Optimal Selling Mechanisms with Buyer Price Search*

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Abstract

We study optimal dynamic selling mechanisms in a two-stage model where the buyer can search for a better price at the second stage. When this outside price is public, the optimal selling mechanism takes the form of a fixed first-stage price with price matching in the second stage. In contrast, when the outside price is the buyer's private information, the optimal mechanism is a menu of two contracts: a first-stage sale at a higher price with immediate delivery, or a first-stage sale at a lower price with second-stage delivery. Thus the optimal form of search deterrence depends on the observability of the buyer's outside option.

Keywords: Dynamic mechanism design; Outside option; Price search; Price matching; Search deterrence.

JEL Classification Numbers: D11, D82, D86.

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1 Introduction

When making a purchase decision, buyers commonly take into account potentially more favorable prices, which can be made available in the future by conducting a search. For example, a passenger approaching a travel agent for an airline ticket can further do an online search. A customer visiting a car dealer has the option of going to the next shop. A buyer facing a door-to-door salesperson, such as a magazine solicitor, has the option of searching at online marketplaces or retail stores later for alternative deals. Anticipating that a better outside price might become available in the future would affect buyers' purchase decisions, which in turn would influence the seller's optimal selling strategy.

In this paper, we explore the revenue-maximizing dynamic selling mechanism for a principal who sells an indivisible object to a buyer in a two-stage setting. The buyer privately learns his value for the object in the first stage, and can further search for an outside option price in the second stage. The outside price and the buyer's private value are independent and continuously distributed. Both the buyer and the seller discount their second-stage payoffs. We consider two scenarios by allowing the second-stage information to be either public or private.

When the outside price is public, we find that the optimal mechanism involves a cutoff value v^*. The seller sets a fixed price at the first stage, which will be taken only by first-stage types above v^*. In the second stage, she sets a price that exactly matches the outside price. First-stage types below v^* have to search for the outside option, and return to buy from the seller in the second stage at the outside price when it is lower than their private values. A posted price coupled with future price matching is widely observed in practice. For example, major retailers, including Walmart, Target, the Home Depot, Lowe's, and Staples, among others, all allow customers to return the goods or present a competitor's advertisement (with a lower price) up to certain number of days after the purchase for a refund of the difference.

When the outside option price is the buyer's private information, a second value cutoff v^{**} ($< v^*$) is involved. The seller offers a menu of two contracts: a first-stage direct sale with immediate delivery at a higher fixed first-stage price or a first-stage direct sale with second-stage delivery at a lower fixed first-stage price. The first-stage prices in both contracts are lower than that in the scenario with a public outside price. First-stage types above v^* will take the first contract, and types between v^* and v^{**} will choose the second. Types below v^{**} never purchase from the seller. A salient feature of this optimal selling mechanism is that it only screens the buyer with respect to the first-stage type, which is atypical in the sequential screening literature. Such contracts are often observed in the real estate market.

A property developer concurrently sells apartments under construction at prices lower than those of the newly finished. Also, online market sellers often use differentiated handling and shipping services with different charges and delivery times to screen buyers.[1] When there is no discounting in the second-stage payoffs, the mechanism with two contracts reduces to a single-contract one, which offers immediate delivery at a fixed first-stage price. This type of posted price selling mechanism is universal in practice.

Our findings contrast with those of Armstrong and Zhou (2016), who investigate the optimal selling mechanism in a setting in which the buyer searches for an independent outside net payoff rather than an outside price. While Armstrong and Zhou study a search model with differentiated products, we consider an environment of searching for homogeneous products. They find that the privateness of the second-stage information does not affect the optimal design, and the optimal mechanism screens buyers in both stages. If we interpret our model from the perspective of Armstrong and Zhou, the equivalent buyer's net outside payoff in our setting—which is the difference between the buyer's private value and the outside price—is correlated with the first-stage private value. Our result thus shows that when the buyer's value and net outside payoff are correlated, the privateness of an orthogonalized signal of this net payoff may affect the optimal design of the selling mechanism.

We would like to emphasize that the correlation between the buyer's value and net outside payoff is not sufficient for the relevance of the privateness of an orthogonalized signal about net payoff. It is, however, necessary, as indicated by the irrelevance result of Armstrong and Zhou (2016).[2] Technically, a sufficient condition to generate the relevance result is as follows: The form of correlation causes the candidate optimal mechanism—which is usually identified by the two-stage pointwise maximization of virtual value function—to fail the monotonicity in the orthogonalized signal, which is necessarily required by the second-stage incentive compatibility.[3] In other words, the relevance result would prevail if the first-order approach fails.

The dynamic mechanism design literature has largely adopted the first-order approach, which relies on the envelope conditions implied by local incentive compatibility constraints and reformulates the seller's objective as a function of allocation rules. In many settings, the first-order-optimal allocation rule, which is obtained by pointwise maximization, can

[1] These differences in selling prices and handling charges are partially due to the different provision costs.

[2] This is consistent with the finding of Esö and Szentes (2007), who assume that the first- and second-stage types are correlated. They show that disclosing an orthogonalized signal about the second-stage type does not affect the mechanism design.

[3] The virtual value function can be derived following the usual "first-order approach" by assuming that the orthogonalized signal is either public or private.

be verified as satisfying the ignored global incentive compatibility conditions. Successful applications of such a first-order approach include Baron and Besanko (1984); Laffont and Tirole (1990, 1996); Courty and Li (2000); Battaglini (2005); Esö and Szentes (2007); Garrett and Pavan (2012); Kakade, Lobel, and Nazerzadeh (2013); Pavan, Segal and Toikka (2014); Halac and Yared (2014); Krähmer and Strausz (2015b); Bergemann and Strack (2015); Armstrong and Zhou (2016); Bergemann, Castro and Weintraub (2018); and Liu and Lu (2018), among many others. In particular, Pavan et al. (2014) adopt a general environment and derive "envelope formulas" for continuous types. Besides providing conditions for the first-order-optimal mechanism derived from these formulas to be incentive compatible, they leave open the question of the general applicability of the first-order approach.

Battaglini and Lamba (2019) find that the "first-order approach" works when the resulting first-order-optimal contract obtained by focusing on local incentive compatibility constraints satisfies a particularly strong form of monotonicity in types. However, the strong form monotonicity and thus global incentive compatibility are generically violated by the first-order-optimal contract in their setting. Our finding is consistent with their insights. In our setting, the first-order-optimal contract violates the required monotonicity in the second-stage information, which is implied by the second-stage global incentive compatibility. The issue arises due to the nonmonotonicity of the virtual value function in the second-stage information.

When the first-order approach fails, there is no generally applicable approach available in the literature. Kakade et al. (2013) identify sufficient conditions under which the first-order approach fails, and they rely on numerical algorithms to search for the optimal mechanisms. Nevertheless, a few studies manage to analytically characterize the optimal contract when the first-order-approach fails. Battaglini and Lamba (2019) have fully solved a three-type two-period example. Krähmer and Strausz (2015a) propose an innovative way to deal with global constraints that works in their environment with N types, and Krasikov and Lamba (2020) successfully identify the optimal dynamic pricing mechanism for the problem they study. In particular, two studies apply the Myerson ironing procedure to fix issues related to monotonicity with respect to types, which cause the failure of incentive compatibility. Mierendorff (2016) applies the procedure in fixing the nonmonotonicity with first-stage information, and Meng and Tian (2019) adopt a slope-ironing technique to ensure that the slope of the allocation with respect to the first-stage type is sufficiently large for incentive compatibility. In our paper, we amend the first-order approach by explicitly taking into account the implications of global incentive compatibility, which allows us to identify the optimal selling mechanism.

Our model can be extended in several ways. First, our approach is extendable to a generalized setting, in which an outside option is not always available. The optimal mechanisms are similar, except that the value cutoff for first-stage sale and second-stage delivery is higher and the prices charged are higher to exploit the possibility of no outside option. Second, we consider a situation in which the outside option is verifiable. We find that even with a private buyer option, verifiability entails an optimal design that coincides with that in the public option case. Third, we examine an extension in which the buyer's outside price is positively correlated with his value. With a private outside price, our approach remains valid under some regularity conditions and the optimal design resembles that in the baseline model. In particular, the feature that the optimal design does not screen the buyer by the second-stage information is robust to correlation between the buyer's private value and the outside price.

Our findings reveal that the privateness of second-stage information, which is independent of the first-stage information in our model, matters for the optimal design. It contrasts with the irrelevance result of Esö and Szentes (2007). Krähmer and Strausz (2015b) consider two scenarios: one with discrete ex ante information and one with continuous ex ante information. They find that the irrelevance of ex post (orthogonalized) information holds if and only if ex ante information is continuously distributed. Kakade et al. (2013) demonstrate environments in which types are continuous but the irrelevance of the privateness of ex post information fails. Li and Shi (2017) find that if instead the second-stage type is correlated with the first-stage type, discriminatory information disclosure can be better than full disclosure. Guo, Li, and Shi (2018) further study optimal information disclosure with correlated information. In our setting with independent types across stages, our relevance result indicates that characterizing the optimal information disclosure policy is not straightforward, since we no longer have the irrelevance result that shows the optimality of full disclosure.

The rest of the paper is organized as follows. In Section 2, we set up the baseline model. In Section 3, we derive the optimal two-stage mechanisms. In Section 4, we study variant settings of our baseline model, and Section 5 concludes. The appendix collects technical proofs.

2 The Model

We consider an environment in which there is a seller who wants to sell an indivisible object (e.g., a laptop computer) to a risk-neutral buyer. The seller's valuation for the object is normalized to 0. There are two stages. At stage 1, the buyer is privately endowed with a value v for the object, which remains the same across the two stages. At stage 2, the buyer can costlessly search for an outside option, which takes the form of an alternative price τ offered by a different source for the same object (e.g., a laptop computer of the same brand and model). We assume that v and τ are independent. Both the seller and the buyer discount their second-stage payoffs by $\delta \in (0, 1]$.

Let F be the cumulative distribution function of the buyer's private value v, and f the corresponding density. We assume that f is continuous and strictly positive on interval $V := [0, \bar{v}]$. Define $\lambda(v) = v - \frac{1-F(v)}{f(v)}$. Throughout the paper, we assume that $\lambda(v)$ is increasing in v. Denote the cumulative distribution function of τ by G, which shares the same support V of value v.[4] The corresponding density g is continuous and positive on V. Both F and G are common knowledge.

According to Myerson (1982), the revelation principle applies. To search for the optimal selling mechanism, the seller can, without loss of generality, commit to a two-stage direct truthful selling mechanism $\mathcal{M} = \{p_1(v), x_1(v); p_2(v, \tau), x_2(v, \tau)\}$. At stage 1, the buyer reports his value to the seller. Upon receiving report v, the seller sells the object with probability $p_1(v)$ and requires payment $x_1(v)$ from the buyer. With probability $1 - p_1(v)$, the object is not sold and the game enters the second stage. At stage 2, the buyer reports his outside option price. Upon receiving report τ, trade takes places with probability $p_2(v, \tau)$ and the buyer pays $x_2(v, \tau)$. With probability $1 - p_2(v, \tau)$, the game ends and the buyer turns to the outside option.

Note that the buyer has the option of not buying from either the seller or the outside source. If a buyer with value v simply waits to search for the outside option at the second stage, his equivalent first-stage expected payoff is $\delta E_\tau \max\{v - \tau, 0\}$, where $\max\{v - \tau, 0\}$ is his second-stage payoff from an outside price τ.

[4] If the upper bound of τ is higher than $\bar{v}$, the buyer may find an outside price he would never take. This case is essentially covered in Section 4.1.

3 The analysis

3.1 Benchmark: The Public Outside Option Price

We first consider a benchmark scenario in which the outside option price is publicly observable. The seller's competitors' advertisements or promotion offers are often publicly observable online or through other types of media. This scenario also applies when the buyer can provide verifiable evidence of outside options, e.g. written price quotes from other merchants.

In this benchmark scenario, we start with the second stage. Recall that the second-stage mechanism is $\{p_2(\cdot,\cdot), x_2(\cdot,\cdot)\}$. If a buyer, who reports v' while his value is v in the first stage, finds an outside option price τ in the second stage, his second-stage expected payoff is

$$\tilde{\pi}(v, v', \tau) = p_2(v', \tau)v + (1 - p_2(v', \tau)) \max\{v - \tau, 0\} - x_2(v', \tau).$$

Since τ is public, there is no need to address the second-stage incentive compatibility. We further relax the problem by ignoring individual rationality constraints in the second stage, both on and off the equilibrium path. We can treat the benchmark scenario as a single-stage design problem at the first stage, with the second-stage outcome fully specified by mechanism $\{p_2(\cdot,\cdot), x_2(\cdot,\cdot)\}$.

We now consider the first stage. Under the first-stage mechanism $\{p_1(\cdot), x_1(\cdot)\}$, if the buyer reports v' while his value is v, then his interim expected payoff is

$$\pi(v, v') = p_1(v')v + (1 - p_1(v'))\delta E_\tau \tilde{\pi}(v, v', \tau) - x_1(v').$$

Denote $\pi(v) = \pi(v, v)$. The first-stage incentive compatibility constraint requires that for all $v, v' \in [0, \bar{v}]$,

$$\pi(v) \geq \pi(v, v'). \tag{3.1}$$

The individual rationality constraint at the first stage requires that for all $v \in [0, \bar{v}]$,[5]

$$\pi(v) \geq \delta E_\tau \max\{v - \tau, 0\}. \tag{3.2}$$

The seller's revenue is the sum of the discounted payments she collects from the buyer in two stages. Hence, under mechanism $\mathcal{M}$, the seller's expected revenue is

$$R = \int_0^{\bar{v}} \Big\{x_1(v) + (1 - p_1(v))\delta E_\tau x_2(v, \tau)\Big\} dF(v).$$

[5]There is no loss of generality to consider full participation in the first stage, since the seller can always set $p_1 = 0$ and $x_1 = 0$ for a nonparticipating type to motivate him to participate.

With public second-stage information τ, the seller's problem is to maximize R subject to constraints (3.1) and (3.2).

Following the standard Myersonian procedure in a single-stage problem, the optimal selling mechanism is presented in Proposition 1. We would like to emphasize that the optimal mechanism is incentive compatible in the first stage and individual rational in both stages.

Proposition 1. *Let v^* be such that $\lambda(v^*) = 0$. In the scenario in which the buyer's outside option price τ is public information, the optimal mechanism is as follows.*

1. *If $v \geq v^*$, the buyer purchases from the seller in the first stage without search at price $(1-\delta)v^* + \delta E_\tau \min\{v^*, \tau\}$.*

2. *If $v < v^*$, the buyer searches for an outside option price and returns to purchase from the seller in the second stage at price τ if $\tau \leq v$; otherwise, he leaves the market without purchasing the object.*

   The object is delivered at the stage in which it is sold.

Note that when the second-stage information is public, the problem is essentially static. The seller screens low-value types (i.e., $v < v^*$) in the first stage as the result of the classic trade-off between efficiency and revenue maximization. In the second stage, the seller screens the low-value types with superior outside option (i.e., $\tau < v$) through price matching. The allocation in the second stage seems surprising. Instead of selling to the buyer with $\tau > v$ at price v in the second stage, the optimal mechanism sells to the buyer with $\tau \leq v$ at price τ. The rationale is as follows. First, the first-stage incentive compatibility implies that low-value types should be excluded based on the classic efficiency versus information rent extraction trade-off. However, the existence of buyer outside options limits rent extraction from higher-value types, since excluded lower-value types can utilize the outside options and the higher-value types can mimic this. By price matching, lower-value types are prevented from using the outside options. The seller could capture additional revenue without changing the rents for higher-value types. Second, if the seller commits to selling at a second-stage price that equals the first-stage reported value whenever the outside price is higher than the reported value, the buyer would have an incentive to underreport in the first stage and obtain the object with a higher probability and at a lower price. By matching the price at the second stage, the seller deters such underreporting behavior of the buyer in the first stage, since a lower reported value means a lower chance of buying from the seller in the second stage.

When $\delta = 1$, the optimal mechanism is not unique.[6] The seller is indifferent between

[6]When $\delta < 1$, the optimal mechanism becomes unique, and it is strictly optimal to sell to types $v \geq v^*$ in the first stage.

selling to types $v \geq v^*$ at price $E_\tau \min\{v^*, \tau\}$ in the first stage and selling to those types at price $\min\{v^*, \tau\}$ in the second stage. This alternative mechanism, together with part 2 of Proposition 1, is thus static in nature: It is implemented only in the second stage after τ is revealed. The seller charges v^* if $\tau > v^*$ and τ otherwise. From this perspective, the benchmark setting reduces to a static monopoly pricing problem in which a public outside price is realized before the monopolist names her own price.

3.2 Private Outside Option Price

We now turn to the original environment with a private outside option price. As mentioned in the introduction, the popular approach to solving the dynamic mechanism design problem with private ex post information in the literature (e.g., Esö and Szentes, 2007 and Pavan et al., 2014) is the first-order approach, which uses only local incentive compatibility conditions to derive the expression of the principal's expected payoff and relies on pointwise maximization to identify the optimal allocation rule. The identified mechanism is indeed optimal and satisfies global incentive compatibility. But in our problem with a private outside price, we will show that the optimal solution by the first-order approach is not globally incentive compatible. Details on why the issue arises and how we solve the problem will be revealed as we proceed with the analysis.

In the second stage, a buyer who observes τ and reports τ' after truthfully reporting his value v in the first stage gets

$$\tilde{\pi}(v, \tau, \tau') = p_2(v, \tau')v + (1 - p_2(v, \tau')) \max\{v - \tau, 0\} - x_2(v, \tau').$$

On the equilibrium path, incentive compatibility in the second stage is equivalent to

$$\tilde{\pi}(v, \tau, \tau) \geq \tilde{\pi}(v, \tau, \tau'), \ \forall v, \tau, \tau'.$$

Let $\tilde{\pi}(v, \tau) \triangleq \tilde{\pi}(v, \tau, \tau)$. The following lemma provides necessary conditions on p_2 for a two-stage mechanism to be incentive compatible in the second stage after a truthful first stage.

Lemma 1. *Suppose the first stage is truthful. The second-stage IC requires that the second-stage allocation rule $p_2(v, \tau)$ be weakly increasing in τ if $\tau \leq v$, and $p_2(v, \tau) \geq p_2(v, v)$ for all $\tau > v$. In addition, we have $\tilde{\pi}(v, \tau) = \tilde{\pi}(v, \tau')$, $\forall \tau,\ \tau' \in [v, \bar{v}]$.*

The proof of Lemma 1 given in Appendix establishes the monotonicity of $p_2(v, \tau)$ in τ on $[0, v]$. But $p_2(v, \tau)$ is not necessarily monotone in τ on $[v, \bar{v}]$. The second-stage incentive

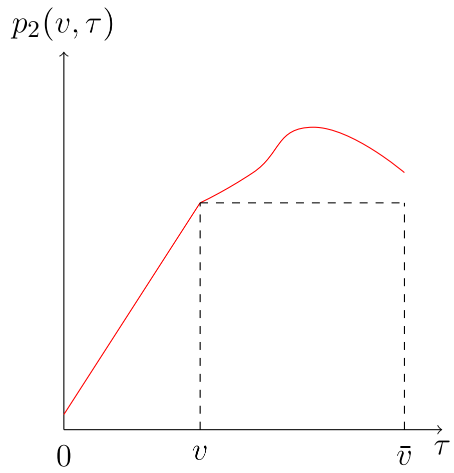

Figure 3.1: Semi-monotonicity of Incentive Compatible Second-stage Allocation Rules

compatibility only requires that $\tilde{\pi}(v,\tau)$ be independent of τ on $[v,\bar{v}]$. For example, consider the type $v=0$. For any given $p_2(0,\tau)$ that is not necessarily monotone, let $x_2(0,\tau)=0$. Clearly, $(p_2(0,\tau),x_2(0,\tau))$ is incentive compatible in stage 2 and induces $\tilde{\pi}(0,\tau)=0$. Figure 3.1 depicts an example of p_2 that satisfies such a semi-monotonicity property.

Note that the buyer's stage-2 payoff is

$$p_2 v + (1-p_2)\max\{v-\tau,0\} - x_2 = \begin{cases} p_2\tau + (v-\tau) - x_2, & \text{if } \tau \le v, \\ p_2 v - x_2, & \text{if } \tau \ge v. \end{cases}$$

The payoff has increasing differences in (p_2,τ) when $\tau < v$. Thus the stage-2 allocation should be weakly increasing in τ for such a pair of (v,τ). The property of increasing difference does not extend to the case with $\tau \ge v$. The allocation can be nonmonotone in τ for $\tau \ge v$. Nevertheless, we must still have $p_2(v,\tau) \ge p_2(v,v)$ if $\tau > v$.

Clearly, the stage-2 allocation of the optimal mechanism in Proposition 1, which is actually decreasing in τ for any given $v \le v^*$, violates the semi-monotonicity of p_2 in Lemma 1. The whole mechanism is thus not implementable when τ is private.

The implications of the above observation are twofold. First, the privateness of second-stage information does make a difference in mechanism design in our setting. Second, to derive the optimal mechanism for the scenario with private τ, the first-order approach must be amended.

We next characterize the buyer's second-stage lie-correction strategy following a first-stage misreport, which is essential for the analysis of the buyer's first-stage incentive and determines a convenient route for amending the first-order approach with the necessary semi-monotonicity conditions on incentive-compatible allocation rules in Lemma 1.

Lemma 2. ***(Optimal lie correction)*** *In an incentive-compatible two-stage selling mechanism, consider a buyer who has value v but reports v' at the first stage. Let $\sigma(v, v'; \tau) = \min\{v, \tau\}$ for $v' \geq \min\{v, \tau\}$ and $\sigma(v, v'; \tau) \in \arg\max_s p_2(v', s)$ for $v' < \min\{v, \tau\}$. Then $\sigma(v, v'; \tau)$ is an optimal second-stage report for the buyer.*

In our model, $\min\{v, \tau\}$ can be interpreted as the buyer's ex post willingness to pay. From this perspective, reporting $\sigma(v, v'; \tau) = \min\{v, \tau\}$ if $v' \geq \min\{v, \tau\}$ fully corrects the buyer's first-stage lie and truthfully reveals his ex post willingness to pay. However, Lemma 2 shows that if the buyer underreports in the first stage, there exist situations (i.e., $v' < \min\{v, \tau\}$) in which he cannot fully correct his first-stage lie because the buyer's ex post willingness to pay, $\min\{v, \tau\}$, has a shifting support that depends on first-stage information. The optimal second-stage report instead simply maximizes the allocation probability. It is mechanism contingent and does not have an explicit solution in general. Therefore, we do not have a "strong truth-telling" property in our setting.

Define

$$u(v, \tau) = \begin{cases} \tau & \text{if } \tau \leq v, \\ \lambda(v) & \text{otherwise.} \end{cases}$$

$u(v, \tau)$ is interpreted as the buyer's second-stage virtual value. The seller's expected revenue derived following the first-order approach in the environment with public τ can alternatively be written as[7]

$$\begin{aligned} R = \int_0^{\bar{v}} p_1(v)\lambda(v) + (1 - p_1(v))\delta\Big[G(v)\lambda(v) \\ - \int_0^v \tau dG(\tau) + \int_0^{\bar{v}} p_2(v, \tau)u(v, \tau)dG(\tau)\Big]dF(v). \end{aligned} \tag{3.3}$$

For simplicity, in our analysis, we follow the route of considering a relaxed problem with a public outside option price while requiring that the second-stage allocation rule satisfy the semi-monotonicity condition in Lemma 1. We solve this relaxed problem and then show that the optimal solution is implementable in the original setting with private τ and achieves the maximal revenue of the relaxed problem.[8]

[7] This revenue expression, in principle, can alternatively be obtained by using the envelope formulas derived in Pavan et al. (2014). The buyer's payoff function involves a term $(v - \tau)^+$, which has a concave kink at τ. As is well known (e.g., Holmström (1979) and Carbajal and Ely (2013)), such kinks can cause the payoff equivalence theorem to fail in some environments. This is not an issue in our setting, since such kink points are of zero measure in the two-dimensional space of (v, τ). Nevertheless, potential issues might arise from the lie-correction strategy in Lemma 2, as we have limited knowledge about its properties.

[8] The simplicity of this procedure also makes it convenient to further analyze, in Section 4.1, the case in which an outside option is not always available.

The optimal solution for the concerned relaxed problem maximizes the revenue expression (3.3) subject to the semi-monotonicity condition in Lemma 1. To solve the relaxed problem, we first find the optimal $p_2(v,\tau)$, which maximizes $\int_0^{\bar{v}} p_2(v,\tau)u(v,\tau)dG(\tau)$ subject to the semi-monotonicity condition. Clearly, we cannot directly apply pointwise maximization to identify the optimal $p_2(v,\tau)$, since $u(v,\tau)$ is not monotone in τ. The solution from pointwise maximization violates the semi-monotonicity condition in Lemma 1.

Let $\omega(v) = \int_0^{\bar{v}} u(v,\tau)dG(\tau)$. Since the second-stage virtual value $u(v,\tau) = \lambda(v)$ for any $\tau \geq v$, we must have $p_2(v,\tau) = p_2(v,v)$ for $\tau \geq v$ at optimum; and since $u(v,\tau) = \tau \geq 0$ for any $\tau < v$, we should have $p_2(v,\tau) = p_2(v,v)$ for $\tau < v$ at optimum. Therefore, the optimal p_2 is independent of τ, and must be equal to 1 if $\omega(v) \geq 0$ and 0 if otherwise. When $\tau < v$, $u(v,\tau) = \tau$ is independent of v. Together with the increasing property of $\lambda(v)$, this implies that $\omega(v)$ is increasing in v. By the continuity and monotonicity of $\omega(v)$ as well as the intermediate value theorem, there exists a unique $v^{**} \in (0, v^*)$ such that $\omega(v^{**}) = 0$.

Given first-stage report v, define $p_2^{**}(v,\cdot) : [0,\bar{v}] \to [0,1]$ as

$$p_2^{**}(v,\tau) = \begin{cases} 1 & \text{if } v \geq v^{**}, \\ 0 & \text{if } v < v^{**}. \end{cases}$$

The following lemma summarizes the optimal allocation probability in the second stage.

Lemma 3. *$p_2^{**}(v,\cdot)$ maximizes $\int_0^{\bar{v}} p_2(v,\tau)u(v,\tau)dG(\tau)$ subject to the semi-monotonicity constraint in Lemma 1.*

By Lemma 3, we thus have

$$\begin{aligned} R \leq &\int_0^{v^{**}} \delta\Big[G(v)\lambda(v) - \int_0^v \tau dG(\tau)\Big] + p_1(v)\Big[\delta\omega(v) + (1-\delta)\lambda(v)\Big]dF(v) \\ &+ \int_{v^{**}}^{\bar{v}} p_1(v)\lambda(v) + (1-p_1(v))\delta\lambda(v)dF(v). \end{aligned} \tag{3.4}$$

If $v < v^{**}$, then $\delta\omega(v) + (1-\delta)\lambda(v) < 0$; if $v \in [v^{**}, v^*)$, then $\lambda(v) \leq \delta\lambda(v) < 0$; if $v \geq v^*$, then $\lambda(v) \geq \delta\lambda(v) \geq 0$. We thus define $p_1^{**}(\cdot) : [0,\bar{v}] \to [0,1]$ as follows:

$$p_1^{**}(v) = \begin{cases} 1 & \text{if } v \geq v^*, \\ 0 & \text{if } v < v^*. \end{cases}$$

Clearly, p_1^{**} maximizes the right side of inequality (3.4), which provides the following revenue upper bound for the relaxed problem:

$$R^* = \delta \int_0^{v^{**}} \Big\{G(v)\lambda(v) - \int_0^v \tau dG(\tau)\Big\}dF(v) + \delta \int_{v^{**}}^{v^*} \lambda(v)dF(v) + \int_{v^*}^{\bar{v}} \lambda(v)dF(v).$$

For the original problem with private τ, R^* provides a valid upper bound for seller revenue. Therefore, if we can identify an implementable mechanism in the original setting that generates revenue R^*, it must be optimal. The following proposition provides such a mechanism.

Proposition 2. *In the scenario in which both v and τ are the buyer's private information, the optimal selling mechanism involves two cutoffs, v^{**} and v^*, which do not depend on $\delta \in (0,1]$.*[9]

1. *If $v \geq v^*$, the buyer purchases from the seller in the first stage at price $(1-\delta)v^* + \delta E_\tau \min\{v^{**}, \tau\}$, and the object is delivered in the first stage.*
2. *If $v \in [v^{**}, v^*)$, the buyer pays $\delta E_\tau \min\{v^{**}, \tau\}$ to the seller in the first stage, and the object is delivered in the second stage.*
3. *If $v < v^{**}$, the buyer conducts a search and buys from the outside source if $\tau \leq v$. There is no purchase from the seller.*

Proposition 2 reveals that when both v and τ are privately observed by the buyer, the seller offers two options at the optimum: a first-stage delivery at a higher price or a second-stage delivery at a lower price.[10] The buyer types with $v \geq v^*$ will take the first option, while the types with $v \in [v^{**}, v^*)$ will choose the second. The low-value types with $v < v^{**}$ will take neither and conduct a search. Payments are always settled in the first stage regardless of whether the delivery happens in the first stage or the second stage. The lower first-stage price is accompanied by a later delivery. Although it lowers the price that could be charged to intermediate-value types, the later delivery would also reduce the rent to high-value types. At optimum, the seller's benefit from lowering the rent to high-value types surpasses her loss from the lower price charged to intermediate-value types.

Compared with the public τ scenario, price matching is now not feasible in this private setting. The seller can't use price matching to collect additional revenue while not changing the rent to high-value types as he does in the public setting. There are two salient features of the optimal mechanism for the private setting. One is that it does not sell to types with $v < v^{**}$ at all; the other is that the second-stage allocation is independent of τ, i.e., it does

[9]When there is no discounting, i.e., $\delta = 1$, the menus for $v \geq v^*$ and $v \in [v^{**}, v^*)$ converge. The seller simply offers a single first-stage posted price $E_\tau \min\{v^{**}, \tau\}$ for all types above v^{**}.

[10]When $\delta = 1$, Proposition 2 says that the optimal mechanism reduces to a single menu contract with a first stage posted price $E_\tau \min\{v^{**}, \tau\}$. The seller might alternatively offer an optimal second-stage posted price, which we find can be either higher or lower than $E_\tau \min\{v^{**}, \tau\}$ depending on distributions $F(\cdot)$ and $G(\cdot)$. Proposition 2 means that the second-stage posted price is dominated by the first-stage posted price in the optimal mechanism.

not screen over the second-stage information. Nevertheless, the incentive compatibility of two-stage mechanisms does not exclude the feasibility of selling to types with $v < v^{**}$ or screening at the second stage, as illustrated by the following lemma that provides sufficient conditions for implementable allocation rules.

Lemma 4. *Allocation rules in a two-stage mechanism $\mathcal{M}$ are implementable if the following conditions hold: (1) $p_1(v)$ increases with v; (2) $p_2(v, \tau)$ increases with v; (3) $p_2(v, \tau)$ is increasing in τ if $\tau \leq v$, and $p_2(v, \tau) = p_2(v, v)$ if $\tau \geq v$.*

We are now ready to illustrate the intuition whereby screening at the second stage is never optimal and selling to first-stage types with $v < v^{**}$ is not optimal. Recall the discussion leading to Lemma 3: For any given v, the virtual value is $\lambda(v)$, $\forall \tau \geq v$ and τ, $\forall \tau < v$. The semi-monotonicity of Lemma 1 means that the optimal second-stage allocation rule that maximizes the expected seller surplus must be independent of the second-stage type. This property satisfies condition (3) in Lemma 4.

For a given v, setting $p_2(v, \tau) = \eta$ leads to a seller surplus of $\eta\delta\omega(v)$ at the second stage, which strictly increases with v. Therefore, there must exist a v-cutoff such that the revenue-maximizing $p_2(v, \tau) = 0$ if and only if v is lower than the cutoff. It turns out that this v-cutoff is exactly v^{**}, since $\omega(v^{**}) = 0$. Such defined optimal $p_2(v, \tau)$ clearly satisfies the conditions in Lemma 4. Therefore, selling to first-stage types with $v < v^{**}$ is not optimal.

3.3 Relevance of the privateness of τ

Comparing the mechanisms in Propositions 1 and 2, we find that the privateness of second-stage information τ is relevant for the seller's optimal design. While information in both stages is used in the public setting, only first-stage information is used in the private setting. Specifically, search deterrence takes the form of price matching in the scenario of public outside price; it takes the form of preemptive first-stage selling prices when the outside price is the buyer's private information.[11]

When there is no discounting, i.e., $\delta = 1$, payments can be collected in different stages in the public τ setting, but this is not the case in the private τ setting. This discrepancy arises because the seller could sell to the buyer at τ in the second stage if τ is observable. In the private setting, the seller cannot commit to such a price match to generate revenue in the second stage. The seller thus would strictly prefer to deter the buyer's search by offering preemptive prices in the first stage.

[11]The first-stage prices are lower than that in the public τ scenario, and they are also lower than what a monopolist would charge in the absence of the outside option.

Note that when outside option τ is verifiable, the seller could costlessly elicit a truthful report at the second stage by requiring the buyer's second-stage report to be supported by verifiable evidence. The optimal design with private but verifiable τ is thus equivalent to that for the public τ setting in Proposition 1. Clearly, the privateness of τ essentially plays the same role as the non-verifiability of τ.

A direct consequence of the differences in selling mechanisms is that the buyer's and seller's expected payoffs change in opposite directions, which is summarized in the following corollary.

Corollary 1. *Compared with the benchmark environment in which the buyer's outside option is public, the privateness of the outside option price benefits the buyer and hurts the seller in expectation.*

Corollary 1 is pretty intuitive. The private outside option in the second stage gives the buyer a larger information advantage over the seller, which would benefit the buyer and hurt the seller. Specifically, a high-value ($v \geq v^*$) buyer purchases the object from the seller at a lower price in the private setting than in the public setting at the first stage. An intermediate-value ($v \in (v^{**}, v^*)$) buyer gets the object from the seller for sure in the second stage, pays a first-stage price $\delta E_\tau \min\{v^{**}, \tau\} (< v)$ in the private setting, and enjoys an expected payoff higher than $\delta E_\tau \max\{v - \tau, 0\}$ (i.e., the payoff he gets in the public setting). A low-value ($v < v^{**}$) buyer has the same expected payoffs in both settings, since he only gets the object when τ turns out to be lower than v and pays the same price.[12] For the seller, the revenue upper bound R^* equals the maximum expected revenue achieved with public τ while imposing the binding semi-monotonicity constraint in Lemma 1. Therefore, the maximum expected revenue with private τ is strictly lower than that in the public setting, which means that the seller is worse off in the private setting.

4 Extensions

In this section, we further study several variant settings of our baseline model. First, we consider a situation in which the outside option is not always available. Second, we investigate an environment in which the buyer's outside option is verifiable. Third, we generalize our baseline analysis to a model that allows positive correlation between the buyer's value and the outside option price.

[12] In the private setting, the low-value buyer gets the object from the outside option at price τ; in the public setting, he gets it from the seller at price τ due to the price-matching feature of the optimal selling mechanism.

4.1 When an outside option is not always available

We first consider a realistic setting in which an outside option is not always available. Let $q_0 \in (0, 1)$ be the probability that an outside option is not available. We assume that the seller knows q_0, but whether an outside option exists is the buyer's private information. We introduce an additional second-stage type ϕ for the buyer, which denotes no outside option. Our analysis in the baseline model can be extended to this setting, and the optimal designs with public and private second-stage information generalize those of Propositions 1 and 2. The following proposition presents the optimal designs.[13]

Proposition 3. *When outside option is not always available, let v_{q_0} be such that $(1 - q_0)\omega(v_{q_0}) + q_0\lambda(v_{q_0}) = 0$.*

1. *If the availability of an outside option and the optional price when it is available are public information, the seller, at optimum, sells to a buyer with $v \geq v^*$ at price $v^* - \delta(1 - q_0)E_\tau \max\{v^* - \tau, 0\}$ in the first stage and sells to a buyer with $v < v^*$ at price τ in the second stage if and only if the buyer finds an outside option price $\tau < v$. The object is delivered at the stage in which it is sold.*

2. *If the availability of an outside option and the optional price when it is available are the buyer's private information, at optimum, the seller sells to a buyer with $v \geq v^*$ at first-stage price $(1 - \delta)v^* + \delta(1 - q_0)E_\tau \min\{v_{q_0}, \tau\} + \delta q_0 v_{q_0}$ and delivers the object in the first stage; she sells to a buyer with $v \in [v_{q_0}, v^*)$ at first-stage price $\delta(1 - q_0)E_\tau \min\{v_{q_0}, \tau\} + \delta q_0 v_{q_0}$ and delivers the object in the second stage.*

The proofs are essentially identical with those of the baseline models. The only difference is that we introduce a new second-stage type to denote the event that an outside option is not available. This event happens with a positive probability mass. No extra difficulty is created by the new second-stage type, and our analysis applies smoothly. The forms of the optimal designs also do not change, and the intuitions behind the optimal designs are similar to those in the baseline model. Compared with the baseline model, when the second-stage information is public, the seller can exploit the possible unavailability of an outside option and charge a higher price in the first stage. When the second-stage information is private, the cutoff for sale, v_{q_0}, is higher than that in the baseline setting (i.e., v^{**}), and the first-stage prices charged are higher. This is intuitive, since the greater informational disadvantage of the buyer due to the possible unavailability of outside options allows the seller to charge higher prices and sell with lower probability.

[13] The proof is omitted to save space and is available from the authors upon request.

4.2 When outside option τ is verifiable

We now turn to a scenario in which the buyer's outside option is verifiable. In other words, the buyer has the capacity of providing verifiable evidence of his second-stage outside option price τ if it is available. Note that through requiring the buyer's second-stage report to be supported by verifiable evidence, the seller could costlessly elicit a truthful report at that stage due to the usual unraveling logic. A report without verifiable evidence is treated as ϕ, i.e., no outside option. That is, the verifiability of τ works as if τ becomes publicly observable. As a result, it is natural that we have the following result.[14]

Proposition 4. *If the buyer's outside option is not always available but is verifiable whenever it exists, the optimal design with private τ is equivalent to that for the public τ setting of Proposition 3.*

Proposition 4 illustrates an important conceptual point: The distinction between public and private outside options is actually identical to that between verifiable and non-verifiable outside options. Thus, the question is whether the buyer can get the competing offer in writing, rather than whether this outside option is directly observable to the seller. The verifiability of second-stage information thus lies at the core of the relevance result of the privateness of τ in optimal designs.

4.3 When outside option τ is correlated with value v

In our baseline model, the buyer's second-stage outside option price τ and his first-stage valuation are independent. There are situations in which a higher value type of buyer is less likely to have a more attractive outside option price τ. For example, a wealthier buyer tends to have higher willingness to pay and lower willingness to search for less costly outside options. In this extension, we consider the case in which v and τ are positively correlated.[15] We allow shifting support for τ. In particular, we assume $\tau \in [0, \bar{\tau}(v)]$, where $\bar{\tau}(v)$ is weakly increasing in v. We focus on the case of private τ, and show that both our approach and the optimal design can be extended to this generalized setting with positively correlated v and τ.[16]

Following the decomposition technique used by Esö and Szentes (2007), we define $s = H_v(\tau)$ as the new information revealed to the buyer at the second stage. We assume

[14]The proof is omitted to save space. It is available from the authors upon request.

[15]For negatively correlated v and τ, the applicability of our procedure in this subsection is not warranted. More conditions need to be imposed on the model primitives for the incentive compatibility of the identified mechanism.

[16]Note that with public τ, full surplus extraction follows by standard argument. We leave this to future work.

that $H_v(\cdot)$ first-order stochastically dominates $H_{v'}(\cdot)$ if $v > v'$, which captures the idea that τ and v are positively correlated. We further impose the following three regularity conditions on $F(\cdot)$ and $H_v(\cdot)$ for the sake of technical tractability.

Assumptions: (i) $\frac{\partial H_v^{-1}(s)}{\partial v} \in (0,1)$; (ii) $\frac{\partial H_v^{-1}(s)}{\partial s}[1-F(v)]$ weakly decreases with v; and (iii) $\frac{\partial H_v^{-1}(s)}{\partial v}\frac{1-F(v)}{f(v)}$ weakly decreases with v.

Assumption (i) implies that the impact of a change in v on the outside option price is not overly large. Assumption (ii) means that the marginal impact of new information s on the outside option price multiplied by $1-F(v)$ decreases in the buyer's first-stage type v. Assumption (iii) states that the marginal impact of the buyer's first-stage type v on the outside option price multiplied by $\frac{1-F(v)}{f(v)}$ decreases in v. These assumptions can be easily satisfied. For example, consider any $F(v)$ and $H_v^{-1}(s) = \alpha[1+F(v)]\beta(s)$, with $\alpha \in (0, (\max_v\{f(v)\})^{-1})$, $\beta(s) \in [0,1]$, and $\beta'(s) > 0$. One can easily verify that Assumptions (i), (ii), and (iii) hold for these examples.

The seller commits to a direct two-stage mechanism $\{(p_1(v), x_1(v)); (p_2(v,s), x_2(v,s))\}$. If the buyer has value v and reports v' at the first stage, his second-stage off-equilibrium-path deviation payoff is

$$\tilde{\pi}(v, v'; s, s') = p_2(v', s')v + (1 - p_2(v', s')) \max\{v - H_v^{-1}(s), 0\} - x_2(v', s').$$

On-equilibrium-path second-stage incentive compatibility requires that

$$\tilde{\pi}(v, v; s, s) \geq \tilde{\pi}(v, v; s, s'), \forall v, s, v'. \tag{4.1}$$

Similar to Lemma 1, second-stage incentive compatibility imposes the following semi-monotonicity condition on the second-stage allocation rule. [17]

Lemma 5. *Let $\hat{s}(v) = H_v(v)$. Second-stage incentive compatibility requires that $p_2(v,s)$ be weakly increasing in s if $s \leq \hat{s}(v)$ and $p_2(v,s) \geq p_2(v, \hat{s}(v))$ if $s > \hat{s}(v)$.*

Define the buyer's virtual value function as

$$J(v,s) = \begin{cases} H_v^{-1}(s) - \frac{\partial H_v^{-1}(s)}{\partial v} \cdot \frac{1-F(v)}{f(v)} & \text{if } s \leq \hat{s}(v), \\ \lambda(v) & \text{if } s > \hat{s}(v). \end{cases}$$

The seller's problem in the relaxed scenario with public s and semi-monotone $p_2(v,s)$ in

[17]The proof is similar to that of Lemma 1. It is available upon request.

Lemma 5 is given by

$$\max_{\{p_1(v),p_2(v,s)\}} R = \int_0^{\bar{v}} \Big\{p_1(v)\lambda(v) + (1-p_1(v))\delta E_s\Big[p_2(v,s)J(v,s) + (\lambda(v) - J(v,s))\Big]\Big\}dF(v),$$

subject to the semi-monotonicity condition in Lemma 5.

As in the baseline setting, $J(v,s)$ is not monotone in s for any given v. In particular, it has a kink point at $s = \hat{s}(v)$. Pointwise maximization will not generate an allocation rule that satisfies the semi-monotonicity requirement of Lemma 5. Following the same procedure as in the baseline setting (Section 3.2), we obtain the optimal mechanism for the case with positively correlated v and τ, which is stated in the following proposition.

Proposition 5. *Define $M(v) = \int_0^1 J(v,s)ds$. Let $\hat{v}^{**}$ be such that $M(v^{**}) = 0$. Under Assumptions (i), (ii), and (iii), when τ is the buyer's private information, the optimal selling mechanism satisfies:*

1. *if $v \geq v^*$, the buyer purchases from the seller at first-stage price $(1-\delta)v^* + \delta E_s \min\{\hat{v}^{**}, H^{-1}_{\hat{v}^{**}}(s)\}$, and the object is delivered in the first stage;*
2. *if $v \in [\hat{v}^{**}, v^*)$, the buyer pays first-stage price $\delta E_s \min\{\hat{v}^{**}, H^{-1}_{\hat{v}^{**}}(s)\}$ to the seller, and the object is delivered in the second stage;*
3. *if $v < \hat{v}^{**}$, the buyer conducts a search and never purchases from the seller.*

The optimal mechanism for the case with correlated information across stages is almost the same as that in our baseline model, except that the sale cutoff $\hat{v}^{**}$ and selling prices now reflect the correlation between v and τ.

5 Conclusion

In this paper, we study optimal two-stage selling mechanisms in an environment in which the buyer can search for an outside optional price that arrives at the second stage. We allow the buyer's second-stage information to be either public or private.

Our analysis shows that the privateness of second-stage information is relevant for the optimal design of selling mechanisms in our setting. With a private outside option price, the seller never screens the buyer over his second-stage information. The optimal selling mechanism takes the form of a menu consisting of two contracts: a first-stage direct sale with immediate delivery at a fixed first-stage price and a first-stage direct sale with second-stage delivery at a lower first-stage price. When the buyer's outside option is public information, we find that second-stage price matching is necessarily a feature of the optimal selling mechanism.

The seller sells at a fixed price immediately to the buyer if his value is above a cutoff; if the buyer's value is lower than the cutoff, he conducts a search and the seller will match the deal the buyer finds.

Our analysis generalizes to variant settings, in which the outside option might not always be available, the outside option can be verifiable, or the outside price might be correlated with the buyer's value. We find that the approach we develop in the baseline setting is still valid, and the optimal designs resemble those in the baseline setting.

The validity of our approach goes beyond the models considered in this paper. For example, consider a two-stage environment in which a buyer's value is the minimum of his (independent) information in two stages. Solving for the optimal selling mechanism using the first-order approach would encounter the same problem of implementability. Moreover, allowing the correlation between buyer value and outside net payoff in Armstrong and Zhou (2016) can also entail the same issue. Our idea of explicitly incorporating the implications of global incentive compatibility in the benchmark model with public ex post information could help resolve the issue and identify the optimal allocation rule that is implementable.

In this paper, we focus on a setting with a single buyer, an exogenous outside option, and full commitment power for the seller. Our analysis can be extended to an environment with multiple buyers but requires an ironing procedure in the second stage. However, analyzing situations with competing sellers and/or without full commitment power is more demanding, although these are interesting and important issues to investigate in the future.[18] Moreover, we only consider revenue-maximizing mechanisms in this paper. Allocative efficiency is also an important and natural goal for mechanism design, which we will leave to future work.

[18] Armstrong and Zhou (2016) provide a substantial analysis of competing sellers. The metholodogy of Garrett, Gomes, and Maestri (2019) is also useful in this regard.

Appendix

Proof of Proposition 1: Following the standard approach of Myerson (1981), the seller's problem can be formulated as

$$\max_{\{p_1(v),p_2(v,\tau)\}} R = \int_0^{\bar{v}} \Big\{p_1(v)\lambda(v) + (1-p_1(v))\delta\Big[G(v)\lambda(v) - \int_0^v \tau dG(\tau) + \int_0^v p_2(v,\tau)\tau dG(\tau) + \int_v^{\bar{v}} p_2(v,\tau)\lambda(v)dG(\tau)\Big]\Big\}dF(v),$$

subject to constraints (3.1) and (3.2). Here, we assume that the IR is binding for the lowest type.

Given value v, the optimal second-stage allocation rule $p_2^*(v,\tau)$ should maximize $\int_0^v p_2(v,\tau)\tau dG(\tau) + \int_v^{\bar{v}} p_2(v,\tau)\lambda(v)dG(\tau)$. By pointwise maximization, it is optimal to set

$$p_2^*(v,\tau) = \begin{cases} 0, & \text{if } \tau > v \text{ and } v < v^*, \\ 1, & \text{otherwise.} \end{cases}$$

With second-stage allocation rule $p_2^*(v,\tau)$, the seller's expected revenue can be written as

$$\begin{aligned} R &= \int_0^{v^*} \Big\{p_1(v)\lambda(v) + (1-p_1(v))G(v)\delta\lambda(v)\Big\}dF(v) \\ &\quad + \int_{v^*}^{\bar{v}} \Big\{p_1(v)\lambda(v) + (1-p_1(v))\delta\lambda(v)\Big\}dF(v). \end{aligned}$$

To maximize R, the seller should set the first-stage allocation rule as

$$p_1^*(v) = \begin{cases} 0, & \text{if } v < v^* \text{ ,} \\ 1, & \text{otherwise.} \end{cases}$$

The corresponding expected revenue for the seller is

$$\bar{R} = \int_0^{v^*} \delta G(v)\lambda(v)dF(v) + \int_{v^*}^{\bar{v}} \lambda(v)dF(v).$$

Set the second-stage payment rule as $x_2^*(v,\tau) = p_2^*(v,\tau)\min\{v,\tau\}$ and first-stage payment rule as $x_1^*(v) = p_1^*(v)\big[(1-\delta)v^* + \delta E_\tau \min\{v^*,\tau\}\big]$. Then the buyer's on-equilibrium-path second-stage payoff is $\tilde{\pi}(v,v,\tau) = \max\{v-\tau,0\}$, and his first-stage truthful payoff is

$$\pi(v) = \begin{cases} \delta E_\tau \max\{v-\tau,0\}, & \text{if } v < v^*, \\ v - (1-\delta)v^* - \delta E_\tau \min\{v^*,\tau\}, & \text{if } v \geq v^*. \end{cases}$$

Clearly, individual rationality holds in both stages. For a type $v(\geq v^*)$ buyer, any deviation to $v'(\geq v^*)$ does not affect his payoff. Any deviation $v'(< v^*)$ would lead to a payoff weakly lower than $\delta E_\tau \max\{v-\tau, 0\}$. Thus, it is not profitable for him to misreport. For a type $v(< v^*)$ buyer, any deviation v' would lead to a payoff weakly lower than $\delta E_\tau \max\{v-\tau, 0\}$. Thus, the first-stage IC holds.

Note that the IR is binding for the lowest type. Thus, maximum revenue from pointwise optimization, $\bar{R}$, is achieved.

Proof of Lemma 1: When the buyer finds outside option price τ and reports τ' after a truthful first stage, his second-stage expected payoff can be rewritten as follows:

$$\tilde{\pi}(v, \tau, \tau') = \tilde{\pi}(v, \tau') + (1 - p_2(v, \tau'))\Big(\min\{v, \tau'\} - \min\{v, \tau\}\Big).$$

The second-stage incentive compatibility condition is equivalent to

$$\tilde{\pi}(v, \tau) \geq \tilde{\pi}(v, \tau') + (1 - p_2(v, \tau'))\Big(\min\{v, \tau'\} - \min\{v, \tau\}\Big). \tag{5.1}$$

Switching τ and τ' in (5.1) gives

$$\tilde{\pi}(v, \tau') \geq \tilde{\pi}(v, \tau) + (1 - p_2(v, \tau))\Big(\min\{v, \tau\} - \min\{v, \tau'\}\Big).$$

Let $\Delta(v, \tau, \tau') = \min\{v, \tau'\} - \min\{v, \tau\}$. Then second-stage IC implies that

$$(1 - p_2(v, \tau))\Delta(v, \tau, \tau') \geq \tilde{\pi}(v, \tau) - \tilde{\pi}(v, \tau') \geq (1 - p_2(v, \tau'))\Delta(v, \tau, \tau').$$

Without any loss, we assume that $\tau' < \tau$. When $0 \leq \tau' < v$, we have $\Delta(v, \tau, \tau') < 0$, which implies that $p_2(v, \tau') \leq p_2(v, \tau)$. When $\tau' \geq v$, we have $\Delta(v, \tau, \tau') = 0$ and $\tilde{\pi}(v, \tau) = \tilde{\pi}(v, \tau')$, i.e., the buyer's utility is independent of τ.

Proof of Lemma 2: Assume that the true value is v and the outside option price is τ. Let σ be the optimal second-stage report after a first-stage deviation v'. The optimality of $\sigma(v, v'; \tau)$ implies that for any τ',

$$\Big(p_2(v', \sigma) - p_2(v', \tau')\Big) \min\{v, \tau\} \geq x_2(v', \sigma) - x_2(v', \tau').$$

If the first-stage true value is v' and the second-stage true outside option price is σ, then second-stage IC implies, for any τ',

$$\Big(p_2(v', \sigma) - p_2(v', \tau')\Big) \min\{v', \sigma\} \geq x_2(v', \sigma) - x_2(v', \tau').$$

If σ exists such that $\min\{v', \sigma\} = \min\{v, \tau\}$, there is full lie correction. This is feasible only when $v' \geq \min\{v, \tau\}$, and we would have optimal $\sigma = \min\{v, \tau\}$.

If $v' < \min\{v, \tau\}$, full lie correction is impossible. Let $\sigma(v, v'; \tau) \in \arg\max_s p_2(v', s)$. We have

$$\Big(p_2(v', \sigma) - p_2(v', \tau')\Big) \min\{v, \tau\} \geq \Big(p_2(v', \sigma) - p_2(v', \tau')\Big) \min\{v', \sigma\} \geq x_2(v', \sigma) - x_2(v', \tau').$$

Proof of Proposition 2: We first show that the mechanism in Proposition 2 is feasible.

First-stage IR. Under the mechanism in Proposition 2, the buyer's first-stage expected payoff is

$$\begin{aligned}
\pi(v, v) =& v p_1^{**}(v) + (1 - p_1^{**}(v))\delta E_\tau \tilde{\pi}(v, v; \tau, \tau) - x_1^{**}(v) \\
=& \begin{cases} v - (1-\delta)v^* - \delta E_\tau \min\{v^{**}, \tau\}, & \text{if } v \geq v^*, \\ \delta v - \delta E_\tau \min\{v^{**}, \tau\}, & \text{if } v \in [v^{**}, v^*), \\ \delta E_\tau \max\{v - \tau, 0\}, & \text{if } v < v^{**}. \end{cases}
\end{aligned}$$

Since $0 < \delta \leq 1$, it follows that for $v \geq v^* (> v^{**})$,

$$v - (1-\delta)v^* - \delta E_\tau \min\{v^{**}, \tau\} \geq \delta\Big(v - E_\tau \min\{v, \tau\}\Big) = \delta E_\tau \max\{v - \tau, 0\}.$$

For $v \in [v^{**}, v^*)$,

$$\delta v - \delta E_\tau \min\{v^{**}, \tau\} \geq \delta\Big(v - E_\tau \min\{v, \tau\}\Big) = \delta E_\tau \max\{v - \tau, 0\}.$$

Thus, $\pi(v, v) \geq \delta E_\tau \max\{v - \tau, 0\}$ for all v; that is, the first-stage IR holds.

First-stage IC. Note that the second-stage allocation rule and payment rule are independent of the second-stage report. Without loss of generality, we can assume that the buyer reports truthfully even off the equilibrium path. The first-stage deviation payoff can then be written as

$$\begin{aligned}
\pi(v, v') =& v p_1^{**}(v') + (1 - p_1^{**}(v'))\delta E_\tau \tilde{\pi}(v, v'; \tau, \tau) - x_1^{**}(v') \\
=& \begin{cases} v - (1-\delta)v^* - \delta E_\tau \min\{v^{**}, \tau\}, & \text{if } v' \geq v^*, \\ \delta v - \delta E_\tau \min\{v^{**}, \tau\}, & \text{if } v \in [v^{**}, v^*), \\ \delta E_\tau \max\{v - \tau, 0\}, & \text{if } v' < v^*. \end{cases}
\end{aligned}$$

When $v \geq v^*$, the buyer's payoff is $\pi(v, v) = v - (1-\delta)v^* - \delta E_\tau \min\{v^{**}, \tau\}$. By the results established above, we have $\pi(v, v) \geq \pi(v, v')$ for any deviation v'.

When $v \in [v^{**}, v^*)$, we have that the buyer's payoff $\pi(v, v) = \delta v - \delta E_\tau \min\{v^{**}, \tau\} \geq \max\Big\{v - (1-\delta)v^* - \delta E_\tau \min\{v^{**}, \tau\}, \delta E_\tau \max\{v - \tau, 0\}\Big\}$. Thus, he cannot benefit from any deviation.

When $v < v^{**}$, we have that the buyer's payoff $\pi(v, v) = \delta E_\tau \max\{v - \tau, 0\} \geq \delta v - \delta E_\tau \min\{v^{**}, \tau\} \geq v - (1-\delta)v^* - \delta E_\tau \min\{v^{**}, \tau\}$. Thus, there is no profitable deviation.

Second-stage IC and IR. Since $p_2^{**}(v, \tau)$ is independent of τ, the second-stage IC holds automatically. Since no payment is required in the second stage, the second-stage IR holds.

Achievability of the revenue bound R^*. Since the allocation rule in the mechanism proposed in Proposition 2 is the same as that in the optimal mechanism for the relaxed problem, the total trade surplus must be the same across the two mechanisms.

R^* is obtained by assuming that the IR is binding for the lowest type in the relaxed problem. In the optimal mechanism for the relaxed problem, the envelope condition is

$$\pi'(v) = p_1^{**}(v) + (1 - p_1^{**}(v))\delta\Big[G(v) + \int_v^{\bar{v}} p_2^{**}(v, \tau) dG(\tau)\Big].$$

For $v \leq v^{**}$, the buyer's first-stage payoff is

$$\pi(v) = \int_0^v \Big\{0 + (1-0)\delta\Big[G(u) + \int_u^{\bar{v}} 0 dG(\tau)\Big]\Big\} du = \delta E_\tau \max\{v - \tau, 0\}.$$

For $v \in (v^{**}, v^*]$, the buyer's first-stage payoff is

$$\begin{aligned}\pi(v) =& \delta E_\tau \max\{v^{**} - \tau, 0\} + \int_{v^{**}}^v \Big\{0 + (1-0)\delta[G(u) + G(\bar{v}) - G(u)]\Big\} du \\ =& \delta(v - E_\tau \min\{v^{**}, \tau\}).\end{aligned}$$

For $v > v^*$, the first-stage payoff is

$$\pi(v) = \delta(v^* - E_\tau \min\{v^{**}, \tau\}) + \int_{v^*}^v 1 du = v - (1-\delta)v^* - \delta E_\tau \min\{v^{**}, \tau\}.$$

We thus have that the buyer's expected payoff in the relaxed problem is the same as that in the mechanism in Proposition 2. Therefore, the mechanism in Proposition 2 achieves the revenue upper bound R^*.

Proof of Lemma 4: The second-stage IC is standard, since the allocation rule is monotone in τ. Therefore, we focus on the first-stage IC. Note that if $\frac{\partial \pi(v,v')}{\partial v'}|_{v'=v} = 0$ and $\frac{\partial^2 \pi(v,v')}{\partial v \partial v'} \geq 0$, the increasing property of $\frac{\partial \pi(v,v')}{\partial v'}$ in v implies $\frac{\partial \pi(v,v')}{\partial v'}|_{v'<v} > 0$ and $\frac{\partial \pi(v,v')}{\partial v'}|_{v'>v} < 0$. That is, the first-stage payoff $\pi(v, v')$ is maximized by the report $v' = v$, which is equivalent to the first-stage IC.

Claim 1. *Under the conditions in Lemma 4,* $\frac{\partial \pi(v,v')}{\partial v'}|_{v'=v} = 0$.

Proof. The partial derivative of the first-stage deviation payoff with respect to v' is

$$\frac{\partial \pi(v,v')}{\partial v} = p_1(v') + (1-p_1(v'))\delta E_\tau \left[\begin{array}{c} \tilde{\pi}_v(v,v',\tau,\sigma(v,v';\tau)) \\ +\tilde{\pi}_\sigma(v,v',\tau,\sigma(v,v';\tau))\sigma_v(v,v';\tau) \end{array} \right],$$

in which $\sigma(v,v';\tau)$ is the optimal second-stage report to first-stage deviation v'. By Lemma 2, condition (3) in Lemma 4 implies that $\sigma(v,v';\tau) = v$ if $v < \min\{v',\tau\}$; and $\sigma(v,v';\tau) = \min\{v',\tau\}$ if $v \geq \min\{v',\tau\}$.

By the optimality of σ, we have $\tilde{\pi}_\sigma(v,v',\tau,\sigma(v,v';\tau))\sigma_v(v,v';\tau) = 0$, since either $\tilde{\pi}_\sigma(v,v',\tau,\sigma(v,v';\tau)) = 0$ (when interior optimal response $\sigma(v,v';\tau) = v$ for $v < \min\{v',\tau\}$) or $\sigma_v(v,v';\tau) = 0$ (when $\sigma(v,v';\tau)$ stays at the bound $\min\{v',\tau\}$ for $v \geq \min\{v',\tau\}$). Therefore,

$$\frac{\partial \pi(v,v')}{\partial v} = p_1(v') + \delta(1-p_1(v'))\Big\{G(v) + \int_v^{\bar{v}} p_2(v',\sigma(v,v';\tau))dG(\tau)\Big\}. \tag{5.2}$$

When $v' = v$ and $\tau \geq v$, we have $\sigma(v,v';\tau) = v$. It follows that

$$\frac{d\pi(v,v)}{dv} = \frac{\partial \pi(v,v')}{\partial v}|_{v'=v} = p_1(v) + \delta(1-p_1(v))\Big[G(v) + p_2(v,v)(1-G(v))\Big].$$

Then the buyer's first-stage payoff $\pi(v,v) = p_1(v)v + \delta(1-p_1(v))E_\tau\tilde{\pi}(v,v;\tau,\tau) - x_1(v)$ can also be written as

$$\pi(v,v) = \pi(0,0) + \int_0^v \{p_1(u) + \delta(1-p_1(u))\,[G(u) + p_2(u,u)(1-G(u))]\}\,du.$$

Set $\pi(0,0) = 0$. We have payment rule

$$\begin{aligned} x_1(v) =& p_1(v)v + \delta(1-p_1(v))E_\tau\tilde{\pi}(v,v;\tau,\tau) \\ &- \int_0^v \{p_1(u) + \delta(1-p_1(u))\,[G(u) + p_2(u,u)(1-G(u))]\}\,du. \end{aligned}$$

Then the buyer's deviation payoff $\pi(v,v') = p_1(v')v + \delta(1-p_1(v'))E_\tau\tilde{\pi}(v,v';\tau,\sigma(v,v';\tau)) - x_1(v')$ can be rewritten as

$$\begin{aligned} \pi(v,v') =& p_1(v')[v-v'] + \delta(1-p_1(v'))[E_\tau\tilde{\pi}(v,v';\tau,\sigma(v,v';\tau)) - E_\tau\tilde{\pi}(v',v';\tau,\tau)] \\ &+ \int_0^{v'} \{p_1(u) + \delta(1-p_1(u))\,[G(u) + p_2(u,u)(1-G(u))]\}\,du. \end{aligned}$$

Taking the partial derivative of $\pi(v,v')$ with respect to v', we obtain

$$
\begin{aligned}
\frac{\partial \pi(v,v')}{\partial v'} =& \frac{dp_1(v')}{dv'}[v - v' - \delta E_\tau \tilde{\pi}(v,v';\tau,\sigma(v,v';\tau)) + \delta E_\tau \tilde{\pi}(v',v';\tau,\tau)] \\
& - p_1(v') + \delta(1-p_1(v'))E_\tau[\frac{\partial \tilde{\pi}(v,v';\tau,\sigma(v,v';\tau))}{\partial v'} - \frac{d\tilde{\pi}(v',v';\tau,\tau)}{dv'}] \\
& + p_1(v') + \delta(1-p_1(v'))(G(v') + p_2(v',v')(1-G(v'))).
\end{aligned}
$$

Since $\frac{d\tilde{\pi}(v',v';\tau,\tau)}{dv'} = \frac{\partial\tilde{\pi}(v,v';\tau,\tau)}{\partial v}|_{v=v'} + \frac{\partial\tilde{\pi}(v,v';\tau,\tau)}{\partial v'}|_{v=v'}$, we have

$$
\begin{aligned}
\frac{\partial \pi(v,v')}{\partial v'}|_{v'=v} =& \delta(1-p_1(v))\Big\{-E_\tau[\frac{\partial\tilde{\pi}(v,v';\tau,\tau)}{\partial v}|_{v=v'}]|_{v'=v} + G(v) + p_2(v,v)(1-G(v))\Big\} \\
=& 0.
\end{aligned}
$$

□

Next, we show that $\frac{\partial^2\pi(v,v')}{\partial v \partial v'} \geq 0$. Differentiating both sides of equation (5.2) with respect to v', we have

$$
\begin{aligned}
\frac{\partial^2 \pi(v,v')}{\partial v' \partial v} =& \frac{dp_1(v')}{dv'}\Big\{1 - \delta\Big[G(v) + \int_v^{\bar{v}} p_2(v',\sigma(v,v';\tau))dG(\tau)\Big]\delta\Big\} \\
& + \delta(1-p_1(v'))\int_v^{\bar{v}}\Big[\frac{\partial p_2(v',\sigma(v,v';\tau))}{\partial v'} + \frac{\partial p_2(v',\sigma(v,v';\tau))}{\partial \sigma}\sigma_{v'}(v,v';\tau)\Big]dG(\tau).
\end{aligned}
$$

By condition (1), the left side of the first line is nonnegative. Since $\sigma(v,v';\tau) = \min\{v',\tau,v\}$, it follows that $\sigma_{v'}(v,v';\tau) \geq 0$. Under conditions (2) and (3), the second line is also nonnegative. We thus have $\frac{\partial^2\pi(v,v')}{\partial v' \partial v} \geq 0$.

In summary, under conditions (1), (2), and (3), the incentive compatibility condition in the first stage holds.

Proof of Proposition 5: We first consider the relaxed problem with public s and semi-monotone $p_2(v,s)$. By the way that virtual value $J(v,s)$ is defined, it has following properties.

Lemma 6. *With Assumptions (i) and (ii), $J(v,s)$ increases in s to the left of $\hat{s}(v) = H_v(v)$; it jumps down to $\lambda(v)$ and stays flat to the right of $\hat{s} = H_v(v)$.*

Proof. By Assumption (ii), we have

$$
\frac{d}{dv}\Big[\frac{\partial H_v^{-1}(s)}{\partial s}(1-F(v))\Big] = \frac{\partial^2 H_v^{-1}(s)}{\partial s \partial v}(1-F(v)) - f(v)\frac{\partial H_v^{-1}(s)}{\partial s} \leq 0,
$$

which is equivalent to

$$
\frac{\partial H_v^{-1}(s)}{\partial s} - \frac{\partial^2 H_v^{-1}(s)}{\partial s \partial v} \cdot \frac{1-F(v)}{f(v)} \geq 0.
$$

Thus, when $s \leq \hat{s}(v)$, we have

$$\frac{\partial J(v,s)}{\partial s} = \frac{\partial H_v^{-1}(s)}{\partial s} - \frac{\partial^2 H_v^{-1}(s)}{\partial s \partial v} \cdot \frac{1-F(v)}{f(v)} \geq 0.$$

Thus, $J(v,s)$ is monotonically increasing in s when $s \leq \hat{s}(v)$.

On the other hand, by Assumption (i), we have

$$\begin{aligned} J(v, H_v(v)) =& \Big[H_v^{-1}(s) - \frac{\partial H_v^{-1}(s)}{\partial v} \cdot \frac{1-F(v)}{f(v)}\Big]|_{s=H_v(v)} \\ \geq& H_v^{-1}(H_v(v)) - \frac{1-F(v)}{f(v)} = \lambda(v). \end{aligned}$$

That is, $J(v,s)$ jumps down to $\lambda(v)$ when s surpasses $\hat{s}(v)$. $\square$

By Lemma 6, pointwise maximization cannot be applied directly, since the resulting allocation rule does not satisfy the semi-monotonicity condition in Lemma 5. With an argument similar to that in our baseline model, the optimal second-stage allocation $p_2^{**}(v,s) = 1$ if and only if $M(v) = \int_0^1 J(v,s)ds \geq 0$.

Claim 2. *With Assumptions (i) and (iii), both $\hat{s}(v)$ and $M(v)$ increase in v.*

Proof. By the definition of $\hat{s}$, we have $H_v^{-1}(\hat{s}(v)) = v$. Totally differentiating both sides with v, we have

$$\frac{\partial H_v^{-1}(\hat{s}(v))}{\partial v} + \frac{\partial H_v^{-1}(\hat{s}(v))}{\partial s}\hat{s}'(v) = 1.$$

By Assumption (i), we have $\frac{\partial H_v^{-1}(\hat{s}(v))}{\partial v} < 1$. Since $\frac{\partial H_v^{-1}(\hat{s}(v))}{\partial s} \geq 0$, we have $\hat{s}'(v) \geq 0$.

Now we investigate the monotonicity of $M(v)$ in v. When $t \leq \hat{s}(v)$, we have

$$\frac{\partial J(v,t)}{\partial v} = \frac{\partial H_v^{-1}(t)}{\partial v} - \Big(\frac{\partial \frac{1-F(v)}{f(v)}}{\partial v} \cdot \frac{\partial H_v^{-1}(t)}{\partial v} + \frac{1-F(v)}{f(v)} \cdot \frac{\partial^2 H_v^{-1}(t)}{\partial v \partial v}\Big)$$

By Assumption (iii), the part inside the bracket is negative. Since $\frac{\partial H_v^{-1}(t)}{\partial v} \geq 0$, we must have $J(v,s)$ weakly increasing in v when $t \leq \hat{s}(v)$. When $t \geq \hat{s}(v)$, we have $J(v,t) = \lambda(v)$, which is also increasing in v. Combining the result that $\hat{s}(v)$ increases in v, $M(v) = \int_0^1 J(v,s)ds$ is increasing in v. $\square$

Notice that

$$M(v) = \int_0^{\hat{s}(v)} \left\{H_v^{-1}(t) - \frac{\partial H_v^{-1}(t)}{\partial v} \cdot \frac{1-F(v)}{f(v)}\right\} dt + \lambda(v)\big(1 - \hat{s}(v)\big).$$

We have $M(0) = \lambda(0) < 0$. Note $H_v^{-1}(0) = 0$ and $\frac{\partial H_v^{-1}(0)}{\partial v} = 0$. We have $H_{v^*}^{-1}(t) - \frac{\partial H_{v=v^*}^{-1}(t)}{\partial v} \cdot \frac{1-F(v^*)}{f(v^*)} > H_{v^*}^{-1}(0) - \frac{\partial H_{v=v^*}^{-1}(0)}{\partial v} \cdot \frac{1-F(v^*)}{f(v^*)} = 0$. Thus, $M(v^*) = \int_0^{\hat{s}(v^*)} \left[H_{v^*}^{-1}(t) - \frac{\partial H_{v=v^*}^{-1}(t)}{\partial v} \cdot \frac{1-F(v^*)}{f(v^*)}\right] dt > 0$. Because of the continuity and monotonicity of $M(v)$ with respect to v, there exists a unique $\hat{v}^{**} \in (0, v^*)$ such that $M(\hat{v}^{**}) = 0$. Thus, the optimal second-stage allocation $p_2^{**}(v,s) = 1$ iff $v \geq \hat{v}^{**}$; otherwise, $p_2^{**}(v,s) = 0$.

Substituting p_2^{**} back into the revenue expression R, the seller's revenue can be written as

$$
\begin{aligned}
R = & \int_0^{\hat{v}^{**}} \Big\{p_1(v)\lambda(v) + (1-p_1(v))\delta E_s\Big[\lambda(v) - J(v,s)\Big]\Big\}dF(v) \\
& + \int_{\hat{v}^{**}}^{\bar{v}} \Big\{p_1(v)\lambda(v) + (1-p_1(v))\delta\lambda(v)\Big\}dF(v) \\
& = \int_0^{\hat{v}^{**}} \Big\{p_1(v)\Big[(1-\delta)\lambda(v) + \delta M_v(1)\Big] + \delta\Big(\lambda(v) - M_v(1)\Big)\Big\}dF(v) \\
& + \int_{\hat{v}^{**}}^{v^*} \Big\{p_1(v)\lambda(v) + (1-p_1(v))\delta\lambda(v)\Big\}dF(v) + \int_{v^*}^{\bar{v}} \Big\{p_1(v)\lambda(v) + (1-p_1(v))\delta\lambda(v)\Big\}dF(v).
\end{aligned}
$$

When $v \leq \hat{v}^{**}$, $(1-\delta)\lambda(v) + \delta M_v(1) \leq 0$; when $v \in (\hat{v}^{**}, v^*]$, $\lambda(v) \leq \delta\lambda(v)$; when $v \geq v^*$, $\lambda(v) \geq \delta\lambda(v)$. Thus, the optimal first-stage allocation can be identified as

$$
p_1^{**}(v) = \begin{cases} 1, & \text{if } v \geq v^*, \\ 0, & \text{if } v < v^*. \end{cases} \tag{5.3}
$$

Lemma 7. *In the relaxed problem with public s and semi-monotonic second-stage allocation rule, the maximum revenue achieved by p_1^{**} and p_2^{**} is*

$$
\begin{aligned}
\bar{R} = & \delta \int_0^{\hat{v}^{**}} \hat{s}(v)\lambda(v) - \int_0^{\hat{s}(v)} \left\{H_v^{-1}(s) - \frac{\partial H_v^{-1}(s)}{\partial v} \cdot \frac{1-F(v)}{f(v)}\right\} dsdF(v) \\
& + \int_{\hat{v}^{**}}^{v^*} \delta\lambda(v)dF(v) + \int_{v^*}^{\bar{v}} \lambda(v)dF(v).
\end{aligned}
$$

Similar to the proof of Proposition 2, it's easy to check that the mechanism in Proposition 5, which has allocation p_1^{**} and p_2^{**}, is implementable (i.e., satisfies the IC and IR conditions in both stages) in the scenario with private second-stage information and achieves the revenue upper bound in Lemma 7.

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