^*(K))$ is

$$V_i(t_i; K) = \begin{cases} 0, & \text{if } t_i \leq t^*(K); \\ (V + NK)F^{N-1}(t_i) - K > 0, & \text{if } t_i > t^*(K), \end{cases}$$

and it is increasing in t_i.

The expected total effort induced by $(\hat{e}^*(K), \mathbf{v}_h^*(K), \mathbf{v}_l^*(K))$ is given by

$$\begin{aligned} R(K) &= N \int_{t^*(K)}^{b} [J(t_i) - t_0]\, V_i(t_i; K) dF(t_i) + t_0 V \\ &= N \int_{t^*(K)}^{b} [J(t_i) - t_0][(V + NK)F^{N-1}(t_i) - K] dF(t_i) + t_0 V \\ &= N \int_{t^*(K)}^{b} [J(t_i) - t_0]\, V F^{N-1}(t_i) dF(t_i) \\ &\quad + NK \int_{t^*(K)}^{b} [J(t_i) - t_0][NF^{N-1}(t_i) - 1] dF(t_i) + t_0 V. \end{aligned}$$

When $K \to +\infty$, $t^*(K)$ goes to b. Therefore, the first part in the last expression goes to zero. The third part is a constant. To show that the expected total effort converges to bV when $K \to +\infty$, it suffices to show that the second part converges to $(b - t_0)V$ when $K \to +\infty$. For the second part, note that when K is large enough, $t^*(K) = \hat{t}(K)$. Thus, $F^{N-1}(t^*(K)) = \frac{NK}{V+NK}$. That leads to

$$\frac{dt^*(K)}{dK} = \frac{NV}{(V + NK)^2 (N-1) F^{N-2}(t^*(K)) f(t^*(K))}.$$

Therefore,

$$\begin{aligned} &\lim_{K \to +\infty} NK \int_{t^*(K)}^{b} [J(t_i) - t_0][NF^{N-1}(t_i) - 1] dF(t_i) \\ &= \lim_{K \to +\infty} N \frac{\int_{t^*(K)}^{b} [J(t_i) - t_0][NF^{N-1}(t_i) - 1] dF(t_i)}{\frac{1}{K}} \\ &= \lim_{K \to +\infty} N \frac{-[J(t^*(K)) - t_0][NF^{N-1}(t^*(K)) - 1] f(t^*(K)) \frac{dt^*}{dK}}{-\frac{1}{K^2}} \quad \text{(by L'Hospital's rule)} \end{aligned}$$

$$
\begin{aligned}
&= \lim_{K\to+\infty} N\frac{[J(t^*(K))-t_0][NF^{N-1}(t^*(K))-1]\frac{NV}{(V+NK)^2(N-1)F^{N-2}(t^*(K))}}{\frac{1}{K^2}}\\
&= \lim_{K\to+\infty} N\frac{[J(b)-t_0][NF^{N-1}(b)-1]\frac{NV}{(V+NK)^2(N-1)F^{N-2}(b)}}{\frac{1}{K^2}}\\
&= \lim_{K\to+\infty} N(b-t_0)(N-1)\frac{NVK^2}{(V+NK)^2(N-1)}\\
&= (b-t_0)V.
\end{aligned}
$$

Hence, the expected total effort $R(K)$ converges to bV.

We now turn to contestants' expected payoffs. Recall that contestants' expected total payoffs are at most the difference between V and the expected total effort costs.[13] Since expected total effort converges to bV, we must have the total effort costs converging to $bV/b = V$, since only those types within a small neighborhood of b would exert positive effort. It follows that contestants' expected total payoffs must converge to zero.

As $K \to +\infty$, almost all types except b obtain zero expected prize and exert zero effort. What remains interesting is how much informational rent type b can obtain in the limit.

Note that $F^{N-1}(t^*(K)) \le F^{N-1}(s) \le 1$, $\forall s \in [t^*(K), b]$. Thus, $(N-1)K \le (V+NK)\times F^{N-1}(s) - K \le (N-1)K + V$. Therefore, recalling that $\tilde{u}_i(\tilde{t}_i, t_i) = t_i \cdot u_i(\tilde{t}_i, t_i)$,

$$
\begin{aligned}
&\lim_{K\to+\infty} \tilde{u}_i(b,b) = \lim_{K\to+\infty} \int_{t^*(K)}^{b} V_i(s;K)ds = \lim_{K\to+\infty} \int_{t^*(K)}^{b} \Big[(V+NK)F^{N-1}(s) - K\Big]ds\\
&= \lim_{K\to+\infty} [b-t^*(K)](N-1)K = \lim_{K\to+\infty} (N-1)\frac{b-t^*(K)}{1/K}\\
&= \lim_{K\to+\infty} (N-1)\frac{dt^*(K)/dK}{1/K^2} = \lim_{K\to+\infty} \frac{N(N-1)VK^2}{(V+NK)^2(N-1)F^{N-2}(t^*(K))f(t^*(K))}\\
&= \frac{V}{Nf(b)}. \qquad \square
\end{aligned}
$$

Proof of Proposition 5. We establish the result by studying a relaxed scenario in which the designer observes contestants' types, but contestants do not know each other's type. We call this relaxed scenario a benchmark scenario. We assume that the type distribution of contestants follows a distribution with a cumulative distribution function $\tilde{F}(\cdot)$ whose support is bounded above by b. In particular, $\tilde{F}(\cdot)$ might or might not admit a density function, or it can be degenerated. Note that when the distribution is degenerated, we do not need to consider the incentive compatibility constraints for contestants in Problem (P). Therefore, whether the distribution is degenerated affects the analysis in Problem (P), but not in the benchmark scenario, in which we do not consider the incentive compatibility constraints in any case.

Let $\tilde{\mathbf{F}}(\mathbf{t})$ denote the joint distribution of $\mathbf{t}$, and $\tilde{\mathbf{F}}_{-i}(\mathbf{t}_{-i})$ denote the joint distribution of $\mathbf{t}_{-i}$. The expected payoff of contestant i with type t_i is:

[13] When the budget constraint is binding, contestants' expected total payoffs are equal to the difference between V and the expected total effort costs.

$$u_i(t_i, t_i) = \int\limits_{\mathbf{t}_{-i}} v_i(t_i, \mathbf{t}_{-i}) d\tilde{\mathbf{F}}_{-i}(\mathbf{t}_{-i}) - \frac{\int_{\mathbf{t}_{-i}} e_i(t_i, \mathbf{t}_{-i}) d\tilde{\mathbf{F}}_{-i}(\mathbf{t}_{-i})}{t_i}$$

$$= V_i(t_i) - \frac{\int_{\mathbf{t}_{-i}} e_i(t_i, \mathbf{t}_{-i}) d\tilde{\mathbf{F}}_{-i}(\mathbf{t}_{-i})}{t_i}.$$

The contest designer's objective can be expressed as:

$$\max_{\{v_i(\cdot), e_i(\cdot), \forall i\}} R = \int\limits_{\mathbf{t}} \left[\sum_i e_i(\mathbf{t}) + t_0 (V - \sum_i v_i(\mathbf{t})) \right] d\tilde{\mathbf{F}}(\mathbf{t}),$$

subject to the following feasibility constraints:

$$u_i(t_i, t_i) \geq 0, \forall t_i, \forall i, \tag{A.21}$$

$$\sum_i v_i(\mathbf{t}) \leq V, \forall \mathbf{t}, \tag{A.22}$$

$$v_i(\mathbf{t}) \geq -K, \forall \mathbf{t}, \forall i, \tag{A.23}$$

$$e_i(\mathbf{t}) \geq 0, \forall \mathbf{t}, \forall i. \tag{A.24}$$

The feasibility constraints consist of four parts: (A.21) is the participation constraint, (A.22) is the designer's budget constraint, (A.23) is the lower bound imposed on prizes, and (A.24) is the nonnegative effort constraint. Note that there is no longer any incentive compatibility constraint, since the designer knows contestants' types. Denote this problem as Problem (P-BM), where "BM" stands for "benchmark." If we compare this problem with the problem in the main text of the paper, the only difference is that there is no incentive compatibility constraint in Problem (P-BM), i.e., the benchmark setting is less constrained. This immediately implies that the expected total effort elicitable when the designer can observe contestants' types cannot be lower than that elicitable when the designer cannot.

It is easy to show that the participation constraint must be binding all the time. Otherwise, the designer can obtain a higher expected effort by simply asking the contestant to put in more effort and maintaining the same prize structure without violating the feasibility constraints. Therefore, we can relax the problem as follows:

$$\max_{\{v_i(\cdot), \forall i\}} R = \int\limits_{\mathbf{t}} \sum_i [t_i - t_0] \, v_i(\mathbf{t}) d\tilde{\mathbf{F}}(\mathbf{t}) + t_0 V,$$

subject to

$$\sum_i v_i(\mathbf{t}) \leq V, \tag{A.25}$$

$$V_i(t_i) \geq 0, \ \forall t_i, \forall i, \tag{A.26}$$

$$v_i(\mathbf{t}) \geq -K, \ \forall \mathbf{t}, \forall i. \tag{A.27}$$

This is a relaxed problem of the original problem. This is because the objective functions are the same in both problems by rewriting the effort with the prize structure, and the constraints are less restrictive than those in the original problem. To see this, constraint (A.26) is implied by the participation constraint (A.21) and nonnegative effort constraint (A.24) directly. Constraints (A.25) and (A.27) are the same as before. Denote this problem as (P-BM-Relax). It follows that the expected total effort from Problem (P-BM) should be no larger than that from Problem (P-BM-Relax).

We next show that the expected total effort equivalent elicitable in problems (P-BM) and (P-BM-Relax) is less than bV. In Problem (P-BM-Relax), note that

$$\int_{\mathbf{t}} \sum_i [t_i - t_0]\, v_i(\mathbf{t}) d\tilde{\mathbf{F}}(\mathbf{t}) = \sum_i \int_{t_i} [t_i - t_0]\, V_i(t_i) d\tilde{F}(t_i).$$

Since $V_i(t_i) \geq 0$, $t_i \leq b$ and $t_0 < b$, we have

$$\begin{aligned}
&\sum_i \int_{t_i} [t_i - t_0]\, V_i(t_i) d\tilde{F}(t_i) + t_0 V \leq [b - t_0] \sum_i \int_{t_i} V_i(t_i) d\tilde{F}(t_i) + t_0 V \\
&= (b - t_0) \int_{\mathbf{t}} [\sum_i v_i(\mathbf{t})] d\tilde{\mathbf{F}}(\mathbf{t}) + t_0 V \leq (b - t_0) \int_{\mathbf{t}} V d\tilde{\mathbf{F}}(\mathbf{t}) + t_0 V \\
&= bV.
\end{aligned}$$

Therefore, bV provides an upper bound for the expected total effort equivalent in Problem (P-BM-Relax) regardless of the distribution $\tilde{F}(\cdot)$, and in turn bV is also an upper bound for the expected total effort equivalent in Problem (P-BM). It follows that in the original setup, in which the principal cannot observe contestants' types, the total expected effort equivalent elicitable cannot go beyond bV. □

Proof of Example 1. As in the main text, let $u_k(t'_k, t_k)$ be contestant k's expected payoff when his type is t_k but he reports t'_k, given that his opponent reports truthfully. We prove the following three claims one by one.

1) There is no incentive for each type's contestant to misreport, given that his opponent reports truthfully. To see this point, let us first calculate type a_1 contestant's expected payoff under truthful report.

$$u_k(a_1, a_1) = q \cdot \frac{V}{2} + (1 - q) \cdot (-\tilde{K}) = 0. \tag{A.28}$$

The expected payoff of type b_1 contestant under truthful report is

$$\begin{aligned}
u_k(b_1, b_1) &= q(V + \tilde{K} - \frac{e^*}{b_1}) + (1 - q)[\frac{1}{2}V - \frac{1}{2} \cdot \frac{e^*}{b_1}] \\
&= q(V + \frac{qV}{2(1 - q)} - \frac{V}{1 - q^2}) + (1 - q)[\frac{1}{2}V - \frac{V}{2(1 - q^2)}] \\
&= \frac{V}{2(1 - q^2)}\{q[2(1 - q^2) + q(1 + q) - 2] + (1 - q)[(1 - q^2) - 1]\} \\
&= \frac{V}{2(1 - q)^2} q[(q - q^2) + (1 - q)(-q)] \\
&= 0.
\end{aligned} \tag{A.29}$$

Now if type a_1 contestant reports b_1, then his expected payoff is

$$u_k(b_1, a_1) = q(V + \tilde{K} - \frac{e^*}{a_1}) + (1 - q)[\frac{1}{2}V - \frac{1}{2} \cdot \frac{e^*}{a_1}]$$

$$
\begin{aligned}
&< q(V + \tilde{K} - \frac{e^*}{b_1}) + (1-q)[\frac{1}{2}V - \frac{1}{2} \cdot \frac{e^*}{b_1}] \\
&= u_k(b_1, b_1) = 0 = u_k(a_1, a_1).
\end{aligned}
$$

Thus, type a_1 contestant will truthfully report his type. Similarly, the expected payoff of type b_1 contestant when he reports a_1 is

$$u_k(a_1, b_1) = q \cdot \frac{V}{2} + (1-q) \cdot (-\tilde{K}) = u_k(a_1, a_1) = 0 = u_k(b_1, b_1).$$

Thus, type b_1 contestant will also report his type truthfully.

2) Both type a_1 and type b_1 contestant's expected payoffs are exactly zero. This observation has been proved in point 1), in equations (A.28) and (A.29).

3) The ex ante expected total effort is the utmost effort level $b_1 V$. To see this, notice that when both contestants are of type a_1, the total effort in this case is 0; when only one contestant's type is b_1, the type b_1 contestant exerts effort e^* and the type a_1 contestant exerts zero effort, so that the total effort in this case is e^*; when both contestants are of type b_1, then the expected effort of each contestant is $e^*/2$, which means that the total effort in this case is again e^*. Therefore, the ex ante expected total effort is $(1-q^2)e^* = b_1 V$, where $1-q^2$ is the probability that there is at least one contestant of type b_1. □

References

Chawla, S., Hartline, J.D., Sivan, B., forthcoming. Optimal crowdsourcing contests. Games Econ. Behav. https://doi.org/10.1016/j.geb.2015.09.001.

Che, Y.-K., Kim, J., 2006. Robustly collusion-proof mechanisms. Econometrica 74, 1063–1107.

Chen, Y.C., Xiong, S.Y., 2013. Genericity and robustness of full surplus extraction. Econometrica 81 (2), 825–847.

Crémer, J., McLean, R., 1988. Full extraction of the surplus in Bayesian and dominant strategy auctions. Econometrica 56, 1247–1257.

Fullerton, R., McAfee, P., 1999. Auctioning entry into tournaments. J. Polit. Econ. 107 (3), 573–605.

Ghosh, A., McAfee, P., 2012. Crowdsourcing with endogenous entry. In: Proceedings of the 21st International Conference on World Wide Web, pp. 999–1008.

Heifetz, A., Neeman, Z., 2006. On the generic (im)possibility of full surplus extraction in mechanism design. Econometrica 74, 213–234.

Kirkegaard, R., 2012. Favoritism in asymmetric contests: head starts and handicaps. Games Econ. Behav. 76, 226–248.

Laffont, J.J., Martimort, D., 2000. Mechanism design with collusion and correlation. Econometrica 68, 309–342.

McAfee, R.P., Reny, P.J., 1992. Correlated information and mechanism design. Econometrica 60, 395–421.

Minor, D., 2012. Increasing Effort Through Rewarding the Best Less (or Not at All). Working paper. Northwestern University.

Moldovanu, B., Sela, A., 2001. The optimal allocation of prizes in contests. Am. Econ. Rev. 91 (3), 542–558.

Moldovanu, B., Sela, A., 2006. Contest architecture. J. Econ. Theory 126 (1), 70–97.

Moldovanu, B., Sela, A., Shi, X., 2007. Contests for status. J. Polit. Econ. 115, 338–363.

Moldovanu, B., Sela, A., Shi, X., 2012. Carrots and sticks: prizes and punishments in contests. Econ. Inq. 50 (2), 453–462.

Myerson, R.B., 1981. Optimal auction design. Math. Oper. Res. 6, 58–73.

Olszewski, W., Siegel, R., 2017. Performance-Maximizing Contests with Many Contestants. Working paper. Northwestern University and Pennsylvania State University.

Peters, M., 2001. Surplus extraction and competition. Rev. Econ. Stud. 68, 613–631.

Polishchuk, L., Tonis, A., 2013. Endogenous contest success functions: a mechanism design approach. Econ. Theory 52 (1), 271–297.

Robert, J., 1991. Continuity in auction design. J. Econ. Theory 55, 169–179.

Taylor, C., 1995. Digging for golden carrots: an analysis of research tournaments. Am. Econ. Rev. 85, 872–890.