# Move orders in Contests: Equilibria and Winning Chances* 

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#### Abstract

This paper studies general two-player sequential-move competitions, accommodating a full spectrum of Tullock contest technology and contestants' asymmetry. We provide necessary and sufficient conditions for a preemptive equilibrium to prevail in both strong-lead and weak-lead contests, and discover a characteristic equation to pin down the players' effort ratio (which fully determines their winning chances) and their effort levels when an interior equilibrium prevails. We find that while the strong player always has a higher winning chance when moving first, simultaneous moves sometimes maximize the weak player's winning odds. We further endogenize the move orders through winning-odd-maximizing coaches' independent choices.


JEL classification: D72, D74, D81.
Keywords: Tullock contests, simultaneous contests, strong-lead/weak-lead sequential contests, interior equilibria, preemptive equilibria, winning chances.

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## 1 Introduction

Many social and economic rent-seeking competitions can be viewed as contests, in which participating players compete for prizes by expending costly and non-refundable effort. Players may expend effort in different orders by nature or by the design of an independent organizer of the contests. In a simultaneous contest, contestants exert effort simultaneously, without knowing the effort levels of others. In a sequential contest, the effort exerted by the leading contestant is sunk and perfectly observable by the follower. Practices of sequential contests are widely instituted in the real world, such as certain procurement processes by large corporations, R\&D and marketing competitions.

This study focuses on a class of two-player Tullock sequential-move contests, ${ }^{1}$ in which two risk-neutral contestants can exhibit any degree of asymmetry in their abilities, and the accuracy of the Tullock contest can vary at any level. General sequential contests are not thoroughly understood in the literature, despite being prevalent in many active economic practices in the real world. For instance, in the global aviation industry, the request for proposal (RFP) process is commonly practiced in aircraft procurement by commercial airline companies. These companies might solicit proposals from a limited number of pre-vetted suppliers, later extending the opportunity to a broader group. This sequential process implies that these supply firms expend efforts sequentially.

Contestants are generally heterogeneous in strength or ability. In our sequential contest model, the asymmetry between contestants is measured by the "ability ratio," which is defined as the leader-follower ratio of their marginal costs of exerting effort. To fully capture this asymmetry, we allow the ability ratio to vary from zero to positive infinity. Consequently, assigning the first mover in a sequential contest (whether it is the strong or weak player) results in different levels of equilibrium effort. Additionally, unlike simultaneous contests where each player always exerts strictly positive effort in equilibrium, ${ }^{2}$ sequential contests introduce the possibility of a preemptive equilibrium, where the leader exerts a sufficiently high level of effort to prompt the follower to respond with zero effort.

For the same set of contestants and their effort entries, variations in the contest's accuracy level can lead to significant differences in their winning chances and equilibrium effort levels.

[^1]This variability can stem from diverse economic environments or competition rules. In the Tullock model (Tullock, 1980) adopted in this paper, the winning technology is given by the Tullock contest success function (CSF), which includes the accuracy level of the contest, denoted by $r$. Contests in practice can exhibit varying accuracy levels. ${ }^{3}$ The breakthrough of this paper is that, for any given asymmetry level of the contestants, we allow the contest's accuracy level $r$ to take any value from zero to positive infinity. ${ }^{4}$

We categorize sequential contests based on the move order of two asymmetric contestants. A contest in which the strong player moves first is termed a strong-lead sequential contest, while the contest in which the weak player is the first mover is referred to as a weak-lead sequential contest. In the context of aircraft procurement by commercial airline companies, the suppliers in the first round of competition are well-known and established aircraft manufacturers and leasing companies, illustrating an example of strong-lead contests. Conversely, in the marketing competition between commercial household plasma televisions and LCD televisions, plasma products face disadvantages such as high prices and poor durability. As a result, its manufacturer, Panasonic, is a weak leader in this sequential contest.

Our analysis reveals that in a general Tullock contest model, there are two crucial factors for determining the equilibrium solution: the ratio of the two contestants' marginal costs and the accuracy level of the contest. In the sequential contest model examined in this paper, the leader-follower marginal cost ratio is denoted by $c$, which measures the asymmetry level between the contestants. In the contest model examined in this paper, considering an arbitrary marginal cost ratio $c$, where $c \in(0, \infty)$, we allow the contest's accuracy level $r$ to vary across all feasible values, $r \in(0, \infty)$, to characterize the equilibria.

We contribute to the literature by completely characterizing the equilibrium set of a twoplayer sequential Tullock contest. Our work represents the final piece of the puzzle in the equilibrium analysis of generalized two-player Tullock contests with complete information. We find that given the players' marginal cost ratio $c$ and the contest's accuracy level $r$, all equilibria are in the form of pure strategy, and an equilibrium is either interior or preemptive.

[^2]In solving for the contestants' equilibrium effort choices, we have identified interior and preemptive solutions, which are the candidates for equilibria. For any given pair of $c$ and $r$, at least one of the two solutions (interior or preemptive) exists, but both may coexist in some cases. In instances where both solutions coexist, the equilibrium is determined as either interior or preemptive, depending on which type of solution yields the higher expected payoff for the leader.

In an interior equilibrium, both players are active, meaning that both players' effort levels are strictly positive. In a preemptive equilibrium, the leader exerts effort so high that the follower would rather choose zero effort. Solving for an interior solution as an equilibrium candidate becomes technically challenging when allowing a general setting of $r \in(0, \infty)$ and $c \in(0, \infty)$, as there is generally no analytical solution. To address this, we provide a characteristic equation for the players' effort ratio, enabling us to fully characterize an interior solution whenever it exists. We demonstrate that when an interior solution exists, the players' effort ratio can be uniquely determined by this characteristic equation. Once this effort ratio is obtained, the contestants' effort levels, winning probabilities, and expected payoffs can be derived straightforwardly.

Specifically, we show that when $r<1$, contests are sufficiently noisy and there exist only interior equilibria and no preemptive equilibria in both weak-lead and strong-lead sequential contests. ${ }^{5}$ Intuitively, there is too much randomness involved in contests with $r<1$, implying that no matter how much effort the leader exerts in stage 1 , the follower always has an incentive to exert some effort in stage 2 . This rules out the possibility of preemptive equilibria. When $r \geq 1$, it is often the case that both interior and preemptive solutions exist. In equilibrium, the leader selects the solution that yields a larger expected payoff.

The equilibrium characterization of sequential contests with $r \geq 1$ is presented as follows. For strong-lead sequential contests, there is a single threshold of $r$ (weakly greater than one) such that the equilibria are interior for $r$ below it and preemptive for $r$ above it. For weaklead sequential contests, the equilibria are more sensitive to the contestants' asymmetry level: when the players' asymmetry level is sufficiently low, there is a bounded interval of $r$, such that the equilibria are preemptive for $r$ in this interval and interior for $r$ outside this interval; this "preemptive" interval shrinks until it disappears, as the asymmetry becomes

[^3]increasingly severe to a certain level, and there are only interior equilibria. Intuitively, in weak-lead sequential contests, it will be too costly for the weak leader to preempt the strong follower when the leader is sufficiently weaker than the follower. This explains why the preemptive interval disappears when the players' asymmetry level is sufficiently high (i.e. when $c$ is sufficiently large).

If we consider the process where $c$ increases from zero to infinity, i.e., when the leader's relative ability varies from super strong to very weak, the regions of $r$ that support preemptive equilibria form a collection of shrinking, nested intervals, reducing from the interval $[1, \infty)$ to a smaller bounded interval and then to an empty set.

The findings from our equilibrium analysis establish a foundation for further exploration into sequential contests. As an immediate application, we investigate scenarios where each contestant is supported by a coach, and these coach-contestant pairs compete for a prize. Contestants aim to maximize their expected payoffs in the competition, while coaches prioritize the winning chances of their supervised contestants. ${ }^{6}$ Our analysis yields the following results. To maximize the strong player's winning chance, the strong-lead sequential contest is optimal among the three contest formats (strong-lead sequential, weak-lead sequential, and simultaneous contests). Conversely, to maximize the weak player's winning chance, the weak-lead sequential contest is optimal only if the contest's accuracy level falls within the preemptive interval and thus the weak player preempts the strong follower in equilibrium; ${ }^{7}$ otherwise, a simultaneous contest is optimal, as it offers the weak player a higher winning chance than either a strong-lead or weak-lead sequential contest.

Moreover, we analyze a two-stage contest model, where the move order of the contestants (in stage 2) is determined by the two coaches' choices (in stage 1): There is a sequential contest if the two coaches choose different actions (between the two actions: Lead and Follow), and a simultaneous contest if they choose the same action. We show that there is either a mixed-strategy equilibrium in which two contestants randomize between the two actions with respective probabilities, allowing all three contest formats to occur, or a pure-

[^4]strategy equilibrium in which both players choose Lead, resulting in only the simultaneous contest format. ${ }^{8}$ As a result, every possible move order (strong-lead, weak-lead, or simultaneous) can be observed as an equilibrium outcome based on the endogenous choices of the winning-odd-maximizing coaches.

We would like to make two remarks about the main results obtained in this paper. First, our findings contribute novelty to the literature, as existing research primarily focuses on sequential Tullock contests with $r=1$. For instance, prior literature with $r=1$ suggests that the leader has an incentive to preempt the follower only when the leader is significantly stronger than the follower, implying that a weak leader never preempts the strong follower in equilibrium. In contrast, we show that the weak leader will preempt the strong follower when the contest's accuracy level falls within a preemptive interval.

Second, while some results of this paper are intuitively understandable, ${ }^{9}$ others are indeed surprising. For instance, a surprising result we find is that: a weak player's winning probability in a weak-lead sequential contest is smaller than that in a corresponding simultaneous contest in interior equilibria. Intuitively, compared to a simultaneous contest, in a weak-lead sequential contest, it is in the weak leader's best interest to play less aggressively to mitigate the strong follower's response, whenever the equilibrium is interior.

The rest of the paper is organized as follows. Section 2 illustrates the connections of this paper to the related literature. Section 3 sets up an original model with two players $A$ (strong) and $B$ (weak), as well as a general sequential contest model with players $L$ (leader) and $F$ (follower). Section 4 provides a complete equilibrium analysis in the sequential contest model. Section 5 analyzes the players' winning chances in scenarios with different move orders of the contestants in a coach-contestant model. ${ }^{10}$ Finally, Section 6 presents the concluding remarks. The technical proofs are relegated to the appendix.

[^5]
## 2 Related literature

Tullock contests were introduced by Tullock (1967), and the standard framework was also proposed by Tullock (1980) to model a class of rent-seeking competitions. ${ }^{11}$ For simultaneous Tullock contests with two asymmetric players, Nti (1999), Wang (2010), and Alcalde and Dahm (2010) conduct a complete equilibrium analysis for a full range of $r$, i.e., $r \in(0, \infty)$ : There exists a pure-strategy equilibrium when $r$ lies in the low range, a mixed-strategy equilibrium when $r$ lies in the middle range, and an all-pay auction equilibrium when $r$ lies in the high range. ${ }^{12}$ Ewerhart (2017a) and Feng and Lu (2017) show the uniqueness of the equilibrium of Wang (2010) using different approaches. Ewerhart (2017b) shows that the all-pay auction equilibrium of Alcalde and Dahm (2010) is unique in the sense that any equilibrium is both payoff equivalent and revenue equivalent to the unique equilibrium of the corresponding all-pay auction.

Dixit (1987) studies contests using a more general CSF, and finds that the strong (resp. weak) leader exerts more (resp. less) effort in a sequential contest than in a simultaneous contest. ${ }^{13}$ Linster (1993) compares sequential and simultaneous contests in a Tullock model with $r=1$. He shows that, depending on the level of asymmetry between players, there are two types of equilibria: interior and preemptive. In an interior equilibrium, both players exert strictly positive effort, while in a preemptive equilibrium, the leader exerts effort high enough to make the follower exert zero effort. Furthermore, a strong-lead (resp. weaklead) sequential contest yields larger (resp. smaller) total effort than a simultaneous contest. In a contest model with $r=1$ and two players who are ex-ante identical (but can be heterogeneous ex-post), Morgan (2003) finds that sequential contests are ex-ante Pareto superior to simultaneous contests for total effort maximization. ${ }^{14}$ Serena (2017) shows that total effort is greater in sequential contests only when contestants are sufficiently symmetric, by correcting an error in Morgan (2003)..$^{15}$ A brief comparison between the results obtained

[^6]Table 1: Comparison: Existing literature v.s. This study

|  | Existing literature | This study |
| :---: | :---: | :---: |
| Scope of the Tullock CSF | CSF with $r=1$ | CSF with $r \in(0, \infty)$ |
| In a two-player sequential Tullock contest, an equilibrium is either interior or preemptive. | Preemptive equilibria are possible in strong-lead contests when the leader is sufficiently stronger (than the follower), but never possible in weak-lead contests. | Preemptive equilibria are not possible for $r<1$, but they are possible for $r>$ 1 in both strong-lead and weak-lead contests, as long as the leader is not excessively weaker. |

in this paper and the existing literature is presented in Table 1.
Kahana and Klunover (2018) and Hinnosaar (2024) develop a technique to characterize the subgame perfect equilibrium in a sequential Tullock contest with $r=1$ and $n$ symmetric contestants. ${ }^{16}$ Hinnosaar (2024) further demonstrates that having information about other players' effort levels strictly increases the total effort. Thus, total effort is maximized when $n$ symmetric players exert effort sequentially in $n$ stages. Xu et al. (2020) compare simultaneous and sequential Tullock contests with three players and $r=1$.

Our paper differs from the above studies in two main aspects. Firstly, we consider sequential contests with a full range of $r \in(0, \infty)$, rather than being restricted to $r=1$ or $r<1 .{ }^{17}$ Secondly, we examine a Tullock contest model with a full range of asymmetry level between players, meaning that they can be asymmetric to any extent. Hinnosaar (2024) demonstrates that in a Tullock contest with $n(\geq 3)$ symmetric players and $r=1$, sequential contests consistently outperform simultaneous contests in maximizing total effort. In contrast, as an application of the equilibrium analysis of this paper, Gao, Lu, and Wang (2024) show that in a two-player Tullock contest with $r \in(0, \infty)$, either a simultaneous or a weak-lead sequential contest can generate more effort than the corresponding strong-lead sequential contest for $r$ (greater than one) in certain intervals. ${ }^{18}$

[^7]Xu, Zenou, and Zhou (2022) obtain general properties of equilibria in a framework in which the contest structure is modeled by a network. Compared with the literature on multi-battle contests (Roberson, 2006; Konrad and Kovenock, 2009; Fu, Lu, and Pan, 2015, Chowdhury et al., 2021) where contest structure is often specialized (e.g., symmetric players) for tractability, ${ }^{19}$ the variational inequality approach of Xu, Zenou, and Zhou (2022) does not rely on player symmetry or certain restrictions on the conflict structure. ${ }^{20}$

## 3 Model setup

We study a contest with two contestants, denoted by players $A$ and $B$, competing for a prize. The value of the prize is normalized to 1 . Let player $i$ 's marginal cost of exerting effort be $c_{i}>0, \forall i \in\{A, B\}$. For player $i$, the cost of exerting effort $x_{i}$ is $c_{i} x_{i}$. A lower marginal cost implies a higher level of ability. Without loss of generality, we assume that player $A$ is the more able player with $c_{A} \leq c_{B}$. Given the two players' effort profile $\left\{x_{A}, x_{B}\right\}$, their winning probabilities are determined by the Tullock contest success function (Tullock, 1980): Player $i$ 's winning probability is $x_{i}^{r} /\left(x_{A}^{r}+x_{B}^{r}\right)$ if $x_{A}+x_{B}>0$ and $1 / 2$ if $x_{A}+x_{B}=0$, in which the discriminatory power denoted by $r$, also known as the accuracy level of the contest, can take values in the interval $(0, \infty)$.

We conduct a complete equilibrium analysis of a general sequential contest model in which either player $A$ or player $B$ could lead in contests. To facilitate the analysis, we let the leader and follower be denoted by players $L$ and $F$, respectively. That is, the leaderfollower pair, $(L, F)$, could be either $(A, B)$ or $(B, A)$ in our analysis. In a sequential contest of Section 4, the leader, referred to as player $L$, exerts effort in stage 1, and the follower, referred to as player $F$, exerts effort in stage 2 after observing the leader's effort level. Given the two players' effort profile $\left(x_{L}, x_{F}\right)$, player $i$ 's winning probability equals $x_{i}^{r} /\left(x_{L}^{r}+x_{F}^{r}\right)$ if $x_{L}+x_{F}>0$ and $1 / 2$ if $x_{L}+x_{F}=0$. In this sequential contest model, we define the leader-follower ability ratio as

$$
\begin{equation*}
c=\frac{c_{L}}{c_{F}}, \tag{1}
\end{equation*}
$$

which measures the asymmetry level between the two players. A sequential contest is referred

[^8]to as a strong-lead contest if $c \in(0,1]$ and a weak-lead contest if $c \in(1, \infty) .{ }^{21}$

## 4 Equilibrium analysis

In this section, we present a complete equilibrium analysis of the sequential contest model in which the leader $(L)$ and the follower $(F)$ exert effort in stages 1 and 2 sequentially. Recall that $c=c_{L} / c_{F}$ measures the level of asymmetry between the two players. Thus, when $c=1$, it indicates a symmetric sequential contest, in which two players are identical ( $c_{L}=c_{F}$ ); when $c<1$, it is a strong-lead sequential contest, in which the leader is more able $\left(c_{L}<c_{F}\right)$; when $c>1$, it is a weak-lead sequential contest, in which the leader is less able $\left(c_{L}>c_{F}\right)$. The sequential contest model we analyze in this paper allows for parameter $c$ to vary within the range of $(0, \infty)$, encompassing all three aforementioned cases.

The leader exerts effort $x_{L}$ in stage 1. After observing $x_{L}$, the follower chooses effort $x_{F}$ in stage 2 to maximize his expected payoff

$$
\begin{equation*}
\pi_{F}=\frac{x_{F}^{r}}{x_{L}^{r}+x_{F}^{r}}-c_{F} x_{F} \tag{2}
\end{equation*}
$$

The follower's optimal choice $x_{F}$ either satisfies the following necessary first-order condition (FOC)

$$
\begin{equation*}
r x_{F}^{r-1} x_{L}^{r}=c_{F}\left(x_{L}^{r}+x_{F}^{r}\right)^{2} \tag{3}
\end{equation*}
$$

where $x_{F}>0$ if $x_{L}$ is sufficiently small, or a corner condition $x_{F}=0$ if $x_{L}$ is sufficiently large.

We define the follower's best response in stage 2 as $x_{F}\left(x_{L}\right)$, which is a function of the leader's effort $x_{L}, \forall x_{L}>0$. Without loss of generality, we assume that given $x_{L}>0$, when the follower is indifferent between exerting zero effort and exerting strictly positive effort, i.e., when both effort levels generate the same level of expected payoff for the follower, he always chooses to exert zero effort. ${ }^{22}$ We call the leader's locally optimal solution $x_{L}$ that induces $x_{F}\left(x_{L}\right)>0$ an interior solution, and the other locally optimal solution $x_{L}$ that induces $x_{F}\left(x_{L}\right)=0$ a preemptive solution, in which the leader preempts the follower.

[^9]Expecting the follower's best response in stage 2, denoted by $x_{F}\left(x_{L}\right)$, the leader chooses $x_{L}$ in stage 1 to maximize his expected payoff

$$
\begin{equation*}
\pi_{L}=\frac{x_{L}^{r}}{x_{F}^{r}\left(x_{L}\right)+x_{L}^{r}}-c_{L} x_{L} . \tag{4}
\end{equation*}
$$

Theoretically speaking, it is possible to observe scenarios where only interior solutions or only preemptive solutions exist for different combinations of $c$ and $r$ values. We verify that the above two cases do occur for certain values of $c$ and $r$. In cases where only preemptive/interior solutions exist, we have preemptive/interior equilibria. It may also hold true that, for certain values of $c$ and $r$, both interior and preemptive solutions coexist. In these cases, the leader chooses the solution that yields a higher expected payoff in equilibrium. We verify that these cases do occur in our subsequent analysis, and find that either the interior or the preemptive solution can be an equilibrium, depending on the values of $c$ and $r$.

The following is a road map for this section. In subsection 4.1, we examine interior and preemptive solutions for each possible ( $r, c$ ) profile. In subsection 4.2, we fully determine the equilibrium for each possible ( $r, c$ ) profile.

### 4.1 Equilibrium candidates: Interior and preemptive solutions

To find the interior and preemptive solutions, we solve the model using backward induction. We start our analysis with the assumption that $x_{L}>0$ in any optimal interior solution, ${ }^{23}$ and then validate this assumption by showing that the leader's expected payoff in this solution is strictly positive, which rules out the possibility of $x_{L}=0$ in any equilibrium. Given $x_{L}$ $(>0)$ exerted by the leader in stage 1 , the follower chooses effort $x_{F}$ in stage 2 to maximize his expected payoff $\pi_{F}$, which is given by equation (2). ${ }^{24}$ Because $\pi_{F}$ is continuous within its domain, given $x_{L}>0$, the follower's effort $x_{F}$, which is his best response to the leader's effort, $x_{L}$, is either strictly positive or zero. When $x_{F}>0$, it is an interior solution, which satisfies the following necessary first-order condition (FOC):

$$
\begin{equation*}
r x_{F}^{r-1} x_{L}^{r}=c_{F}\left(x_{L}^{r}+x_{F}^{r}\right)^{2} \tag{5}
\end{equation*}
$$

[^10]When $x_{F}=0$, it is a corner solution.
By analyzing the follower's best response function $x_{F}\left(x_{L}\right)$, we obtain the following results.
Lemma 1. Consider the follower's best response $x_{F}\left(x_{L}\right)$ to the leader's effort $x_{L}>0$. (i) When $r<1, x_{F}\left(x_{L}\right)$ is determined by the unique solution of (5) and $x_{F}\left(x_{L}\right)>0$ for any $x_{L}>0$; there exist only interior solutions, in which $x_{L} \in\left(0,1 / c_{L}\right)$, and preemptive solutions do not exist. (ii) When $r \geq 1$, for any $x_{L} \in\left(0, \hat{x}_{L}\right), x_{F}\left(x_{L}\right)=\sqrt{x_{L} / c_{F}}-x_{L}$ if $r=1$, while $x_{F}\left(x_{L}\right)$ is determined by the larger of the two solutions of equation (5) if $r>1$; for any $x_{L} \in\left[\hat{x}_{L}, \infty\right), x_{F}\left(x_{L}\right)=0$, where

$$
\hat{x}_{L}= \begin{cases}\frac{1}{c_{F}} & \text { if } r=1,  \tag{6}\\ \frac{1}{c_{F}}\left(\frac{1}{r}\right)(r-1)^{\frac{r-1}{r}} & \text { if } r>1,\end{cases}
$$

which is the minimum effort required from the leader to preempt the follower.
In the appendix (A.1), we provide the proof of this Lemma as well as the analysis of the valid equilibrium candidates. More specifically, in the proof of Lemma 1, we first establish that the solutions to both players' first-order conditions provide us with the unique best response of the follower. When $r \leq 1$, the solution $x_{F}^{*}$ obtained from the first-order condition is the follower's unique global optimum. However, when $r>1$, there exist two solutions to the first-order condition, denoted as $x_{F 1}$ and $x_{F 2}$, where $x_{F 1}<x_{F 2}$. We show that $x_{F 1}$ can be ruled out as it represents a local minimum for the follower. Consequently, the follower possesses a well-defined best response function, denoted as $x_{F}\left(x_{L}\right)$, where $x_{L} \in\left(0, \hat{x}_{L}\right]$. In the appendix (A.2), we further demonstrate that given the follower's best response function, the leader has a unique optimal choice among all possible effort levels that induce positive effort from the follower. Thus, for any pair of $c$ and $r$ resulting in the existence of interior solutions, the interior solution must be unique, where each player adopts a pure strategy. Therefore, we only need to focus on constructing pure-strategy interior solutions.

Expecting the follower's best response in stage 2, $x_{F}\left(x_{L}\right)$, which is given by Lemma 1 , the leader chooses $x_{L}$ in stage 1 to maximize his expected payoff (4). When $r<1$, preemptive solutions do not exist, leading to the absence of preemptive equilibria. Because the contestants will not choose effort levels that yield negative expected payoffs, it must hold that $x_{L} \in\left(0,1 / c_{L}\right)$ in any possible interior solution. Intuitively, when $r<1$, there is too much randomness involved in the contest: No matter how much effort the leader exerts in
stage 1, the follower always has an incentive to exert some positive effort in stage 2. If an equilibrium exists for $r<1$, it must be an interior equilibrium.

When $r \geq 1$, there exist preemptive solutions in which the leader exerts minimum preemptive effort $x_{L}=\hat{x}_{L}$ and the follower's best response is to exert zero effort, provided that the leader's expected payoff is positive, i.e., $\pi_{L}^{p r e}=1-c_{L} \hat{x}_{L} \geq 0$, which is the leader's participation constraint. When interior solutions exist for $r \geq 1$, we must have $x_{L}<\hat{x}_{L}$ and $x_{F}\left(x_{L}\right)>0$. In summary, when $r \geq 1$, we have $x_{L} \in\left(0, \hat{x}_{L}\right]$ in any possible equilibrium.

Based on the analysis so far, we conclude that, for any given values of $c>0$ and $r \geq 1$, there are three possible scenarios: (i) Only a preemptive solution exists, resulting in a preemptive equilibrium. (ii) Only an interior solution exists, resulting in an interior equilibrium. (iii) Both preemptive and interior solutions exist, and the type of equilibrium depends on which solution generates a higher level of expected payoff for the leader. In fact, we will show that when $r \geq 1$, either interior equilibria in which $x_{L}<\hat{x}_{L}$ and $x_{F}\left(x_{L}\right)>0$, or preemptive equilibria in which $x_{L}=\hat{x}_{L}$ and $x_{F}(\hat{x})=0$ indeed occur for certain values of $c$ and $r$.

To facilitate the analysis, we introduce the quotient $t$ as the follower-leader effort ratio, where

$$
\begin{equation*}
t=\frac{x_{F}}{x_{L}} . \tag{7}
\end{equation*}
$$

In equilibrium, $t$ can be seen as a function of $x_{L}$, as $x_{F}$ is essentially a function of $x_{L} .{ }^{25}$
If $t>0$ occurs in equilibrium, we have $x_{F}>0$, which corresponds to an interior solution where both players exert strictly positive effort. In this case, using (5), we derive that $t$ must satisfy

$$
\begin{equation*}
r t^{r-1}=c_{F} x_{L}\left(1+t^{r}\right)^{2} \tag{8}
\end{equation*}
$$

which is the FOC for the follower. Using (7) and (8), we further derive that

$$
\begin{equation*}
x_{L}=\frac{r t^{r-1}}{c_{F}\left(1+t^{r}\right)^{2}} \text { and } x_{F}=\frac{r t^{r}}{c_{F}\left(1+t^{r}\right)^{2}} . \tag{9}
\end{equation*}
$$

If $t=0$ in an equilibrium, it must be the case that $x_{F}=0$, as we have shown that $x_{L} \in\left(0, \hat{x}_{L}\right]$ in an equilibrium. This corresponds to a preemptive solution in which the leader exerts sufficiently large effort $\left(x_{L}=\hat{x}_{L}\right)$ to preempt the follower $\left(x_{F}\left(\hat{x}_{L}\right)=0\right)$.

The interior and preemptive solutions are two candidates for an equilibrium. When both

[^11]solutions exist, the leader chooses the one with a greater expected payoff in equilibrium. Recall that we have assumed that the leader always chooses a preemptive solution when he is indifferent between the two solutions.

In the following analysis of interior solutions, we demonstrate that for any pair of $c$ and $r$ that leads to an interior solution, there exists a unique value of $t>0$, denoted by $t^{*}$, such that it satisfies the FOCs of both players and ensures the fulfillment of their participation constraints at the same time.

## Interior solutions

In an interior solution, the follower chooses an interior best response, $x_{F}>0$, given the leader's effort $x_{L}>0$. In this case, equation (8) is satisfied and the leader chooses $x_{L}$ to maximize his expected payoff $\pi_{L}$. Using (4), we obtain that

$$
\pi_{L}=\frac{1}{1+t^{r}\left(x_{L}\right)}-c_{L} x_{L}
$$

The necessary FOC for the leader is

$$
\begin{equation*}
-\frac{r t^{r-1} t^{\prime}}{\left(1+t^{r}\right)^{2}}-c_{L}=0 \tag{10}
\end{equation*}
$$

where $t^{\prime}$ denotes the first-order derivative of $t$ with respect to $x_{L}$. By combining equations (8) and (10), we construct a characteristic equation that must be satisfied by a valid $t$ for given values of $c$ and $r$. This characteristic equation plays a vital role in identifying and characterizing the interior solutions, and in assisting us to determine the types of equilibria when both the interior and preemptive solutions coexist.

Lemma 2. (Characteristic equation) If a sequential contest with given values of $c$ and $r$ has an interior equilibrium, the effort ratio of the players, denoted as $t$, must satisfy the following characteristic equation:

$$
\begin{equation*}
c-t=c r\left(\frac{2}{1+t^{r}}-1\right) \tag{11}
\end{equation*}
$$

In the appendix (A.3), we show that the characteristic equation (11) is derived from the FOCs of both players' optimization problems. We denote the effort ratio in a valid interior solution by $t^{*}$. As a valid interior solution, $t^{*}$ should satisfy (11). Moreover, an arbitrary
player $i$ 's corresponding effort choice $x_{i}\left(t^{*}\right)$ can be obtained using (9), and his expected payoff must have a global maximum at $x_{i}=x_{i}\left(t^{*}\right) .{ }^{26}$

To facilitate the analysis, we define two functions as follows:

$$
\begin{align*}
& L(t)=c-t \\
& R(t)=c r\left(\frac{2}{1+t^{r}}-1\right), \tag{12}
\end{align*}
$$

where $L(t)$ and $R(t)$ represent the $L H S$ and $R H S$ of equation (11), respectively. These functions are essential for understanding the properties of the equation.

If an interior solution $t^{*}$ exists, it must represent a point of intersection between the functions $z=L(t)$ and $z=R(t)$ on the $t z$-plane. In the appendix (A.4), we provide further details regarding the characteristics of functions $z=L(t)$ and $z=R(t)$ in relation to a valid solution $t^{*}$. To further investigate $t^{*}$, we divide the values of $c$ and $r$ into different regions and analyze the solutions of equation (11) for each of these cases. In all these cases, we show that: for any given values of $c$ and $r$, a valid effort ratio $t^{*}$ can be uniquely determined among all possible solutions of (11) provided that such solutions exist. Once $t^{*}$ is determined uniquely, equation (9) can be used to calculate the effort levels of the players, $x_{L}^{*}$ and $x_{F}^{*}$, which constitute an interior solution in a sequential contest.

In the following Propositions 1, 2 and 3, we fully characterize $t^{*}$ by considering three exhaustive cases, respectively. We start with the case of $r \in(0,1)$, in which there is a unique solution to equation (11).

Proposition 1. When $r \in(0,1), t^{*}$ is the unique solution to equation (11) for any $c \in$ $(0, \infty)$. The following results can be obtained: (i) When $r \in(0,1)$ and $c \in(0,1], t^{*} \leq c \leq 1$, and $t^{*}$ decreases in $r$. (ii) When $r \in(0,1)$ and $c \in(1, \infty), 1<c<t^{*}$, and $t^{*}$ increases in $r$.

Note that in simultaneous contests, it always holds true that the players' effort ratio equals their ability ratio in any pure-strategy equilibrium, i.e., $\frac{x_{F}}{x_{L}}=\frac{c_{L}}{c_{F}}$. However, in sequential contests with $r<1$, Proposition 1 says that with interior solutions, $\frac{x_{F}}{x_{L}}<\frac{c_{L}}{c_{F}}$ (as $t^{*}<c$ ) when the leader is the strong player, and $\frac{x_{F}}{x_{L}}>\frac{c_{L}}{c_{F}}\left(\right.$ as $\left.t^{*}>c\right)$ when the leader is the weak player. We summarize the results obtained with $r<1$ as follows. The strong player exerts

[^12]more effort than the weak player in both sequential and simultaneous contests. Furthermore, the relative effort level of the strong player, represented by the effort ratio between the strong and weak players, is greater in a (strong-lead or weak-lead) sequential contest compared to a simultaneous contest. Moreover, in sequential contests, the strong-weak effort ratio increases as the contest's accuracy improves.

Intuitively, in a sequential contest, when the strong player leads the move, he is better motivated as the first mover compared with his situation in a simultaneous contest, because he anticipates that with $r<1$, the follower will not respond aggressively to an increase in his effort. In contrast, when the weak player is the leader, being the first mover discourages him because he anticipates that with $r<1$, the strong follower, who has an ability advantage, will respond aggressively to an increase in the weak leader's effort. The above arguments explain why in cases with $r<1$, the strong-weak effort ratio is greater in a (strong-lead or weak-lead) sequential contest than in a simultaneous contest.

Next, we analyze the case where $r \geq 1$ and $c>1$, and equation (11) has two solutions for these values of $r$ and $c$. In the appendix (A.6), we demonstrate that the solution of the smaller value cannot be a valid interior solution, because the follower's expected payoff derived from this smaller solution is strictly negative, which violates the individual participation constraint, whereas it is positive for the larger solution.

Proposition 2. When $r \in[1, \infty)$ and $c \in(1, \infty)$, there are exactly two solutions to equation (11), and $t^{*}$ is the one with greater value. Moreover, $1<c<t^{*}$, and $t^{*}$ increases in $r$.

In weak-lead contests where $c \in(1, \infty), r \geq 1$ corresponds to a higher level of accuracy compared to $r<1$. This higher level of accuracy, all else being equal, gives the follower a stronger incentive to exert effort. Thus, in interior solutions, when $r$ increases, the strong follower tends to respond more aggressively when the weak leader increases his effort. This reduces the weak leader's incentive to exert effort in stage 1. In this case, we find that the strong follower always exerts more effort than the weak leader $\left(t^{*}>c>1\right)$, and when the contest becomes more accurate, the strong follower will exert relatively more effort than the leader, which corresponds to the result that $t^{*}$ increases in $r .{ }^{27}$

Finally, we consider the case where $r \in[1, \infty)$ and $c \in(0,1]$. This case is complex because equation (11) may yield zero, one, or two solutions.

[^13]Proposition 3. (i) When $r \in[1, \infty)$ and $c \in\left(0, \frac{1}{2}\right]$, there exists a unique $r_{s}$ that is determined by the unique solution to the system of equations involving (11) and $\frac{d R}{d t}=\frac{d L}{d t}$, with $r_{s}>\sqrt{2} .{ }^{28}$ It can be shown that: When $r \in\left[1, r_{s}\right), t^{*}$ does not exist. When $r \in\left[r_{s}, \infty\right), t^{*}$ is the unique solution to (11) for $r=r_{s}$, and is the greater of the two solutions for $r \in\left(r_{s}, \infty\right)$; we can show that $t^{*}>1>c$, and $t^{*}$ increases in $r .{ }^{29}$
(ii) When $r \in[1, \infty)$ and $c \in\left(\frac{1}{2}, 1\right]$, there exist $r_{s}^{1}$ and $r_{s}^{2}$, where $r_{s}^{1} \leq r_{s}^{2}$, determined by the two solutions of the system of equations involving (11) and $\frac{d L}{d t}=\frac{d R}{d t}$. It can be shown that: $r_{s}^{1}<\sqrt{2}<r_{s}^{2}$ for $c \in\left(\frac{1}{2}, 1\right)$, and $r_{s}^{1}=\sqrt{2}=r_{s}^{2}$ for $c=1$. When $r \in\left(r_{s}^{1}, r_{s}^{2}\right)$, $t^{*}$ does not exist. When $r \in\left[1, r_{s}^{1}\right]$, $t^{*}$ is the unique solution to (11) if $r=1$ and $r=r_{s}^{1}$, and is the greater of the two solutions if $r \in\left(1, r_{s}^{1}\right) ; t^{*} \leq c \leq 1$, and $t^{*}$ decreases in $r$. When $r \in\left[r_{s}^{2}, \infty\right)$, $t^{*}$ is the unique solution if $r=r_{s}^{2}$ and the greater of the two solutions if $r \in\left(r_{s}^{2}, \infty\right)$; we can show that $t^{*} \geq 1 \geq c$ and $t^{*}$ increases in $r$.

We offer two remarks about the results of Proposition 3.
Remark 1. In a strong-lead sequential contest with $c \in\left(0, \frac{1}{2}\right]$ and $r \geq r_{s}$, the strong leader exerts less effort than the weak follower (shown by $t^{*}>1>c$ ); the strong-weak effort ratio decreases when the contest becomes more accurate (implied by the fact that $t^{*}$ increases in $r$ ) whenever interior solutions exist. This result is surprising because it is the opposite of our results obtained when $r<1$ and the findings in the literature on Tullock contests with $r=1$, where a strong player always exerts more effort in equilibrium.

Here is an intuitive explanation for the above result. When $r$ is sufficiently large, the contest is sufficiently accurate, the follower has a stronger incentive to exert effort than that with a smaller $r$, i.e., when $r$ gets larger, the follower will respond more aggressively to a given amount of increase in the leader's effort. This gives a second-mover advantage to the follower, because a more aggressive response by the follower in stage 2 , which is expected by the leader, will decrease the leader's incentive to exert effort in stage 1. Moreover, when $r$ gets larger, the strong leader chooses to reduce his relative effort level (compared to the weak follower), which lowers the intensity of the competition and benefits himself in the interior solutions.

We now look at the more complicated case with $c \in\left(\frac{1}{2}, 1\right]$. When $r$ is sufficiently large $\left(r \geq r_{s}^{2}\right)$, the follower's second-mover advantage outweighs the leader's ability advantage in

[^14]strong-lead sequential contests. As a result, the follower's aggressive response diminishes the leader's incentive to exert effort, leading to a situation where the strong leader exerts less effort ( $t^{*} \geq 1 \geq c$ ). When $r$ is sufficiently small $\left(r \in\left[1, r_{s}^{1}\right]\right)$, the follower's second-mover advantage, though still present, is smaller compared to cases with larger $r$. Consequently, the leader's ability advantage prevails, implying that the weak follower does not respond aggressively to an increase in the leader's effort. Thus, in interior solutions, the strong leader's effort remains greater $\left(t^{*} \leq c \leq 1\right)$.

Remark 2. For $r$ in a moderate range, interior solutions may not exist. For instance, when $c \in\left(\frac{1}{2}, 1\right]$, $t^{*}$ does not exist for $r \in\left(r_{s}^{1}, r_{s}^{2}\right)$. We find that in these cases, given the follower's best response $x_{F}\left(x_{L}\right)$, the leader's marginal benefit of increasing effort $x_{L}$ is always larger than his marginal cost $c_{L}$ for any $x_{L}<\hat{x}_{L}$. The opposite is true for $x_{L} \geq \hat{x}_{L}$, implying that his marginal benefit becomes zero when $x_{L} \geq \hat{x}_{L}$ because he already wins for certain at $x_{L}=\hat{x}_{L}$. Thus, the strong leader always chooses to preempt the weak follower by exerting preemptive effort (6).

Intuitively, in these cases with moderate $r$, the follower will respond to an increase in the leader's effort conservatively, which gives the leader an incentive to increase effort until it reaches the preemptive level. That is to say, there exists no $x_{L} \in\left(0, \hat{x}_{L}\right)$ that corresponds to an interior local maximum for the leader, with only $x_{L}=\hat{x}_{L}$ being the unique global maximum for all $x_{L} \in(0, \infty)$.

We have shown that for given values of $c$ and $r$, solution $t^{*}$ in Propositions 1 to 3 is unique, which implies that if an interior solution exists, it must hold that $t=t^{*}$. The following corollary shows that for each $t^{*}$ determined in Propositions 1 to 3 , the corresponding effort levels $x_{L}^{*}$ and $x_{F}^{*}$ ensure strictly positive expected payoffs for both players.

Proposition 4. At each $t^{*}$ that is determined in Propositions 1 to 3, the players' expected payoffs are

$$
\begin{align*}
& \pi_{L}^{i n t}\left(t^{*} ; c, r\right)=\frac{c^{2}-\left(c r-t^{*}\right)^{2}}{4 c r t^{*}}>0  \tag{13a}\\
& \pi_{F}^{i n t}\left(t^{*} ; c, r\right)=\frac{t^{* 2}-c^{2}(r-1)^{2}}{4 c^{2} r}>0 \tag{13b}
\end{align*}
$$

which are strictly positive. Furthermore, for any given $t^{*}$ that is determined in Propositions 1 to 3, the corresponding effort levels $x_{L}^{*}$ and $x_{F}^{*}$ (which are given by equation (9) and $t^{*}$ ) constitute a valid interior solution.

## Preemptive solutions

For given values of $c$ and $r$, in a preemptive solution, the leader exerts preemptive effort $x_{L}=\hat{x}_{L}$, which forces $x_{F}=0$ to be the follower's best response. ${ }^{30}$

We have shown that when $r<1$, there exist no preemptive solutions; when $r \geq 1$, there exists an $\hat{x}_{L}$ given by (6), which is the leader's minimal effort that preempts the follower. At $x_{L}=\hat{x}_{L}$, the follower is indifferent between a corner response $x_{F}^{c o r}=0$ and an interior response $x_{F}^{i n t}>0$, as both responses lead to zero expected payoff. Without loss of generality, we assume that in any equilibrium, the leader always chooses a corner response rather than an interior response when he is indifferent.

Using (6), the leader's expected payoff in a preemptive solution is given by

$$
\pi_{L}^{p r e}=1-c_{L} \hat{x}_{L}= \begin{cases}1-c & \text { if } r=1  \tag{14}\\ 1-\frac{c}{r}(r-1)^{\frac{r-1}{r}} & \text { if } r>1\end{cases}
$$

By (6) and (14), the leader's effort converges to $\hat{x}_{L}=\frac{1}{c_{F}}$ and his expected payoff converges to $1-c$, as $r$ goes to $1^{+}$. We obtain the following results by analyzing (14).

Proposition 5. (i) There is no preemptive solution in sequential contests for all values of $c$ when $r<1$. (ii) Preemptive solutions are possible when $r \geq 1$. In cases where preemptive solutions exist, the leader's effort $\hat{x}_{L}$ is given by (6); $\pi_{L}^{\text {pre }}$ strictly increases with $r$ when $r \in[1,2)$, and strictly decreases with $r$ when $r \in(2,+\infty)$. $\pi_{L}^{\text {pre }}$ reaches its global maximum at $r=2$ for $r \in[1, \infty)$, with $\pi_{L}^{p r e}(r=2)=1-\frac{c}{2}$, and $\lim _{r \rightarrow 1^{+}} \pi_{L}^{p r e}=\lim _{r \rightarrow \infty} \pi_{L}^{p r e}=1-c$.

A caveat is that Proposition 5 is valid for all $r \in[1, \infty)$ and $c \in(0, \infty)$, provided that preemptive solutions exist. This proposition will be used in later analysis when the leader compares his expected payoffs of the two solutions, preemptive and interior. The outcome of this comparison will determine whether the equilibrium is preemptive or interior.

Furthermore, the above proposition implies that when the leader is the strong player, i.e., when $c \in(0,1]$, there always exists a preemptive solution for any $r, r \geq 1$. In contrast, when $r \geq 1$ but the leader is the weak player, i.e., when $c>1$, a preemptive solution exists if and only if $c$ is not overly large, otherwise the leader's expected payoff would become negative. Preemptive solutions do not exist for large $c$, when the leader is too weak compared with

[^15]the follower. Preempting the strong follower becomes too costly for the weak leader when their ability difference gets large (i.e., when $c$ increases). The weak leader would receive a negative expected payoff in this case. For instance, there exist no preemptive solutions when $c>2$, because $\pi_{L}^{p r e}<0$ for any $c>2$. In other words, it is never optimal for the leader to preempt when his marginal cost is more than two times larger than that of the follower's.

### 4.2 Equilibria in sequential contests

Formally, we refer to the equilibria where interior solutions are chosen as "interior equilibria" and the ones where preemptive solutions are chosen as "preemptive equilibria." In the subsequent analysis (Theorems 1, 2, 3 and 4), we study the different regions of $(r, c)$ that correspond respectively to the cases studied in Propositions 1, 3 and 2, and pin down the equilibrium type for each case. If an interior equilibrium prevails, the equilibrium effort levels $x_{L}^{*}$ and $x_{F}^{*}$ are given by equation (9) and the corresponding $t^{*}$. If a preemptive equilibrium prevails, then the leader's effort $\hat{x}_{L}$ is given by (6).

### 4.2.1 The case with $r \in(0,1)$ and $c \in(0, \infty)$

The case with $r \in(0,1)$ is simple because there exist only interior solutions by Lemma 1 . We show in Theorem 1 that for any $r \in(0,1)$ and $c \in(0, \infty)$, equation (11) has a unique solution and this interior solution is indeed an interior equilibrium.

Theorem 1. In a sequential contest in which $r \in(0,1)$, for a given $c>0$, there always exists a unique interior equilibrium. For different values of $c$, the equilibrium effort ratio $t^{*}$ varies with $r$ :
(i) When the strong player leads, i.e., when $c \in(0,1), t^{*}<c<1$ and $t^{*}$ is decreasing in $r$.
(ii) When the weak player leads, i.e., when $c \in(1, \infty), 1<c<t^{*}$ and $t^{*}$ is increasing in $r$.
(iii) When the two players are symmetric, i.e., when $c=1, t^{*}=c=1$.

### 4.2.2 The case with $r \in[1, \infty)$ and $c \in(0,1]$

This is the case in which the leader is the strong player, as $c \leq 1$. Proposition 3 states that with $r \geq 1$ and $c \in\left(0, \frac{1}{2}\right]$, there exists a threshold $r_{s}$, such that interior solutions exist for $r \in\left[r_{s}, \infty\right)$, but do not exist for $r \in\left[1, r_{s}\right)$. In the appendix (A.11), we show that when
$c \in\left(0, \frac{1}{2}\right]$, for $r \in\left[r_{s}, \infty\right)$, a preemptive solution always dominates an interior solution, which further implies that all equilibria are preemptive for $r \geq 1$.

Theorem 2. In the strong-lead sequential contests where $r \in[1, \infty)$ and $c \in\left(0, \frac{1}{2}\right]$, the leader always chooses to preempt the follower in equilibrium, i.e., all equilibria are preemptive.

When $r \geq 1$ and $c \leq \frac{1}{2}$, we have shown that the leader always chooses to preempt the follower in equilibrium. However, when $r \geq 1$ and $c>\frac{1}{2}$, the follower becomes relatively more able as $c$ increases. Now, it is possible that the leader's payoff from a preemptive solution is lower than that from a corresponding interior solution. Recall that we have shown that the leader's expected payoff may even turn negative when $c>2$. In the case of $r \geq 1$ and $c>\frac{1}{2}$, the leader compares his expected payoffs from the two solutions and chooses the one with the higher expected payoff in equilibrium.

The following lemma characterizes the leader's expected payoff function in interior solutions for $r \geq 1$. This characterization, along with Proposition 5, which characterizes the leader's expected payoff function in preemptive solutions, will play an important role in comparing the leader's expected payoffs between the two types of solutions. It will also assist us in identifying the value intervals of $r$ that support a specific type of equilibrium.

Lemma 3. Consider interior solutions in sequential contests in which $r \in[1, \infty)$ and $c \in$ $\left(\frac{1}{2}, \infty\right)$. (i) In strong-lead sequential contests with $c \in\left(\frac{1}{2}, 1\right]$ : For $r \in\left[1, r_{s}^{1}\right]$, there are interior solutions with $t^{*} \leq 1$, and there exist $\bar{c}_{1}$ and $\bar{c}_{2}$, where $\frac{1}{2}<\bar{c}_{1}<\bar{c}_{2}<1$, such that when $c \in\left(\frac{1}{2}, \bar{c}_{1}\right]$ the leader's expected payoff $\pi_{L}^{\text {int }}$ increases in $r$; when $c \in\left(\bar{c}_{1}, \bar{c}_{2}\right]$, $\pi_{L}^{\text {int }}$ first decreases and then increases in $r$; when $c \in\left(\bar{c}_{2}, 1\right], \pi_{L}^{i n t}$ decreases in $r$. For $r \in\left[r_{s}^{2}, \infty\right)$, there are interior solutions with $t^{*}>1$, and $\pi_{L}^{i n t}$ always decreases in $r$. (ii) In weak-lead sequential contests with $c \in(1, \infty)$ : For $r \in[1, \infty)$, there are interior solutions with $t^{*}>1$ and $\pi_{L}^{i n t}$ decreases in $r$, with $\lim _{r \rightarrow \infty} \pi_{L}^{i n t}=0$.

We provide an intuitive explanation for Lemma 3. In sequential contests with $c \in\left(\frac{1}{2}, 1\right]$, the strong leader's relative ability advantage is measured by the magnitude of $c$, and the leader's ability advantage gets smaller when $c$ increases in this interval. Additionally, recall that in interior solutions, there is a second-mover advantage for the follower when $r$ is sufficiently large, and this advantage gets larger when $r$ increases. When $r \in\left[1, r_{s}^{1}\right]$ and $c \in$ $\left(\frac{1}{2}, 1\right]$, the leader's ability advantage always outweighs the follower's second-mover advantage, which explains why the stronger leader exerts greater effort in these cases. As $r$ increases,
the leader's overall advantage diminishes because the follower's second-mover advantage gets larger. Therefore, as $r$ increases in the interval $\left[1, r_{s}^{1}\right]$ that supports interior equilibria, the competition becomes fiercer. This explains the result of Lemma 3, which says that the leader's expected payoff decreases in $r$ for a larger $c$.

When $c \in\left(\frac{1}{2}, 1\right]$, for sufficiently large $r, r \in\left[r_{s}^{2}, \infty\right)$, the leader's ability advantage is dominated by the follower's second-mover advantage. This explains why the strong leader exerts less effort than the weak follower (demonstrated by $t^{*}>1$ ) and the leader's expected payoff decreases with $r$. In sequential contests with $c \in(1, \infty)$, the leader is the weak player. The follower now has both the ability advantage and the second-mover advantage, for $r \in[1, \infty)$. Thus, it is easy to understand why the leader exerts less effort and his expected payoff decreases with $r$ in this case.

We now proceed to conduct an equilibrium analysis with $r \in[1, \infty)$ and $c \in\left(\frac{1}{2}, 1\right]$, using the result in Proposition 3 (ii). We obtain the following results by comparing the leader's expected payoffs from the two solutions, $\pi_{L}^{p r e}$ and $\pi_{L}^{i n t}$.

Theorem 3. In strong-lead sequential contests in which $r \in[1, \infty)$ and $c \in\left(\frac{1}{2}, 1\right]$, there exists a critical value $\hat{r}_{s}$, which is determined by the unique solution of equation $\pi_{L}^{\text {int }}\left(t^{*} ; c, r\right)=$ $\pi_{L}^{p r e}(c, r)$, such that the equilibria are interior for all $r<\hat{r}_{s}$ and preemptive for all $r \geq \hat{r}_{s}$. Moreover, it can be shown that $\hat{r}_{s} \in\left[1, r_{s}^{1}\right)$.


Figure 1: The leader's expected payoff as a function of $r$, for $c \in\left(\frac{1}{2}, 1\right]$. Here it is illustrated by $c=0.8$.

In Figure 1, we present a specific example of Theorem 3 by comparing the leader's
expected payoffs (on the vertical axis) from both interior and preemptive solutions, with $r$ running from zero to infinity (on the horizontal axis). First, for any $r<1$, an interior solution exists, but no preemptive solution exists, as indicated by Lemma 1. Second, for any $r \geq 1$, there always exists a preemptive solution, but no interior solution for $r$ in a specific range $\left(r \in\left(r_{s}^{1}, r_{s}^{2}\right)\right)$ according to Proposition $3 .{ }^{31}$ Third, for any $r \geq 1$, the leader's expected payoff with an interior solution is greater than that with a corresponding preemptive solution if $r$ is sufficiently small $\left(r<\hat{r}_{s}\right)$; otherwise, if $r$ is sufficiently large ( $r \geq \hat{r}_{s}$ ), either only a preemptive solution exists, or both solutions coexist but the preemptive solution dominates its corresponding interior solution, because the expected payoff of the preemptive solution is strictly larger than that of the interior solution.

In strong-lead sequential contests, when the difference in players' abilities is relatively large, $c \in\left(0, \frac{1}{2}\right]$, the strong leader always prefers a preemptive solution over an interior solution by Theorem 2. In these cases, there are preemptive equilibria for $r \in[1, \infty)$. In contrast, Theorem 3 states that when the ability difference is relatively small, $c \in\left(\frac{1}{2}, 1\right]$, the strong leader chooses a preemptive solution over an interior solution only when the contest is sufficiently accurate, $r \geq \hat{r}_{s}$, where $\hat{r}_{s}>1$. The difference between the results of Theorems 2 and 3 can be explained from a new perspective: in strong-lead sequential contests, the leader's preemptive effort decreases when the follower gets weaker, which implies that a preemptive equilibrium is more likely to occur when $c$ decreases.

### 4.2.3 The case with $r \in[1, \infty)$ and $c \in(1, \infty)$

This is the case in which the leader is the weak player, as $c>1$. For these weak-lead sequential contests with $r \geq 1$, preemptive solutions only exist when $r$ is sufficiently small, as demonstrated by the expression for the leader's expected payoff in Proposition 5. We compare the leader's expected payoffs of the two solutions when they coexist.

Lemma 4. In weak-lead sequential contests with $r \in[1, \infty)$ and $c \in(1, \infty)$, there exists a threshold of $c$, denoted by $\hat{c}^{h}$, where $\hat{c}^{h} \approx 1.983$, such that for any $c>\hat{c}^{h}$, it is never optimal for the leader to preempt the follower for any $r \in[1, \infty) .{ }^{32}$

The following theorem summarizes our findings on the two types of equilibria (interior

[^16]and preemptive) in weak-lead sequential contests.
Theorem 4. When $c \in\left(1, \hat{c}^{h}\right]$, where $\hat{c}^{h}$ is given by Lemma 4, there exist $\hat{r}_{w}^{1}$ and $\hat{r}_{w}^{2}$, where $1<\hat{r}_{w}^{1} \leq \hat{r}_{w}^{2}$, which are the two solutions of equation $\pi_{L}^{\text {int }}\left(t^{*} ; c, r\right)=\pi_{L}^{\text {pre }}(c, r)$ for $c \in\left(1, \hat{c}^{h}\right)$, and $\hat{r}_{w}^{1}=\hat{r}_{w}^{2}$ is the unique solution to equation $\pi_{L}^{\text {int }}\left(t^{*} ; c, r\right)=\pi_{L}^{p r e}(c, r)$ for $c=\hat{c}^{h}$, such that the leader preempts the follower in equilibrium if and only if $r \in\left[\hat{r}_{w}^{1}, \hat{r}_{w}^{2}\right] \subset \mathbb{R}^{+}$. This "preemptive" interval $\left[\hat{r}_{w}^{1}, \hat{r}_{w}^{2}\right]$ shrinks to a point as c increases to $\hat{c}^{h}$. When $c>\hat{c}^{h}$, there are only interior equilibria.


2(a) The leader's expected payoffs with $c_{1}$ 2(b) The leader's expected payoffs with $c_{2}$

Figure 2: The leader's expected payoffs from interior and preemptive solutions with respect to $r$, in which $1<c_{1}=1.4<c_{2}=1.5<\hat{c}^{h}$.

Theorem 4 indicates that in weak-lead sequential contests, the weak leader chooses to preempt when $r$ falls within the preemptive interval $\left[\hat{r}_{w}^{1}, \hat{r}_{w}^{2}\right]$. Moreover, a sufficiently small difference in the players' abilities, with $c \in\left(1, \hat{c}^{h}\right]$, guarantees the existence of such an interval. This preemptive interval shrinks when ability difference $c$ increases in the interval $\left(1, \hat{c}^{h}\right]$. In particular, it shrinks to a point as $c$ reaches $\hat{c}^{h}$. When the ability difference is sufficiently large $\left(c>\hat{c}^{h}\right)$, there exists no preemptive interval, i.e., there are only interior equilibria for $r \geq 1$. Intuitively, in these cases $\left(c>\hat{c}^{h}\right)$, the leader is too weak and it is never optimal for him to preempt follower who is much stronger, as doing so would lead to a negative payoff. Figure 2 (a) and (b) illustrate two specific examples of Theorem 4, with $c_{1}=1.4<c_{2}=1.5<\hat{c}^{h}$. We observe that the preemptive interval shrinks as $c$ increases.

Finally, we present a summary of the equilibrium types in the sequential contests for all possible values of $c$ and $r$, as illustrated in Figure 3. In strong-lead sequential contests with $c \leq 1$, there are interior equilibria when $r$ is below a threshold and preemptive equilibria when


Figure 3: Equilibrium types for given values of $c$ and $r$.
$r$ is above the threshold. This threshold, denoted by $\hat{r}_{s} \geq 1$, (weakly) increases with $c$. As the difference in players' abilities becomes larger, the strong leader is more likely to preempt the weak follower in the sense that the preemptive interval gets larger, which is shown by the fact that $\hat{r}_{s}$ decreases when $c$ decreases. In weak-lead sequential contests with $c>1$, the weak leader chooses to preempt the strong follower for $r \in\left[\hat{r}_{w}^{1}, \hat{r}_{w}^{2}\right]$. The preemptive interval decreases as $c$ increases from $c=1$; when the ability difference is sufficiently large such that $c>\hat{c}^{h}$, the preemptive interval disappears and only interior equilibria exist for all $r$. In these cases, intuitively, it is too costly for the weak leader to preempt the strong follower.

Once we comprehend the equilibrium sets of the two-player sequential Tullock contests for all possible values of $c$ and $r$, it is natural to investigate the implications of these equilibria. In the next section, we analyze the optimal move order of the contestants that maximizes the winning probabilities for either player in equilibrium.

## 5 Winning chances and decisions on move orders

Our previous equilibrium analysis shows that, given the asymmetry level of the two contestants and the accuracy level of the contest, denoted by $c$ and $r$ respectively, different move
orders result in distinct equilibrium outcomes. While the leader is assured of victory in a preemptive equilibrium, how does the winning probability of a contestant vary with the move order of the contestants in interior equilibria?

The answer to the above question can be important in many real-world situations. For instance, in sports activities such as tennis or boxing, there can be a coach (or sponsor) for each contestant. These coaches (or sponsors) are primarily concerned with maximizing their contestants' winning chances in the contest. Beyond sports, consider a scenario where senior managers from two companies supervise their respective R\&D scientists in an innovation contest. In this case, managers and scientists, akin to coaches and contestants in sports examples, form teams to compete, with managers striving to maximize their teams' chances of winning. Therefore, understanding how a contestant's winning probability varies with different move orders of contestants becomes crucial, especially in interior equilibria where every team has a positive winning chance.

As a direct application of our equilibrium characterization, we present an analysis of a modified two-stage contest model. In this setup, contestants choose effort levels to maximize their expected payoffs in a contest. However, before this stage, their winning-odd-maximizing coaches independently and simultaneously choose the time (between two strategies: lead or follow) for their contestants to move. ${ }^{33}$ Therefore, the contestants' move order arises from the coaches' choices. Given a particular move order (which may correspond to a simultaneous contest, strong-lead sequential contest, or weak-lead sequential contest), the contestants select the equilibrium effort levels, an issue we have addressed in previous sections. In this two-stage game, we investigate the optimal strategies of the winning-odd-maximizing coaches. In particular, we study whether all possible move orders (i.e., whether the three contest formats) can arise as an equilibrium outcome.

Consider an original two-player Tullock contest model presented at the beginning of Section 3, in which players $A$ and $B$ compete for a unit prize. The two players have constant marginal costs, denoted by $c_{A}$ and $c_{B}$, respectively. Without loss of generality, player $A$ is

[^17]assumed to be the strong player, with $c_{A} \leq c_{B} .{ }^{34}$ To facilitate the analysis, we define
$$
\tilde{c}=\frac{c_{A}}{c_{B}}
$$
where $\tilde{c} \in(0,1]$. The contest can be simultaneous, strong-lead sequential, or weak-lead sequential, depending on the move order of the contestants. We first rank the three contest formats based on the equilibrium winning probabilities of one of the two players, say player $A$. The ranking for player $B$ follows naturally.

In a strong-lead sequential contest, player $A$ is the leader. In this case, $\tilde{c}=c_{A} / c_{B}$, defined in the current model, is equivalent to $c=c_{L} / c_{F}$, defined in the sequential contest model studied in previous sections, since here $A$ is the leader and $B$ is the follower. Let $t_{A}=x_{B} / x_{A}$ be the solution of the characteristic equation

$$
\begin{equation*}
\tilde{c}-t=\tilde{c} r\left(\frac{2}{1+t^{r}}-1\right) \tag{15}
\end{equation*}
$$

which is obtained from (11) by replacing $c$ with $\tilde{c}$. In a strong-lead sequential contest, player $A$ has an equilibrium winning probability given by

$$
\begin{equation*}
p_{S L}=\frac{1}{1+t_{A}^{r}} \tag{16}
\end{equation*}
$$

In a weak-lead sequential contest, player $A$ is the follower. In this case, $\tilde{c}=c_{A} / c_{B}$ is equivalent to $1 / c$, where $c=c_{L} / c_{F}$ is defined in the general sequential contest model, since here $B$ is the leader and $A$ is the follower. Let $t_{B}=x_{A} / x_{B}$ be the solution of the characteristic equation

$$
\frac{1}{\tilde{c}}-t=\frac{r}{\tilde{c}}\left(\frac{2}{1+t^{r}}-1\right),
$$

which is obtained from (11) by replacing $c$ with $1 / \tilde{c}$. The above characteristic equation can be further rewritten as

$$
\begin{equation*}
\tilde{c}-\tilde{c}^{2} t=\tilde{c} r\left(\frac{2}{1+t^{r}}-1\right) \tag{17}
\end{equation*}
$$

[^18]In a weak-lead sequential contest, player $A$ has an equilibrium winning probability given by

$$
\begin{equation*}
p_{W L}=\frac{t_{B}^{r}}{1+t_{B}^{r}} . \tag{18}
\end{equation*}
$$

As to simultaneous contests, we leverage the findings of Nti (2004), Wang (2010), and Alcalde and Dahm (2010) to establish the following lemma.

Lemma 5. Consider a simultaneous Tullock contest in which players $A$ and $B$ compete for a unit prize, with the players' marginal costs denoted as $c_{A}$ and $c_{B}$, and $\tilde{c}=c_{A} / c_{B} \in(0,1]$. For any $r \in(0, \infty)$, the equilibrium winning probability of player $A$ is given by

$$
p_{\text {Simu }}= \begin{cases}\frac{1}{1+\tilde{c}^{r}} & \text { for } r \in(0, \bar{r}]  \tag{19}\\ 1-\frac{\tilde{c}}{r}(r-1)^{\frac{r-1}{r}} & \text { for } r \in[\bar{r}, 2] \\ 1-\frac{\tilde{c}}{2} & \text { for } r \in[2, \infty)\end{cases}
$$

where $\bar{r}$ is uniquely determined by

$$
\begin{equation*}
\tilde{c}^{\bar{r}}=\bar{r}-1, \tag{20}
\end{equation*}
$$

and it can be shown that $\bar{r} \in(1,2]$.
The following proposition states that the strong player (player $A$ ) has a smaller chance of winning only when the weak player (player $B$ ) is the leader who chooses to preempt in equilibrium; otherwise, the strong player (player $A$ ) always has a greater chance of winning, regardless of him being the leader or the follower.

Proposition 6. Consider the ranking order of player A's winning probabilities in the three contest formats, i.e., the strong-lead sequential, weak-lead sequential, and simultaneous contests, where player A's winning chances are denoted as $p_{S L}, p_{W L}$, and $p_{\text {Simu }}$, respectively. ${ }^{35}$ (i) When $\tilde{c} \in\left(0,1 / \hat{c}^{h}\right)$, where $\hat{c}^{h}=1.983$ (Lemma 4), weak-lead sequential contests have interior equilibria for all $r \in(0, \infty)$, and the ranking order is: $p_{S L}>p_{W L}>p_{\text {Simu }}>1 / 2$.
(ii) When $\tilde{c} \in\left[1 / \hat{c}^{h}, 1\right)$, weak-lead sequential contests have interior equilibria for $r \in\left(0, \hat{r}_{w}^{1}\right) \cup$ $\left(\hat{r}_{w}^{2}, \infty\right)$, in which the same ranking order is maintained: $p_{S L}>p_{W L}>p_{\text {Simu }}>1 / 2$.
(iii) When $\tilde{c} \in\left[1 / \hat{c}^{h}, 1\right)$, weak-lead sequential contests have preemptive equilibria for $r \in$

[^19]$\left[\hat{r}_{w}^{1}, \hat{r}_{w}^{2}\right]$, where $A$ is preempted, and the ranking order is: $p_{S L}>p_{\text {Simu }}>1 / 2>p_{W L}=0$.
(iv) When $\tilde{c}=1$, the two players are symmetric, the ranking orders are: $p_{S L}=p_{W L}=$ $p_{\text {Simu }}=1 / 2$ for $r \in\left(0, \hat{r}_{s}\right)$, and $p_{S L}=1>p_{\text {Simu }}=1 / 2>p_{W L}=0$ for $r \geq \hat{r}_{s}$, where $\hat{r}_{s} \approx 1.0789 .{ }^{36}$

When the two contestants are sufficiently asymmetric in ability (i.e., when $\tilde{c}$ is small), the strong player (player $A$ ) consistently maintains a dominant winning probability for all values of $r$. Notably, the strong player's winning probability in the strong-lead sequential contest is the highest among the three (strong-lead, weak-lead, and simultaneous) contest formats. Intuitively, serving as the first mover in a strong-lead sequential contest compels the strong player, who has an ability advantage, to adopt a more aggressive strategy, aiming to suppress the weak player who moves second. It is intriguing to observe that, even in the weak-lead sequential contest, the strong player's winning probability surpasses that in the simultaneous contest. In this scenario, where the players exhibit significant asymmetry, the weak leader, who has an ability disadvantage, always refrains from preempting the strong follower; instead, it is in the weak leader's best interest to play less aggressively in the first stage, aiming to mitigate the strong follower's response in the second stage.

When the ability levels of the two contestants are sufficiently close (i.e., when $\tilde{c}$ is large), the weak player chooses to preempt the strong follower in a weak-lead sequential contest for $r$ in a preemptive interval. In these preemptive equilibria, the weak player wins the contest with certainty, and the strong player's winning probability is zero (i.e., $p_{W L}=0$ ). Meanwhile, in an interior equilibrium where the weak leader does not choose to preempt the strong follower, the logic from the preceding paragraph still applies, and it remains the case that $p_{S L}>p_{W L}>p_{\text {Simu }}$. Therefore, in a setting with two asymmetric contestants, we obtain the following corollary.

Corollary 1. Consider a setting with two asymmetric players, i.e., with $\tilde{c} \in(0,1)$. Let the coaches of player $A$ and $B$ be denoted as coaches $A$ and $B$ respectively, and each coach aims to maximize the winning chance of his contestant:
(i) Coach A always prefers the strong-lead sequential contest among the three contest formats (namely, the strong-lead sequential, weak-lead sequential, and simultaneous contests).

[^20](ii) Coach B prefers the weak-lead sequential contest only if the contest's accuracy level falls within the preemptive interval (this interval disappears when the two players are sufficiently asymmetric, according to Theorem 4); otherwise, coach B prefers the simultaneous contest among the three contest formats.

When the two players are symmetric in ability, neither of them has an ability advantage. For $r<1$, only interior equilibria exist. In a sequential contest, compared to a simultaneous contest, a leader has no incentive to play more or less aggressively, as there is no ability advantage or disadvantage. In other words, the leader exerts the same level of effort in a sequential contest as in a simultaneous contest. However, for $r \geq 1$, a leader in a sequential contest compares two solutions, interior and preemptive, and chooses the one with a higher payoff. It turns out that when the contest's accuracy level is sufficiently high ( $r \geq \hat{r}_{s} \approx$ 1.0789), the leader always chooses to preempt the follower in a sequential contest, resulting in the leader's winning chance being one and the follower's winning chance being zero.

Corollary 2. Consider a setting with two symmetric players, i.e., with $\tilde{c}=1$ : each coach, aiming to maximize his contestant's winning chance, is indifferent between the three contest formats for $r<\hat{r}_{s}$, where $\hat{r}_{s} \approx 1.0789$ (Theorem 3), as the two players' equilibrium effort levels are the same across all contest formats. For $r \geq \hat{r}_{s}$, preemptive equilibria emerge in sequential contests; in this case, each coach prefers the sequential contest where his contestant leads and wins the contest with certainty, as the opponent is preempted in equilibrium.

With the objective of maximizing a particular player's winning chance, Corollaries 1 and 2 imply the following results. The strong player always has a first-mover advantage, in the sense that his winning chance in the strong-lead sequential contest is the highest among all three contest formats. In contrast, the weak player has a first-mover advantage only if the equilibrium is preemptive in the weak-lead sequential contest; when the equilibrium in the weak-lead sequential contest is interior, the weak player prefers the simultaneous contest. In sum, while the strong player always has a first-mover advantage, neither player has a second-mover advantage in all cases.

Corollaries 1 and 2 reveal how a dictator would choose the move order of the players if $\mathrm{s} / \mathrm{he}$ aims to maximize a given contestant's winning chance. Next, we investigate a situation in which the move order of the contestants is rather determined jointly by the coaches of the two players through the following non-cooperative game.

Consider a two-stage contest game between two teams, $A$ and $B$, where each team consists of a coach and a contestant. The contestants have asymmetric abilities, and their cost ratio is denoted by $\tilde{c}$. The contest has an accuracy level denoted by $r$. In stage 1 , namely the "coach-decision stage," the coaches independently and simultaneously choose a strategy between two actions: Lead and Follow, in order to maximize their own team's winning chances. Subsequently, in stage 2, named the "contest stage", the contestants compete in a contest where their move order is determined by the outcome in the coach-decision stage a sequential-move contest occurs only when one coach chooses Lead and the other coach chooses Follow; otherwise, a simultaneous-move contest ensues.

The equilibria in the coach-decision stage are presented in the following theorem. Recall that we assume player $A$ is stronger than player $B$, with $\tilde{c}=c_{A} / c_{B} \leq 1$.

Theorem 5. (i) When $\tilde{c} \in\left(0,1 / \hat{c}^{h}\right)$, for any $r$, there is a unique mixed-strategy equilibrium in the coach-decision stage game, in which coach A randomizes between the two actions, Lead and Follow, with respective probabilities $x$ and $1-x$, while coach $B$ randomizes with respective probabilities $y$ and $1-y$, where

$$
\begin{equation*}
x=\frac{p_{W L}-p_{\text {Simu }}}{p_{S L}+p_{W L}-2 p_{\text {Simu }}} \text { and } y=\frac{p_{S L}-p_{\text {Simu }}}{p_{S L}+p_{W L}-2 p_{\text {Simu }}} . \tag{21}
\end{equation*}
$$

Thus, the three contest formats (simultaneous, strong-lead sequential, and weak-lead sequential) occur in equilibrium with positive probability in stage 2.
(ii) When $\tilde{c} \in\left[1 / \hat{c}^{h}, 1\right)$ and $r \in\left(0, \hat{r}_{w}^{1}\right) \cup\left(\hat{r}_{w}^{2}, \infty\right)$, weak-lead sequential contests have interior equilibria. In stage 1, there is a unique mixed-strategy equilibrium, the same as in (i). Thus, any of the three contest formats can emerge in stage 2.
(iii) When $\tilde{c} \in\left[1 / \hat{c}^{h}, 1\right)$ and $r \in\left[\hat{r}_{w}^{1}, \hat{r}_{w}^{2}\right]$, weak-lead sequential contests have preemptive equilibria. In stage 1, there is a unique pure-strategy equilibrium where both coaches choose Lead. Thus, a simultaneous contest is the only contest format in stage 2.
(iv) When $\tilde{c}=1$, players $A$ and $B$ are symmetric in ability, the coaches choose arbitrary randomization between Lead and Follow for $r \in\left(0, \hat{r}_{s}\right)$, where $\hat{r}_{s} \approx 1.0789$, and any of the three contest formats can occur in stage 2; while for $r \geq \hat{r}_{s}$, both coaches choose Lead in a unique pure-strategy equilibrium in stage 1, leading to a simultaneous contest in stage 2.

The above theorem indicates that in a two-stage game, where the move order of the contestants in stage 2 is determined by the outcome of stage 1 , in various competitive envi-
ronments described by different pairs of $\tilde{c}$ and $r$, there is either a mixed-strategy equilibrium in which players randomize between the two actions (Lead and Follow), allowing all three contest formats to occur as an equilibrium outcome of the winning-odd-maximizing coaches' independent choices, or a pure-strategy equilibrium in which both players choose Lead, resulting in only the simultaneous contest format.

When the contestants' asymmetry level is sufficiently high, we always have a mixedstrategy equilibrium for any $r$. In these cases, it is optimal for the strong player to choose a different strategy as his opponent (the weak player) does; while it is optimal for the weak player to choose the same strategy as his opponent (the strong player) does. Given the above best responses of the two players, a pure-strategy equilibrium is never possible. Additionally, we can derive that in the unique mixed-strategy equilibrium, the strong player chooses Lead with a higher probability (since $y>x$ in Theorem 5), implying that a strong-lead sequential contest is more likely to occur compared to a weak-lead sequential contest in equilibrium.

Conversely, when the contestants' asymmetry level is sufficiently low, we will have a pure-strategy equilibrium for $r$ in a preemptive interval (and this interval gets larger when the contestants' asymmetry level gets lower). ${ }^{37}$ In these cases, the weak player is strong enough to preempt the strong player in a weak-lead sequential contest, so choosing Lead is a dominant strategy for either player, leading to a simultaneous contest with certainty.

## 6 Concluding remarks

In this paper, we start by fully characterizing equilibria in a general two-player sequential Tullock contest, allowing the two players to have arbitrarily asymmetric ability levels, with $c \in(0, \infty)$, and considering the entire range of contest accuracy levels, with $r \in(0, \infty)$. Incorporating the full ranges of $c$ and $r$ presents technical challenges in searching for equilibrium solutions in sequential contests. To overcome this challenge, we derive a nonlinear higher-order characteristic equation based on the first-order conditions of the players in interior solutions. This equation plays a crucial role in characterizing interior equilibria, given the fact that explicit interior solutions are not available in general. ${ }^{38}$

Our examination of the general sequential contest model (Section 4), along with our ex-

[^21]ploration of the relationship between the move order of the contestants and their winning chances (Section 5), contributes novel findings and new insights to the existing literature. For instance, prior literature that focuses on the case of $r=1$ suggests that the leader has the incentive to preempt the follower only when the leader is significantly stronger than the follower, implying that a weak leader will never preempt the strong follower in equilibrium. However, we show that a weak leader will preempt a strong follower when the contest's accuracy level falls within a preemptive interval. Conversely, we find that the strong player's winning chance in a strong-lead sequential contest is the highest among the three contest formats, while the weak player's winning chance in a simultaneous contest is larger than that in a weak-lead sequential contest when the equilibrium is interior. These investigations further allow us to study how different move orders of contestants are endogenized by winning-odd-maximizing coaches' independent choices as equilibrium outcomes.

As an application of the equilibrium analysis, we investigate the optimal move order of the contestants that maximizes a player's winning chance in this paper (Section 5). A natural and closely related question concerns the optimal move order that maximizes the total effort of the contestants. Due to space constraints, we provide a detailed analysis of this question in a separate paper (Gao, Lu, and Wang, 2024). ${ }^{39}$

[^22]
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## A Appendix: The Proofs

## A. 1 Proof of Lemma 1

Proof. In the first paragraph of Section 4.1, we have shown that $x_{L}=0$ cannot occur in any equilibrium. Thus, we analyze the follower's best response to the leader's effort $x_{L}>0$. To facilitate the analysis, given $x_{L}>0$, we define the follower's marginal benefit of increasing his effort $x_{F}$ as $\alpha\left(x_{F} ; x_{L}\right)$, where

$$
\begin{equation*}
\alpha\left(x_{F} ; x_{L}\right)=r \frac{x_{F}^{r-1} x_{L}^{r}}{\left(x_{L}^{r}+x_{F}^{r}\right)^{2}} \tag{22}
\end{equation*}
$$

Equations (2) and (22) imply that $\frac{\partial \pi_{F}}{\partial x_{F}}=\alpha\left(x_{F} ; x_{L}\right)-c_{F}$.
We first consider the case in which $r<1$. With $x_{L}>0$, we can derive that $\alpha\left(x_{F} ; x_{L}\right)$ (and $\frac{\partial \pi_{F}}{\partial x_{F}}=\alpha-c_{F}$ ) always decreases with $x_{F}$, for $x_{F} \in(0, \infty)$. In particular, $\alpha\left(x_{F} ; x_{L}\right)$ goes to infinity when $x_{F}$ goes to zero, and $\alpha\left(x_{F} ; x_{L}\right)$ goes to zero when $x_{F}$ goes to infinity. Thus, for given values of $x_{L}>0$ and $c_{F}>0$, there must be a unique solution to equation $\alpha\left(x_{F} ; x_{L}\right)=c_{F}$. Let $x_{F}^{*}$ be the unique solution of $\alpha\left(x_{F} ; x_{L}\right)=c_{F}$. For a given $x_{L}>0$, the follower's best response $x_{F}\left(x_{L}\right)$ is simply equal to $x_{F}^{*}$.

Next, we show that $\forall x_{L}>0$, the follower's payoff at the unique global maximum, denoted as $\pi_{F}\left(x_{F}^{*}\left(x_{L}\right) ; x_{L}\right)$, strictly decreases in $x_{L}$. To see this, using the envelope theorem, we have

$$
\begin{aligned}
\frac{d \pi_{F}\left(x_{F}^{*}\left(x_{L}\right) ; x_{L}\right)}{d x_{L}} & \left.=\frac{\partial \pi_{F}\left(x_{F} ; x_{L}\right)}{\partial x_{L}} \right\rvert\, x_{F}=x_{F}^{*}\left(x_{L}\right) \\
& =-\frac{x_{F}^{r}}{\left(x_{L}^{r}+x_{F}^{r}\right)^{2}} r x_{L}^{r-1}<0
\end{aligned}
$$

It is clear that $\lim _{x_{L} \rightarrow 0^{+}} \pi_{F}\left(x_{F}^{*}\left(x_{L}\right) ; x_{L}\right)>0$. Despite the fact that $\pi_{F}\left(x_{F}^{*}\left(x_{L}\right) ; x_{L}\right)$ decreases with $x_{L}$, we show that $\pi_{F}\left(x_{F}^{*}\left(x_{L}\right) ; x_{L}\right)>0, \forall x_{L}>0$. To see this, using the fact that $x_{F}^{*}$ is the unique solution of equation $c_{F}=\alpha\left(x_{F} ; x_{L}\right)$, we derive that $c_{F} x_{F}^{*}=r\left(\frac{x_{F}^{* r}}{x_{L}^{r}+x_{F}^{* r}}\right)\left(\frac{x_{L}^{r}}{x_{L}^{r}+x_{F}^{* r}}\right)$. Substituting it into (2), we obtain that for $r<1$,

$$
\begin{aligned}
& \pi_{F}\left(x_{F}^{*} ; x_{L}\right)=\frac{x_{F}^{* r}}{x_{L}^{r}+x_{F}^{* r}}-c_{F} x_{F}^{*}=\frac{x_{F}^{* r}}{x_{L}^{r}+x_{F}^{* r}}\left(1-r \frac{x_{L}^{r}}{x_{L}^{r}+x_{F}^{* r}}\right) \\
& \quad>\frac{x_{F}^{* r}}{x_{L}^{r}+x_{F}^{* r}}\left(1-\frac{x_{L}^{r}}{x_{L}^{r}+x_{F}^{* r}}\right) \\
& \quad=\left(\frac{x_{F}^{* r}}{x_{L}^{r}+x_{F}^{* r}}\right)^{2}>0
\end{aligned}
$$

in which the first inequality holds since $r<1$. Notice that $\pi_{F}\left(x_{F}^{*} ; x_{L}\right)>0$ for any $x_{L}>0$ means that for $r<1$, it is impossible for the leader to preempt the follower by exerting any finite amount of effort. Thus, there exist no preemptive solutions for $r<1$.

We now move on to the case in which $r>1$. We first show that with $r>1, \frac{\partial \pi_{F}}{\partial x_{F}}=$ $\alpha\left(x_{F} ; x_{L}\right)-c_{F}$ is inverse U-shaped on $x_{F} \in[0, \infty)$ with an interior maximum at $\breve{x}_{F}\left(x_{L}\right)>0$. Using (22), we derive that

$$
\begin{aligned}
\frac{\partial^{2} \pi_{F}}{\partial x_{F 2}} & =\frac{\partial \alpha}{\partial x_{F}}=r x_{L}^{r} \frac{(r-1) x_{F}^{r-2}}{\left(x_{L}^{r}+x_{F}^{r}\right)^{2}}-r x_{L}^{r} 2 \frac{x_{F}^{r-1} r x_{F}^{r-1}}{\left(x_{L}^{r}+x_{F}^{r}\right)^{3}} \\
& =r x_{L}^{r} \frac{x_{F}^{r-2}}{\left(x_{L}^{r}+x_{F}^{r}\right)^{3}}\left[(r-1) x_{L}^{r}-(r+1) x_{F}^{r}\right] .
\end{aligned}
$$

Thus, we can derive that $\frac{\partial \alpha}{\partial x_{F}}>0$ if $x_{F}<\breve{x}_{F}\left(x_{L}\right), \frac{\partial \alpha}{\partial x_{F}}=0$ if $x_{F}=\breve{x}_{F}\left(x_{L}\right)$, and $\frac{\partial \alpha}{\partial x_{F}}<0$ if $x_{F}>\breve{x}_{F}\left(x_{L}\right)$, where

$$
\breve{x}_{F}\left(x_{L}\right)=\left(\frac{r-1}{r+1}\right)^{\frac{1}{r}} x_{L}>0
$$

In other words, $\alpha\left(x_{F} ; x_{L}\right)$ (and $\frac{\partial \pi_{F}}{\partial x_{F}}$ ) is inverse U -shaped on $x_{F} \in[0, \infty)$ with an interior maximum $\breve{x}_{F}\left(x_{L}\right)>0$. Also, it is clear that $\frac{\partial}{\partial x_{F}} \pi_{F}\left(x_{F}=0 ; x_{L}\right)=\alpha\left(x_{F}=0 ; x_{L}\right)-c_{F}=$ $-c_{F}<0$ and $\frac{\partial}{\partial x_{F}} \pi_{F}\left(x_{F}=\breve{x}_{F}\left(x_{L}\right) ; x_{L}\right)=\alpha\left(\breve{x}_{F}\left(x_{L}\right) ; x_{L}\right)-c_{F}>-c_{F}$, since $\alpha\left(x_{F}=0 ; x_{L}\right)=0$ and $\alpha\left(\breve{x}_{F}\left(x_{L}\right) ; x_{L}\right)>0$.

It can be further shown that $\frac{\partial}{\partial x_{F}} \pi_{F}\left(\breve{x}_{F}\left(x_{L}\right) ; x_{L}\right)=\alpha\left(\breve{x}_{F}\left(x_{L}\right) ; x_{L}\right)-c_{F}$ strictly decreases in $x_{L}, \lim _{x_{L} \rightarrow 0^{+}} \alpha\left(\breve{x}_{F}\left(x_{L}\right) ; x_{L}\right)=+\infty$, and $\lim _{x_{L} \rightarrow \infty} \alpha\left(\breve{x}_{F}\left(x_{L}\right) ; x_{L}\right)=0$, since $\alpha\left(\breve{x}_{F}\left(x_{L}\right) ; x_{L}\right)$ decreases with $x_{L}$, where

$$
\alpha\left(\breve{x}_{F}\left(x_{L}\right) ; x_{L}\right)=\left(\frac{r}{x_{L}}\right)\left(\frac{r+1}{2 r}\right)^{2}\left(\frac{r-1}{r+1}\right)^{\frac{r-1}{r}} .
$$

Let

$$
\bar{x}_{L}=\left(\frac{r}{c_{F}}\right)\left(\frac{r+1}{2 r}\right)^{2}\left(\frac{r-1}{r+1}\right)^{\frac{r-1}{r}}
$$

such that $\frac{\partial}{\partial x_{F}} \pi_{F}\left(\breve{x}_{F}\right)=0$ at $x_{L}=\bar{x}_{L}$. From the above results and the FOC for an interior local optimum of the follower, we can obtain the following results: when $x_{L} \in\left(0, \bar{x}_{L}\right)$, there always exist two threshold values, denoted $x_{F 1}$ and $x_{F 2}$, where $0<x_{F 1}<x_{F 2}$, which are the two solutions to equation $\frac{\partial \pi_{F}}{\partial x_{F}}=\alpha\left(x_{F} ; x_{L}\right)-c_{F}=0$, such that the follower's expected payoff $\pi_{F}\left(x_{F}\right)$-starting from $\pi_{F}\left(x_{F}\right)=0$ at $x_{F}=0$, which is always a local maximum of $\pi_{F}\left(x_{F}\right)$-first decreases with $x_{F}$ for $x_{F} \in\left[0, x_{F 1}\right]$, then increases with $x_{F}$ for $x_{F} \in\left(x_{F 1}, x_{F 2}\right]$, and decreases with $x_{F}$ for $x_{F} \in\left(x_{F 2}, \infty\right)$-i.e., $\pi_{F}\left(x_{F}\right)$ reaches its unique local minimum $\pi_{F}\left(x_{F 1}\right)<0$ at $x_{F}=x_{F 1}$, and $\pi_{F}\left(x_{F}\right)$ reaches its unique local maximum $\pi_{F}\left(x_{F 2}\right)$ at $x_{F}=$
$x_{F 2}$. In this case, with the other (corner) local maximum being $\left(\pi_{F}(0)=0, x_{F}=0\right)$, the interior local maximum $\left(\pi_{F}\left(x_{F 2}\right), x_{F 2}\right)$ is a unique global maximum if and only if $\pi_{F}\left(x_{F 2}\right)>0$. When $x_{L} \in\left[\bar{x}_{L}, \infty\right), \pi_{F}\left(x_{F}\right)$ always decreases with $x_{F}$, for any $x_{F} \geq 0$.

Formally, we summarize the results obtained so far as follows: (1) when $x_{L} \in\left(0, \bar{x}_{L}\right)$, $\pi_{F}\left(x_{F} ; x_{L}\right)$ has a unique interior local maximum $x_{F 2}\left(x_{L}\right)>\breve{x}_{F}\left(x_{L}\right)$ and $\lim _{x_{L} \rightarrow \bar{x}_{L}^{-}} x_{F 2}\left(x_{L}\right)=$ $\breve{x}_{F}\left(\bar{x}_{L}\right) ;(2)$ when $x_{L} \geq \bar{x}_{L}, \pi_{F}\left(x_{F} ; x_{L}\right)$ has no interior local maximum, and thus $\pi_{F}\left(x_{F} ; x_{L}\right)$ decreases with $x_{F} ;(3) \pi_{F}\left(x_{F} ; x_{L}\right)$ always has a local maximum at $x_{F}=0$. Also, it is clear that the leader's payoff decreases in $x_{L}$ when $x_{L} \geq \bar{x}_{L}$, since the follower best response to the leader's effort $x_{L} \geq \bar{x}_{L}$ is always zero. Therefore, we conclude that in any equilibrium, either interior or preemptive, the leader's effort must fall in $\left(0, \bar{x}_{L}\right)$. We thus focus on $x_{L} \in\left(0, \bar{x}_{L}\right)$ in the following equilibrium analysis.

In the case with $r>1$, we show the following result using a similar method (adopted in the case with $r<1): \forall x_{L} \in\left(0, \bar{x}_{L}\right)$, the follower's payoff at the unique interior local maximum $\pi_{F}\left(x_{F 2}\left(x_{L}\right) ; x_{L}\right)$ strictly deceases in $x_{L}$. Moreover, $\lim _{x_{L} \rightarrow \bar{x}_{L}} \pi_{F}\left(x_{F 2}\left(x_{L}\right) ; x_{L}\right)<0$, and $\lim _{x_{L} \rightarrow 0^{+}} \pi_{F}\left(x_{F 2}\left(x_{L}\right) ; x_{L}\right)>0$. To see this, by the envelope theorem, we have $\frac{d \pi_{F}\left(x_{F 2}\left(x_{L}\right) ; x_{L}\right)}{d x_{L}}<$ 0 . From continuity of $\pi_{F}\left(x_{F 2}\left(x_{L}\right) ; x_{L}\right)$ and $x_{F 2}\left(x_{L}\right)$, it can be derived that

$$
\lim _{x_{L} \rightarrow \bar{x}_{L}^{-}} \pi_{F}\left(x_{F 2}\left(x_{L}\right) ; x_{L}\right)=\pi_{F}\left(\breve{x}_{F}\left(\bar{x}_{L}\right) ; \bar{x}_{L}\right),
$$

which is strictly smaller than $\pi_{F}\left(x_{F}=0 ; \bar{x}_{L}\right)=0$. Clearly, $\lim _{x_{L} \rightarrow 0^{+}} \pi_{F}\left(x_{F 2}\left(x_{L}\right) ; x_{L}\right)>0$.
Define $\hat{x}_{L} \in\left(0, \bar{x}_{L}\right)$ such that $\pi_{F}\left(x_{F 2}\left(\hat{x}_{L}\right) ; \hat{x}_{L}\right)=0$. The expression of $\hat{x}_{L}$, which is given by (6), is obtained using (2) and (5) with $\pi_{F}=0$. Recall that we have shown that for $x_{L} \in\left(0, \bar{x}_{L}\right)$, the follower's expected payoff at its unique interior local maximum-i.e., $\pi_{F}\left(x_{F 2}\right)$ - always decreases when $x_{L}$ gets larger. We further obtain that $\pi_{F}\left(x_{F 2}\left(x_{L}\right) ; x_{L}\right)>0$ when $x_{L}<\hat{x}_{L}$, and $\pi_{F}\left(x_{F 2}\left(x_{L}\right) ; x_{L}\right)<0$ when $x_{L}>\hat{x}_{L}$. In this paper, without loss of generality, we assume that the follower will pick zero effort if he is indifferent between exerting zero effort or strictly positive effort. The above implies that the follower's best response is $x_{F}\left(x_{L}\right)=x_{F 2}\left(x_{L}\right)>0$ if $x_{L}<\hat{x}_{L}$, and $x_{F}\left(x_{L}\right)=0$ if $x_{L} \geq \hat{x}_{L}$. Thus, $\hat{x}_{L}$ is the minimum preemptive effort of the leader. When $x_{L} \in\left[\hat{x}_{L}, \bar{x}_{L}\right)$, we still have an interior local maximum at $x_{F 2}$, with $\pi_{F}\left(x_{F 2}\right) \leq \pi_{F}(0)=0$, which implies that $\pi_{F}\left(x_{F}\right) \leq 0$ for any $x_{F} \geq 0$-i.e., the follower's best response is always zero; when $x_{L} \in\left[\bar{x}_{L}, \infty\right), \pi_{F}\left(x_{F}\right)$ always decreases in $x_{F}$, which also implies that the follower's best response is zero effort.

Lastly, we consider the case in which $r=1$. It is straightforward to show that $\partial \pi_{F} / \partial x_{F}=$ $x_{L} /\left(x_{L}+x_{F}\right)^{2}-c_{F}$ and

$$
\frac{\partial^{2} \pi_{F}}{\partial x_{F 2}}=\frac{\partial \alpha}{\partial x_{F}}=\frac{-2 x_{L}}{\left(x_{L}^{r}+x_{F}^{r}\right)^{3}}<0 .
$$

In this case, the follower's best response is simply that $x_{F}\left(x_{L}\right)=\sqrt{x_{L} / c_{F}}-x_{L}$ for $x_{L} \in$
$\left(0,1 / c_{F}\right]$, and $x_{F}\left(x_{L}\right)=0$ for $x_{L}>1 / c_{F}$. Thus, in any equilibrium, it must be the case that $x_{L} \in\left(0,1 / c_{F}\right]$. Note that by (6), we have $\hat{x}_{L}=1 / c_{F}$ at $r=1$, which implies that the relevant results in part (ii) with $r=1$ are correct.

## A. 2 Further analysis of equilibrium candidates

In the proof of Lemma 1, we show that when $r<1$, given any $x_{L}>0$, there always exists a unique value of $x_{F}$, denoted by $x_{F}^{*}>0$, such that the follower's expected payoff $\pi_{F}\left(x_{F}\right)$-starting from $\pi_{F}\left(x_{F}\right)=0$ at $x_{F}=0$-first increases with $x_{F}$ for $x_{F} \leq x_{F}^{*}$ and then decreases with $x_{F}$ for $x_{F} \geq x_{F}^{*}$. This implies that $\pi_{F}\left(x_{F}\right)$ reaches its unique global maximum at $x_{F}=x_{F}^{*}$. In this case, the follower's best response $x_{F}\left(x_{L}\right)$ is simply $x_{F}^{*}$. Because of $\pi_{F}\left(x_{F}=0\right)=0$, we obtain that $\pi_{F}\left(x_{F}^{*}\right)>0$ for certain. The above case with $r<1$ and $x_{L}>0$ is illustrated by Figure A.1.


Figure A.1: Follower's expected payoff when $r<1$ and $c<1$. Here it is illustrated by $r=0.8$ and $c=0.8$.

When $r>1$, the follower's expected payoff varies with $x_{F}$ in a more complex fashion. Specifically, in the proof of Lemma 1, we show that when $r>1$, given that $x_{L}$ is not sufficiently large in the sense that $x_{L} \in\left(0, \bar{x}_{L}\right)$, there always exist two threshold values, denoted by $x_{F 1}$ and $x_{F 2}$, where $0<x_{F 1}<x_{F 2}$, which are the two solutions to equation (5) (i.e., the follower's first-order condition) such that the follower's expected payoff $\pi_{F}\left(x_{F}\right)$ —starting from $\pi_{F}\left(x_{F}\right)=0$ at $x_{F}=0$, which is always a (corner) local maximum of $\pi_{F}\left(x_{F}\right)$-first decreases with $x_{F}$ for $x_{F} \in\left[0, x_{F 1}\right]$, then increases with $x_{F}$ for $x_{F} \in\left(x_{F 1}, x_{F 2}\right]$, and decreases with $x_{F}$ for $x_{F} \in\left(x_{F 2}, \infty\right)$-i.e., $\pi_{F}\left(x_{F}\right)$ reaches its unique local minimum $\pi_{F}\left(x_{F 1}\right)<0$ at
$x_{F}=x_{F 1}$, and reaches its unique interior local maximum at $x_{F}=x_{F 2}$. In this case, with the other local maximum being a corner solution, $\left(\pi_{F}(0)=0, x_{F}=0\right)$, the interior local maxi$\operatorname{mum}\left(\pi_{F}\left(x_{F 2}\right), x_{F 2}\right)$ is a unique global maximum if and only if $\pi_{F}\left(x_{F 2}\right)>0$. The above case is illustrated in Figure A. 2 (a). In the proof, we further show that the follower's expected payoff at $x_{F}=x_{F 2}$-i.e., $\pi_{F}\left(x_{F 2}\right)$-always decreases when $x_{L}$ gets larger. This implies that there exists a critical value of $x_{L}$, denoted by $\hat{x}_{L}$, where $\hat{x}_{L}<\bar{x}_{L}$ such that the follower's best response is $x_{F}\left(x_{L}\right)=0$ if $x_{L} \geq \hat{x}_{L}$, and $x_{F}\left(x_{L}\right)=x_{F 2}$ if $x_{L}<\hat{x}_{L}$. The case with $x_{L}=\hat{x}_{L}$ is illustrated by Figure A. 2 (b). When $x_{L} \in\left[\hat{x}_{L}, \bar{x}_{L}\right.$ ), there are still two local maximums, one corner and the other interior, where $\pi_{F}\left(x_{F 2}\right) \leq \pi_{F}(0)=0$, which means that $\pi_{F}\left(x_{F}\right) \leq 0$ for any $x_{F} \geq 0$. When $x_{L} \in\left[\bar{x}_{L}, \infty\right), \pi_{F}\left(x_{F}\right)$ always decreases with $x_{F}$ for any $x_{F} \geq 0$. The case with $x_{L} \in\left[\bar{x}_{L}, \infty\right)$ is illustrated by Figure A. 2 (c). To sum up, the follower's best response is $x_{F}\left(x_{L}\right)=x_{F 2}$ if $x_{L}<\hat{x}_{L}$, and $x_{F}\left(x_{L}\right)=0$ if $x_{L} \geq \hat{x}_{L}$, where $x_{F 2}$ is the larger of the two solutions of equation (5).


Figure A.2: Follower's expected payoff when $r>1$. Here it is illustrated by $r=1.8, c=1$, and from left to right $x_{L}=0.4, x_{L}=\hat{x}_{L} \approx 0.504$, and $x_{L}=0.56$, respectively.

Expecting the follower's best response in stage $2, x_{F}\left(x_{L}\right)$, where $x_{L} \in\left(0, \hat{x}_{L}\right]$, the leader chooses $x_{L}$ in stage 1 to maximize his expected payoff (4). Using Lemma 1, which characterizes the follower's best response to the leader's effort, we analyze the leader's optimal choice of $x_{L}$. The following results are obtained.

Claim 1. Consider an arbitrary equilibrium candidate, either an interior solution or a preemptive solution, in which the leader exerts effort $x_{L}>0$ and the follower exerts effort $x_{F} \geq 0$. (i) When $r<1$, there exist no preemptive solutions; when interior solutions exist, we have $x_{L} \in\left(0, \frac{1}{c_{L}}\right)$. (ii) When $r \geq 1$, there exist preemptive solutions in which the leader exerts minimum preemptive effort $x_{L}=\hat{x}_{L}$ and the follower's best response is $x_{F}\left(\hat{x}_{L}\right)=0$,
provided that the leader's expected payoff is positive-i.e., $\pi_{L}^{\text {pre }}=1-c_{L} \hat{x}_{L} \geq 0$; when interior solutions exist, we have $x_{L}<\hat{x}_{L}$ and $x_{F}\left(x_{L}\right)>0$. In summary, when $r \geq 1$, we have $x_{L} \in\left(0, \hat{x}_{L}\right]$ in any equilibrium.

Proof of Claim 1. We first consider the case with $r<1$. In the proof of Lemma 1, we have shown that the follower's best response $x_{F}\left(x_{L}\right)$ is always strictly positive for any $x_{L}>0$, which implies that there exist no preemptive solutions (and thus no preemptive equilibria) in which the follower is preempted by the leader. With the unit prize, the leader has no incentive to bid more than $1 / c_{L}$, because the leader's probability of winning the contest is always strictly smaller than 1 with $x_{F}\left(x_{L}\right)>0, \forall x_{L}>0$. Thus, we have $x_{L} \in\left(0,1 / c_{L}\right)$.

Next, we move to the case where $r \geq 1$. In the proof of Lemma 1, we have shown that when $r \geq 1$, there always exists a preemptive solution in which the leader exerts effort $\hat{x}_{L}$, which is given by (6), and the follower exerts zero effort. Certainly, there may exist an interior solution, in which the leader exerts effort $x_{L}<\hat{x}_{L}$ and the follower exerts effort $x_{F}\left(x_{L}\right)>0$. In any equilibrium, either preemptive or interior, the leader's effort must fall in ( $0, \hat{x}_{L}$ ], where $\hat{x}_{L} \in\left(0, \bar{x}_{L}\right)$, because the follower's best response to any $x_{L}$ greater than $\hat{x}_{L}$ is always zero, and thus the leader's expected payoff decreases with $x_{L}$ when $x_{L} \geq \hat{x}_{L}$.

Claim 1 (i) says that when $r<1$ there exist no preemptive solutions, which further implies that there exists no preemptive equilibrium. Claim 1 (ii) states that when $r \geq 1$, there exist preemptive solutions in which the leader exerts effort $\hat{x}_{L}$ and the follower is preempted, as long as the leader's participation constraint holds.

We now verify that, when an interior solution exists for given values of $c$ and $r$, this interior solution is unique and it must be a pure-strategy solution. Therefore, we only need to focus on the construction of pure-strategy interior solutions.

Claim 2. Suppose an interior solution exists for given values of $c$ and $r$, then the interior solution is unique, in which every player adopts a pure strategy. In solving for interior solutions: (i) the restriction of the leader's first-order condition guarantees that the solution of the follower's first-order condition is the follower's local optimum; (ii) the leader's firstorder condition has either one or two solutions, which can be denoted by $x_{L 1}$ and $x_{L 2}$ with $0<x_{L 1} \leq x_{L 2}$, and $x_{L 2}$ is ruled out for a valid interior solution when $x_{L 2}>x_{L 1}$.

Proof of Claim 2. As shown in the proof of Lemma 1, the first- and second-order derivatives of $\pi_{F}$ with respect to $x_{F}$, treating $x_{L}$ as a constant, are given by

$$
\frac{d \pi_{F}}{d x_{F}}=\frac{r x_{F}^{r-1} x_{L}^{r}}{\left(x_{L}^{r}+x_{F}^{r}\right)^{2}}-c_{F},
$$

and

$$
\frac{d^{2} \pi_{F}}{d x_{F}^{2}}=\frac{r x_{F}^{r-2} x_{L}^{r}}{\left(x_{L}^{r}+x_{F}^{r}\right)^{3}}\left[(r-1) x_{L}^{r}-(r+1) x_{F}^{r}\right] .
$$

In an interior solution (when it exists), the first-order derivative is zero and the second-order derivative should be negative. It can be derived that for general values of $x_{L}$ and $x_{F}$, when $r \leq 1$, the solution of the FOC must be the follower's local maximum. However, when $r>1$, its solution could be either a local minimum or a local maximum.

Using equation (5), the follower's best response $x_{F}$ can be treated as a function of $x_{L}-$ i.e., $x_{F}\left(x_{L}\right)$. Notice that equation (5) defines a one-dimensional smooth manifold of variable $x_{L}$ and $x_{F}$, and the derivative $d x_{F} / d x_{L}$ is piece-wisely well defined. Taking the derivative of both sides of (5) with respect to variable $x_{L}$ and using the formula

$$
\begin{equation*}
\frac{d \cdot}{d x_{L}}=\frac{\partial \cdot}{\partial x_{L}}+\frac{\partial \cdot}{\partial x_{F}} \frac{d x_{F}}{d x_{L}}, \tag{23}
\end{equation*}
$$

we derive that

$$
r(r-1) x_{F}^{r-2} x_{L}^{r} \frac{d x_{F}}{d x_{L}}+r^{2} x_{F}^{r-1} x_{L}^{r-1}=2 c_{F}\left(x_{L}^{r}+x_{F}^{r}\right)\left(r x_{L}^{r-1}+r x_{F}^{r-1} \frac{d x_{F}}{d x_{L}}\right) .
$$

Multiplying both sides of the above equation by $\left(x_{L}^{r}+x_{F}^{r}\right)$ and using (5) to get rid of $c_{F}$ yields

$$
\left(r(r-1) x_{F}^{r-2} x_{L}^{r} \frac{d x_{F}}{d x_{L}}+r^{2} x_{F}^{r-1} x_{L}^{r-1}\right)\left(x_{L}^{r}+x_{F}^{r}\right)=2 r x_{F}^{r-1} x_{L}^{r}\left(r x_{L}^{r-1}+r x_{F}^{r-1} \frac{d x_{F}}{d x_{L}}\right),
$$

which further implies that

$$
\begin{equation*}
\frac{d x_{F}}{d x_{L}}=\frac{r x_{F}\left(x_{L}^{r}-x_{F}^{r}\right)}{x_{L}\left((r-1) x_{L}^{r}-(r+1) x_{F}^{r}\right)}, \tag{24}
\end{equation*}
$$

where the RHS is a function of variable $x_{L}$ only.
Next, we set up the FOC of the leader's optimization problem:

$$
r x_{L}^{r-1} x_{F}^{r-1}\left(x_{F}-x_{L} \frac{d x_{F}}{d x_{L}}\right)=c_{L}\left(x_{L}^{r}+x_{F}^{r}\right)^{2},
$$

which, by using (24), can be rewritten as

$$
\begin{equation*}
-\frac{r x_{L}^{r-1} x_{F}^{r}}{(r-1) x_{L}^{r}-(r+1) x_{F}^{r}}=c_{L}\left(x_{L}^{r}+x_{F}^{r}\right) . \tag{25}
\end{equation*}
$$

A necessary condition that ensures the existence of a solution of the leader's FOC is
that the denominator is negative-i.e., $(r-1) x_{L}^{r}-(r+1) x_{F}^{r}<0$, which implies that the second-order derivative of the follower is negative. This proves Claim (i).

We examine the second-order condition of the leader's optimization problem. We take the derivative of the following expression concerning $x_{L}$, using (23) and (24) repeatedly,

$$
\frac{d \pi_{L}}{d x_{L}}=-\frac{r x_{L}^{r-1} x_{F}^{r}}{\left((r-1) x_{L}^{r}-(r+1) x_{F}^{r}\right)\left(x_{L}^{r}+x_{F}^{r}\right)}-\frac{1}{c_{L}} .
$$

Also, using (24), it can be further derived that

$$
\frac{d^{2} \pi_{L}}{d x_{L}^{2}}=-\frac{r x_{L}^{r-2} x_{F}^{r}}{\left((r-1) x_{L}^{r}-(r+1) x_{F}^{r}\right)^{3}\left(x_{L}^{r}+x_{F}^{r}\right)}\left[(r-1) x_{L}^{2 r}+2\left(r^{2}-1\right) x_{L}^{r} x_{F}^{r}-(r+1) x_{F}^{2 r}\right] .
$$

The sign of this second-order derivative is the same as the sign of the expression in the square bracket. The expression in the square bracket can be rewritten as:

$$
\begin{aligned}
& {\left[(r-1) x_{L}^{2 r}+2\left(r^{2}-1\right) x_{L}^{r} x_{F}^{r}-(r+1) x_{F}^{2 r}\right] } \\
= & 2 r^{2} x_{L}^{r} x_{F}^{r}+\left((r-1) x_{L}^{r}-(r+1) x_{F}^{r}\right)\left(x_{L}^{r}+x_{F}^{r}\right) \\
= & 2 r^{2} x_{L}^{r} x_{F}^{r}-\frac{r}{c_{L}} x_{L}^{r-1} x_{F}^{r}=2 r^{2} x_{L}^{r-1} x_{F}^{r}\left(x_{L}-\frac{1}{2 r c_{L}}\right),
\end{aligned}
$$

in which the second equality holds using (25), which is the leader's FOC. Combining the result from Claim 1, we can derive that, as $x_{L}$ increases from 0 to $\frac{1}{2 r_{c}}$ and then to $\hat{x}_{L},{ }^{40}$ the leader's second-order derivative changes from negative to positive. Thus, the leader's first order condition has at most two solutions, which can be denoted by $x_{L 1}$ and $x_{L 2}$, with $0<x_{L 1} \leq x_{L 2}$. Consider the case where $x_{L 1}<x_{L 2}$, i.e., there are exactly two solutions. Because the leader's payoff from $x_{L 2}$ is a local minimum and cannot exceed the payoff from preemption, we conclude that $x_{L 2}$ cannot be a valid interior solution, so $x_{L 1}$ is the only valid interior solution when it exists, and it is sufficient to compare this payoff with that of the corresponding preemptive solution. This concludes the proof.

As shown by the results of Lemma 1 and Claim 2, the solution of both players' firstorder conditions gives us the follower's unique best response, which means that in Lemma 1 , when $r \leq 1, x_{F}^{*}$ is the unique global optimum of the follower; when $r>1$, there are at most two solutions of the first-order condition, and the smaller one is ruled out since it is a local minimum of the follower. Thus, the follower has a well-defined best response function,

[^23]$x_{F}\left(x_{L}\right)$, for $x_{L}, x_{L} \in\left(0, \hat{x}_{L}\right]$. Claim 2 (ii) shows that given the follower's best response function, the leader has a unique optimal choice of effort among all of his effort choices that induce positive effort from the follower. That is to say, for any pair of $c$ and $r$ such that an interior solution exists, the interior solution is unique.

## A. 3 Proof of Lemma 2

Proof. Using (4), we derive that

$$
\pi_{L}=\frac{1}{1+t^{r}\left(x_{L}\right)}-c_{L} x_{L}
$$

The necessary FOC for the leader is

$$
-\frac{r t^{r-1} t^{\prime}}{\left(1+t^{r}\right)^{2}}-c_{L}=0
$$

where $t^{\prime}$ denotes the first-order derivative of $t$ with respect to $x_{L}$. Using (9), the above equation can be rewritten as

$$
\begin{equation*}
-x_{L} t^{\prime}=c \tag{26}
\end{equation*}
$$

Differentiating both sides of (8) with respect to $x_{L}$, we have

$$
r(r-1) t^{r-2} t^{\prime}=c_{F}\left(1+t^{r}\right)^{2}+2 c_{F} x_{L}\left(1+t^{r}\right) r t^{r-1} t^{\prime} .
$$

Multiplying both sides of the above equation by $-x_{L}$ and using (26), we get

$$
c r(r-1) t^{r-2}=-x_{L} c_{F}\left(1+t^{r}\right)^{2}+2 c_{L} x_{L}\left(1+t^{r}\right) r t^{r-1} .
$$

Simplifying the above equation with (9), we obtain the required characteristic equation for the ratio of players' effort when an interior equilibrium exists:

$$
c-t=c r\left(\frac{2}{1+t^{r}}-1\right)
$$

## A. 4 Further analysis of the characteristic equation

The effort ratio in an interior solution $t^{*}$, when it exists, must be a crossing point of $z=L(t)$ and $z=R(t)$ on the $t z$-plane. By analyzing $L(t)$ and $R(t)$ on the $t z$ plane, we obtain four
key observations, which are presented in the following claim.
Claim 3. On the $t z$-plane, consider two functions $z=L(t)$ and $z=R(t), t \in[0, \infty)$, where $L(t)$ and $R(t)$ are defined by (12):
(i) $z=L(t)$ is a straight line that crosses both the $z$-axis and the $t$-axis on two fixed points $(c, 0)$ and $(0, c)$.
(ii) The curve $z=R(t)$, which crosses the two axes on two fixed points $(1,0)$ and $(0, c r)$, decreases from $R(t=0)=c r$ to $\lim _{t \rightarrow \infty} R(t)=-c r$ as $t$ increases.
(iii) The curve $z=R(t)$ rotates clockwise around the fixed point $(1,0)$ when $r$ increases, because $\frac{\partial R}{\partial r} \geq 0$ (resp. $\frac{\partial R}{\partial r} \leq 0$ ) when $t \leq 1$ (resp. when $t \geq 1$ ).
(iv) When $r \leq 1, R(t)$ is a convex function; when $r>1, R(t)$ is a concave function for $0 \leq t<\tilde{t}$ and a convex function for $t \geq \tilde{t}$, where

$$
\begin{equation*}
\tilde{t}=\left(\frac{r-1}{r+1}\right)^{\frac{1}{r}} \tag{27}
\end{equation*}
$$

It can be shown that $\tilde{t}<1$ and $\tilde{t}$ increases in $r$.
Proof of Claim 3. It is clear that $t^{*}$, when it exists, must be a crossing point of $z=L(t)$ and $z=R(t)$ on the $t z$-plane.
(i) On the $t z$-plane, it is clear to derive that $L(t)=c-t$ is a straight line, which crosses the $z$-axis and $t$-axis on two fixed points $(c, 0)$ and $(0, c)$, respectively.
(ii) $R(t)$ decreases in $t$, since

$$
\begin{equation*}
\frac{\partial R(t)}{\partial t}=-\frac{2 c r^{2} t^{r-1}}{\left(1+t^{r}\right)^{2}}<0 \tag{28}
\end{equation*}
$$

Clearly, when $t$ increases from zero to infinity, $R(t)$ decreases from $R(t=0)=c r$ to $\lim _{t \rightarrow \infty} R(t)=-c r$.
(iii) The curve defined by $z=R(t)$, which crosses the $t$-axis and $z$-axis on two fixed points $(1,0)$ and $(0, c r)$, respectively, rotates clockwise around the fixed point $(1,0)$ when $r$ increases, because it can be shown that $\frac{\partial R}{\partial r} \geq 0$ (resp. $\frac{\partial R}{\partial r} \leq 0$ ) when $t \leq 1$ (resp. when $t \geq 1$ ), using

$$
\frac{\partial R(t)}{\partial r}=\frac{\left(1-t^{2 r}-2 r t^{r} \log t\right) c}{\left(1+t^{r}\right)^{2}}
$$

(iv) For $r \leq 1, R(t)$ is always a convex function; for $r>1, R(t)$ is a concave function when $0 \leq t<\tilde{t}$, where $\tilde{t}$ is given by (27), and $R(t)$ turns into a convex function when $t \geq \tilde{t}$. To see this, using (28), we obtain that

$$
\frac{\partial^{2} R}{\partial t^{2}}=\frac{2 c r^{2} t^{r-2}}{\left(1+t^{r}\right)^{3}}\left[t^{r}+1+\left(t^{r}-1\right) r\right]
$$

which further implies that for $r \in(0,1], \frac{\partial^{2} R}{\partial t^{2}}>0$; for $r>1, \frac{\partial^{2} R}{\partial t^{2}} \leq 0\left(\operatorname{resp} . \frac{\partial^{2} R}{\partial t^{2}} \geq 0\right)$ if $t \leq \tilde{t}$ (resp. $t \geq \tilde{t}$ ). It can be shown that $\tilde{t}$ increases in $r$ and $\tilde{t}<1$ for any finite $r>0$, which verifies the results in (iv).

## A. 5 Proof of Proposition 1

Proof. Based on the four key observations in Claim 3, we analyze two cases with different values of $r$ and $c$ : Case $(i) r \in(0,1)$ and $c \in(0,1]$; Case $(i i) r \in(0,1)$ and $c \in(1, \infty)$.

In the case in which $r \in(0,1)$ and $c \in(0,1]$, we derive that $L(0)=c \geq R(0)=c r$, $L^{\prime}(t)=-1, R(t)$ is a convex function that decreases in $t$, and recall that $\lim _{t \rightarrow \infty} R(t)>$ $\lim _{t \rightarrow \infty} L(t)$. Thus, we can conclude that there exists a unique solution $t^{*}$, where $t^{*} \leq c \leq 1$, and $t^{*}$ decreases from $c$ to 0 when $r$ increases from 0 to 1 .

In the case in which $r \in(0,1)$ and $c \in(1, \infty)$, we still have $L(0)=c \geq R(0)=c r$, $L^{\prime}(t)=-1, R(t)$ is a convex function that decreases in $t$. Thus, we can conclude that there exists a unique solution $t^{*}$, where $1<c<t^{*}$, and $t^{*}$ increases in $r$.

## A. 6 Proof of Proposition 2

Proof. Based on the four observations in Claim 3, we analyze the case in which $r \in[1, \infty)$ and $c \in(1, \infty)$. In this case, since $L(0)=c<R(0)=c r, L(1)=c-1>R(1)=0$, and $\lim _{t \rightarrow \infty} R(t)>\lim _{t \rightarrow \infty} L(t)$, it is safe to conclude that there are exactly two solutions to the equation (11) -one is smaller than 1 and the other is greater than $c$.

Notice that when $r \geq 1$, it is possible that there are two solutions to (11). Next, we show that $t^{*}$ is the one with the greater value when there are two solutions. Using (9), we obtain that

$$
\pi_{F}^{i n t}(t ; c, r)=\frac{t^{r}}{1+t^{r}}-\frac{r t^{r}}{\left(1+t^{r}\right)^{2}}
$$

At any solution $t^{*}$ that is determined by Lemma 2, (11) is satisfied, and we further derive that

$$
\pi_{F}^{i n t}\left(t^{*} ; c, r\right)=\frac{\left(t^{*}\right)^{2}-c^{2}(r-1)^{2}}{4 c^{2} r}
$$

Let $t_{1}$ and $t_{2}$ denote the two solutions to (11), whenever they exist. Without loss of generality, we assume $t_{1}<t_{2}$. Using the above expression of $\pi_{F}^{i n t}$, it can be shown that at the smaller solution, $t_{1}, \pi_{F}^{i n t}\left(t_{1} ; c, r\right)<0$, since $t_{1}^{2}<c^{2}\left(r^{2}-1\right)$; at the larger solution, $t_{2}, \pi_{F}^{i n t}\left(t_{2} ; c, r\right)>0$, since $t_{2}^{2}>c^{2}\left(r^{2}-1\right)$. The proof for $t_{1}^{2}<c^{2}\left(r^{2}-1\right)$ and $t_{2}^{2}>c^{2}\left(r^{2}-1\right)$ is provided in the following paragraph.

Because $L^{\prime}(t)=-1$ and $t_{1}$ is the left-most crossing point of $z=L(t)$ and $z=R(t)$, we
derive that $R^{\prime}\left(t_{1}\right)<-1$. Taking the derivative of $R(t)$ with respect to $t$, we find

$$
\begin{equation*}
R^{\prime}(t)=-\frac{2 c r^{2} t^{r-1}}{\left(1+t^{r}\right)^{2}}=-\frac{2 c r^{2}}{t}\left(\frac{1}{1+t^{r}}-\frac{1}{\left(1+t^{r}\right)^{2}}\right) . \tag{29}
\end{equation*}
$$

Also, equation (11) can be rewritten as

$$
\begin{equation*}
\frac{1}{1+\left(t^{*}\right)^{r}}=\frac{1}{2}\left(1+\frac{c-t^{*}}{c r}\right) . \tag{30}
\end{equation*}
$$

Using $-R^{\prime}\left(t_{1}\right)>1$, (29), and (30), we derive that

$$
\begin{equation*}
\frac{c r^{2}}{2 t_{1}}\left[1-\frac{\left(c-t_{1}\right)^{2}}{c^{2} r^{2}}\right]>1 \tag{31}
\end{equation*}
$$

which further implies that $t_{1}^{2}<c^{2}\left(r^{2}-1\right)$ for $r>1$. Using a similar approach, we can also show that $t_{2}^{2}>c^{2}\left(r^{2}-1\right)$. Thus, we get that $\pi_{F}^{i n t}\left(t_{1} ; c, r\right)<0$ and $\pi_{F}^{i n t}\left(t_{2} ; c, r\right)>0$ whenever there are two solutions $t_{1}$ and $t_{2}$. This rules out the possibility of $t_{1}$ being a valid interior solution.

## A. 7 Proof of Proposition 3

Proof. We analyze the case with $r \in[1, \infty)$ and $c \in(0,1]$. Observe that $R(0)=c r>c=$ $L(0)$ and $\lim _{t \rightarrow \infty} R(t)>\lim _{t \rightarrow \infty} L(t)$. Recall that $L(t)$ is a straight line with $L^{\prime}(t)=-1$. Also, $R(t)$ is decreasing in $t$. It is concave when $0 \leq t<\tilde{t}$ and convex when $t \geq \tilde{t}$. This implies that $\left|R^{\prime}(t)\right|$ first increases from $\left|R^{\prime}(t=0)\right|=0$ to some positive value, then decreases to zero when $t$ goes to infinity, that is, $\lim _{t \rightarrow \infty}\left|R^{\prime}(t)\right|=0$. Based on the above results, we derive that in this case the number of solutions to equation (11) can be zero, one, or two, which depends on the values of $c$ and $r$.

More specifically, we first show that (i) when $c \in\left(0, \frac{1}{2}\right]$, there exists a unique $r_{s}>\sqrt{2}$ such that with $r=r_{s}, R(t)$ and $L(t)$ are tangent at $t=t_{s}^{*}>1$. From the four observations in Claim 3, it can be derived that when $c \in\left(0, \frac{1}{2}\right], R(t)>L(t)$ for any $t \in(0,1]$ if $R^{\prime}(t=$ $0 ; r=1) \geq-1$. Using (12), we obtain that $R^{\prime}(t=0 ; r=1)=-2 c$, which implies that if $R^{\prime}(t=0 ; r=1)=-2 c \geq-1$ (i.e., if $\left.c \in\left(0, \frac{1}{2}\right]\right)$, we always have $R(t)>L(t)$ for any $t \in(0,1]$, which implies that $R(t)$ and $L(t)$ have no crossing point for $t \in(0,1]$. Also, from the four observations in Claim 3, we show that there exists a unique $r_{s}$ such that with $r=r_{s}, R(t)$ and $L(t)$ are tangent at $t=t_{s}^{*}>1$.

We now conduct a general analysis of the case in which $L(t)$ and $R(t)$ are tangent at some unique $t$. For a given $r, L(t)$ and $R(t)$ being tangent at $t$ implies that this pair of $r$ and $t$ solves the system of equations involving (11) and $L^{\prime}(t)=R^{\prime}(t)$. The second equality
is equivalent to

$$
-1=R^{\prime}(t)=-\frac{2 c r^{2}}{t}\left(\frac{1}{1+t^{r}}-\left(\frac{1}{1+t^{r}}\right)^{2}\right)
$$

Using the above equation and $1 /\left(1+t^{r}\right)=(c-t+c r) / 2 c r$, which is obtained by (11), we derive that at the tangent point, we have $t=c \sqrt{r^{2}-1}$. Thus, it is clear that with $r=r_{s}$, $R(t)$ and $L(t)$ are tangent at $t_{s}^{*}=c \sqrt{r_{s}^{2}-1}$.

Given $c \in\left(0, \frac{1}{2}\right], t_{s}^{*}>1$ implies that $\sqrt{r_{s}^{2}-1}>2$, which further implies that $r_{s}^{2}>\sqrt{2}$. Again, by the four observations in Claim 3, which state that $L(t)$ is a straight line and $R(t)$ is a curve that rotates clockwise around the fixed point $(1,0)$ when $r$ increases, we derive that when $r$ is (strictly) greater than $r_{s}^{2}, t^{*}>1$, and $t^{*}$ increases in $r$. In summary, we show that when $r \in\left[1, r_{s}\right), t^{*}$ does not exist, since there is no solution to equation (11); when $r \in\left[r_{s}, \infty\right)$, there exist two solutions to equation (11) and $t^{*}$ is the larger solution. ${ }^{41}$ Moreover, using the four observations in Claim 3, it can be shown that $t^{*}>c>1$, and $t^{*}$ increases in $r$.

Next, we show that part (ii) is true. When $c \in\left(\frac{1}{2}, 1\right]$, there exist $r_{s}^{1}$ and $r_{s}^{2}$, such that they are the two solutions of $r$ to the system of equations involving (11) and $R^{\prime}(t)=L^{\prime}(t)=-1$. From the above analysis of (i), it is clear that with $c \in\left(\frac{1}{2}, 1\right], R^{\prime}(t=0 ; r=1)=-2 c<-1$, which implies that with $r=1, R(t)$ and $L(t)$ must have a unique crossing point for $t \in(0,1]$. By the four observations in Claim 3, we know that when $r$ increases, $R(t)$ rotates clockwise around the fixed point $(1,0)$. The number of solutions to equation (11) is two when $r$ exceeds 1 but is (strictly) smaller than some threshold, say $r_{s}^{1}$; when $r=r_{s}^{1}$, the number of solutions to (11) is one -i.e., $L(t)$ and $R(t)$ are tangent at $t$, denoted by $t_{s}^{1}$, where $t_{s}^{1} \leq c$; when $r$ exceeds $r_{s}^{1}$ but is (strictly) smaller than some threshold, say $r_{s}^{2}$, the number of solutions is zero-i.e., $L(t)$ and $R(t)$ have no crossing point; when $r=r_{s}^{2}$, the number of solutions to (11) is one again-i.e., $L(t)$ and $R(t)$ are tangent at $t$, denoted by $t_{s}^{2}$, where $t_{s}^{2} \geq c$; when $r$ exceeds $r_{s}^{2}$, the number of solutions to (11) is two.

Based on the above analysis, we obtain the following results: First, there exists a unique $r_{s}^{1}$ such that with $r=r_{s}^{1}, R(t)$ and $L(t)$ are tangent to each other at $t^{*}=t_{s}^{1}$, where $t_{s}^{1}<c<1$. Second, there exists a unique $r_{s}^{2}, r_{s}^{2}>r_{s}^{1}$, such that at $r=r_{s}^{2}, R(t)$ and $L(t)$ are tangent to each other at $t^{*}=t_{s}^{2}$, where $t_{s}^{2}>c>1$. Next, we seek to show that $r_{s}^{1}<\sqrt{2}<r_{s}^{2}$ for $c \in(0,1)$. From our previous analysis of the case in which $L(t)$ and $R(t)$ are tangent, we have $t=c \sqrt{r^{2}-1}$ at the tangent point, which implies that $t_{s}^{1}=c \sqrt{\left(r_{s}^{1}\right)^{2}-1}$ and $t_{s}^{2}=$ $c \sqrt{\left(r_{s}^{2}\right)^{2}-1}$. When $c \in(0,1), t_{s}^{1}<c<1$ implies that $\sqrt{\left(r_{s}^{1}\right)^{2}-1}<1$, which further implies that $r_{s}^{1}<\sqrt{2} ; t_{s}^{2}>c>1$ implies that $\sqrt{\left(r_{s}^{2}\right)^{2}-1}>1$, which further implies that $r_{s}^{2}>\sqrt{2}$. Thus, we have $r_{s}^{1}<\sqrt{2}<r_{s}^{2}$.

[^24]From the four observations in Claim 3, it is clear that when $c \in\left(\frac{1}{2}, 1\right)$, for any $r \in\left[1, r_{s}^{1}\right)$, there are two solutions to equation (11) and $t^{*}$ is the larger one (because the smaller one leads to a strictly negative payoff for the follower), and for $r$ in this region, $t^{*}<c<1$ and $t^{*}$ decreases in $r$; for $r=r_{s}^{1}, t^{*}$ is the unique solution to equation (11); for any $r \in\left(r_{s}^{1}, r_{s}^{2}\right)$, there is no solution to equation (11) and thus $t^{*}$ does not exist; for $r=r_{s}^{2}, t^{*}$ is the unique solution to equation (11); for any $r \in\left(r_{s}^{2}, \infty\right)$, there are two solutions to equation (11) and $t^{*}$ is the one with the greater value, and for $r$ in this region, $t^{*}>1>c$ and $t^{*}$ increases in $r$.

Lastly, when $c=1$, following a similar procedure, we show that there exists a unique $r_{s}^{\prime}=\sqrt{2}$ such that for $r \in\left[1, r_{s}^{\prime}\right), t^{*}<c=1$, and $t^{*}$ decreases in $r$; for $r=r_{s}^{\prime}$, we have $t^{*}=c=1$; for $r \in\left(r_{s}^{\prime}, \infty\right)$, there are two solutions to equation (11), and $t^{*}$ is the one with greater value, in this case we have $c=1<t^{*}$ and $t^{*}$ increases in $r$. Alternatively, the case with $c=1$ can be seen as the previous case with $c$ approaching $1^{-}$, both $r_{s}^{1}$ and $r_{s}^{2}$ converge to the same value $r_{s}^{\prime}=\sqrt{2}$, and the interval $\left(r_{s}^{1}, r_{s}^{2}\right)$ shrinks to a point.

## A. 8 Proof of Proposition 4

Proof. At solution $t^{*}$ that is determined in Propositions 1 to 3, the leader has an expected payoff

$$
\pi_{L}^{i n t}\left(t^{*} ; c, r\right)=\frac{1}{1+\left(t^{*}\right)^{r}}-\frac{c r\left(t^{*}\right)^{r-1}}{\left(1+\left(t^{*}\right)^{r}\right)^{2}}
$$

Using the above equation and (11), we can further derive that

$$
\begin{equation*}
\pi_{L}^{i n t}\left(t^{*} ; c, r\right)=\frac{c^{2}-\left(c r-t^{*}\right)^{2}}{4 c r t^{*}} \tag{32}
\end{equation*}
$$

Rewrite $c^{2}-\left(c r-t^{*}\right)^{2}$ as $\left(c+c r-t^{*}\right)\left(c-c r+t^{*}\right)$ and observe that $z=-c r$ is an asymptotic line for $z=R(t)$ on the $t z$-plane. Because $t^{*}$ cannot exceed the intersection of $z=L(t)$ and $z=-c r$, it always holds true that $t^{*}<c(r+1)$ and thus $c+c r-t^{*}>0$. In an interior solution, $t^{*}>0$, it holds true that $t^{*}>c(r-1)$ for all $r \leq 1$, which is equivalent to $c-c r+t^{*}>0$ for all $r \leq 1$. Next, we verify that $c-c r+t^{*}>0$ for all $r>1$.

Because $L^{\prime}(t)=-1, \lim _{t \rightarrow \infty} R^{\prime}(t)=0$, and $t^{*}$ is the right-most crossing point of $z=L(t)$ and $z=R(t)$, we have $-1 \leq R^{\prime}\left(t^{*}\right)<0$. Using a similar procedure that shows $t_{2}^{2}>c^{2}\left(r^{2}-1\right)$ in the proof of Proposition 2, we can derive that $t^{* 2} \geq c^{2}\left(r^{2}-1\right)$. Now, both $t^{*}<c(r+1)$ and $t^{* 2} \geq c(r+1) c(r-1)$ imply $t^{*}>c(r-1)$-because otherwise, if $t^{*} \leq c(r-1)$, we would have $\left(t^{*}\right)^{2} \geq c(r+1) c(r-1)$. This verifies that $c^{2}-\left(c r-t^{*}\right)^{2}>0$ for all $r>1$. Thus, we have $\pi_{L}^{\text {int }}\left(t^{*} ; c, r\right)>0$ for all $r>0$.

Next, we show that $\pi_{F}^{i n t}\left(t^{*} ; c, r\right)>0$. Using (9), we obtain that

$$
\begin{equation*}
\pi_{F}^{i n t}\left(t^{*} ; c, r\right)=\left(1-\frac{1}{1+t^{* r}}\right)\left(1-\frac{r}{1+t^{* r}}\right) \tag{33}
\end{equation*}
$$

Clearly, $\pi_{F}^{i n t}\left(t^{*} ; c, r\right)>0$ when $r \leq 1$. When $r>1$, using (11) and (33), we derive that

$$
\pi_{F}^{i n t}\left(t^{*} ; c, r\right)=\frac{1}{4 c^{2} r}\left[t^{*}+c(r-1)\right]\left[t^{*}-c(r-1)\right]>0
$$

because we have shown that $t^{*}>c(r-1)$ when $r>1$. Thus, $\pi_{F}^{\text {int }}\left(t^{*} ; c, r\right)>0$ for all $r>0$.
We have just shown that for each $t^{*}, \pi_{L}^{i n t}\left(t^{*} ; c, r\right)>0$ and $\pi_{F}^{i n t}\left(t^{*} ; c, r\right)$. Notice that for each $t^{*}$ that is determined in Propositions 1 to 3 , the corresponding $x_{L}^{*}$ and $x_{F}^{*}$ can be obtained using (9). Recall that $t^{*}$ solves equation (11), which implies that each player's FOC is satisfied with the corresponding $x_{L}^{*}$ and $x_{F}^{*}$. From the follower's perspective, given $x_{L}=x_{L}^{*}$, as shown in the proof of Lemma 1 , when $r>1$ and for $x_{L} \in\left[0, \hat{x}_{L}\right)$, there exist two threshold values, denoted by $x_{F 1}$ and $x_{F 2}$, where $0<x_{F 1}<x_{F 2}$, which are the two solutions that satisfy the follower's FOC. Indeed, we have shown that in these cases, $\pi_{L}$ reaches a local maximum at $x_{F}=x_{F 1}$ and reaches a local maximum at $x_{F}=x_{F 2}$, with $\pi_{F}\left(x_{F 1}\right)<0$ and $\pi_{F}\left(x_{F 2}\right)>0$, for any given $x_{L} \in\left[0, \hat{x}_{L}\right)$. The fact that $\pi_{F}^{i n t}\left(t^{*} ; c, r\right)>0$ rules out the possibility that $x_{F}^{*}=x_{F 1}$, which thus further implies that $x_{F}^{*}=x_{F 2}$ is a global maximizer for the follower, given $x_{F} \in[0, \infty)$, because $\pi_{F}\left(x_{F 2}\right)$ reaches its unique local maximum, given $x_{F} \in(0, \infty)$, and $\pi_{F}\left(x_{F 2}\right)>\pi_{F}(0)=0$.

Next, we seek to show that given the follower's best response $x_{F}\left(x_{L}\right), x_{L}=x_{L}^{*}$ is a global maximizer that maximizes the leader's expected payoff (4) for all $x_{L} \in\left[0, \hat{x}_{L}\right)$. Notice that for any $t^{*}$ considered here, we have the uniqueness of $t^{*}$ for any given pair of $c$ and $r$. This implies that the corresponding $\pi_{L}^{i n t}\left(x_{L}^{*}\right)$ must be either a unique local maximum or a unique local minimum at $x_{L}=x_{L}^{*}$, given $x_{L} \in\left[0, \hat{x}_{L}\right)$. Notice that $\pi_{L}^{\text {int }}\left(t^{*} ; c, r\right)>0$ rules out the possibility that $\pi_{L}^{i n t}\left(x_{L}^{*}\right)$ is a unique local minimum-because otherwise, when $\pi_{L}^{i n t}$ reaches a unique local minimum at $x_{L}=x_{L}^{*}, \pi_{L}^{i n t}\left(x_{L}=x_{L}^{*}\right)<0$ since $\pi_{L}^{i n t}\left(x_{L}=x_{L}^{*}\right)<\pi_{L}^{i n t}\left(x_{L}=0\right)=0$, which contradicts $\pi_{L}^{i n t}\left(t^{*} ; c, r\right)>0$. Thus, $\pi_{L}^{i n t}$ reaches a unique local maximum at $x_{L}=x_{L}^{*}$, given $x_{L} \in\left[0, \hat{x}_{L}\right)$. Because $\pi_{L}^{i n t}\left(x_{L}\right)$ is a continuous function in $x_{L}$ and $\pi_{L}^{i n t}\left(x_{L}\right)$ reaches its unique local maximum at $x_{L}=x_{L}^{*}$, we conclude that $\pi_{L}^{i n t}\left(x_{L}^{*}\right)$ is indeed the unique global maximum, given $x_{L} \in\left[0, \hat{x}_{L}\right)$.

## A. 9 Proof of Proposition 5

Proof. It can be derived that

$$
d\left((r-1)^{\frac{r-1}{r}}\right) / d r=\frac{(r-1)^{\frac{r-1}{r}}}{r^{2}}[\ln (r-1)+r]
$$

It is easy to further derive that

$$
d\left(\frac{(r-1)^{\frac{r-1}{r}}}{r}\right) / d r=\frac{(r-1)^{\frac{r-1}{r}}}{r^{3}} \ln (r-1) .
$$

To facilitate the analysis, define

$$
\begin{equation*}
h(r)=\frac{(r-1)^{\frac{r-1}{r}}}{r}, \tag{34}
\end{equation*}
$$

where $r \in(1, \infty)$. Thus, the leader's expected payoff in preemptive solutions can be rewritten as $\pi_{L}^{p r e}=1-c \cdot h(r)$. Note that function $h(r)$, which is defined on the interval of $(1, \infty)$, is strictly decreasing in $r$ until its global minimum at $r=2$ and strictly increasing with $r$ when $r>2$. It is easy to derive that $h(r=2)=1 / 2$.

We now extend $h(r)$ to $r=1$ and $r \rightarrow \infty$. Because

$$
\begin{aligned}
\lim _{r \rightarrow 1}(r-1)^{\frac{r-1}{r}} & =\lim _{r \rightarrow 1} \exp \left(\frac{r-1}{r} \ln (r-1)\right) \\
& =\lim _{r \rightarrow 1} \exp \left(\frac{\ln (r-1)}{\frac{r}{r-1}}\right)=e^{0}=1,
\end{aligned}
$$

by L'Hospital's rule, we obtain that $\lim _{r \rightarrow 1} h(r)=1$. As to $r \rightarrow \infty$,

$$
\lim _{r \rightarrow \infty} \frac{(r-1)^{\frac{r-1}{r}}}{r}=\lim _{r \rightarrow \infty}\left(\frac{r-1}{r}\right) \frac{1}{(r-1)^{\frac{1}{r}}}=1
$$

Then, using $\pi_{L}^{p r e}=1-c \cdot h(r)$, the results of this lemma are obtained.

## A. 10 Proof of Theorem 1

Proof. Theorem 1 follows from the results of Propositions 1 and 4.

## A. 11 Proof of Theorem 2

Proof. In a sequential contest, the leader compares the expected payoffs of an interior solution and a preemptive solution when they both exist, and he chooses the one that yields a higher
expected payoff. The follower's best response $x_{F}$ is determined once $x_{L}$ is chosen. Here, we focus on the case in which $c \in\left(0, \frac{1}{2}\right]$.

Recall that from Lemma 3, when $r \in(0,1], R(t)$ is a convex function, while for $r \in$ $(1, \infty), R(t)$ is a concave function when $0 \leq t<\tilde{t}$, and turns to a convex function when $t \geq \tilde{t}$, where $\tilde{t}(<1)$ is given by (27). When $r \in(0,1]$, because $L(0)=c \geq R(0)=c r$, $L(1)=c-1<R(1)=0, L^{\prime}(t)=-1$, and $R^{\prime}(t)$ is strictly increasing, there is a unique solution $t^{*}$, where $t^{*}<c$. The solution $t^{*}$ decreases to zero continuously when $r$ increases to one; $t^{*}=0$ at $r=1$ means that the leader "preempts" the follower at $r=1$-i.e., an interior solution coincides with the preemptive solution when $r=1$ for any $c \in\left(0, \frac{1}{2}\right]$.

We show that for $c \in\left(0, \frac{1}{2}\right]$, the leader preempts the follower for all $r \geq 1$, by proving the following claim: (i) for any $r \in(1, \sqrt{2})$, there exists no $t^{*}$; (ii) for $r \in[\sqrt{2}, \infty)$, there may exist a solution $t>1$, which however is not a valid candidate for equilibrium, because the leader's expected payoff in the preemptive solution is always greater than that in its corresponding interior solution. ${ }^{42}$

Suppose part (i) of the above claim is not true - i.e., there exists a solution $t^{*}<1$ when $r \in(1, \sqrt{2})$ and $c \in\left(0, \frac{1}{2}\right]$. Recall that the curve defined by $z=R(t)$ rotates clockwise around fixed point $(t=1, z=0)$ as $r$ increases on $t z$-plane. Because $z=R(t)$ is convex around the fixed point $(1,0)$ and the interior solution $t^{*}<1$ is the rightmost crossing point of $z=L(t)$ and $z=R(t)$ to the left of the fixed point $(1,0), z=R(t)$ is convex at $t=t^{*}$, given $r>1$. In this situation, the number of the crossing points between this convex curve and the straight line could be 0,1 , and 2 . The number equals 1 if and only if the straight line is a supporting line. It is easy to check that when $r=1$, the number of crossing points is 0 , because when $c \in\left(0, \frac{1}{2}\right]$ and $r=1$, (11) implies that $t=2 c-1<0$. Once $t^{*}$ exists, there must exist a pair of $t$ and $r$, where $t<1$ and $r \in(1, \sqrt{2})$, such that $L(t)$ and $R(t)$ are tangent at this $t$-i.e., they solve equations (11) and

$$
-1=R^{\prime}(t)=-\frac{2 c r^{2}}{t}\left(\frac{1}{1+t^{r}}-\frac{1}{\left(1+t^{r}\right)^{2}}\right) .
$$

Using the above equation and $1 /\left(1+t^{r}\right)=(c-t+c r) / 2 c r$ obtained by (11), we derive that at the tangent point, $t=c \sqrt{r^{2}-1}$. Let $\varepsilon=\sqrt{r^{2}-1}$. Then $t=c \sqrt{r^{2}-1}$ implies that $t=c \varepsilon$, and $\varepsilon \in(0,1)$ as $r \in(1, \sqrt{2})$. By plugging $t=c \varepsilon$ into equation (11), we can solve for $t$, $\varepsilon$, and $r$ as functions of $c$.

Next, we show that (11) does not hold for any $\varepsilon \in(0,1)$, which leads to a contradiction.

[^25]Using (11), we derive that

$$
\begin{aligned}
L(t) & =c-t=c(1-\varepsilon) \\
R(t) & =c \sqrt{1+\varepsilon^{2}}\left(\frac{2}{1+c^{r} \varepsilon^{r}}-1\right)>c\left(\frac{2}{1+c^{r} \varepsilon^{r}}-1\right) \\
& \geq c\left(\frac{2}{1+\frac{\varepsilon^{r}}{2^{r}}}-1\right)>c\left(\frac{2-\varepsilon}{2+\varepsilon}\right) .
\end{aligned}
$$

Because $(1-\varepsilon)(2+\varepsilon)=2-\varepsilon-\varepsilon^{2}<2-\varepsilon$, we always have $L(t)<R(t)$, which indicates that there exists no $t^{*}<1$ for $r \in(1, \sqrt{2})$.

Part (ii) of the claim focuses on the case in which $c \in\left(0, \frac{1}{2}\right]$ and $r \in[\sqrt{2},+\infty)$. In this case, when $c=1 / 2$, we can derive that the smallest $r$ that ensures the existence of $t^{*}$ is $\underline{r} \approx 3.3$, at which $L(t)$ and $R(t)$ are tangent to each other. Thus, in this case we obtain that $t^{*}=c \sqrt{\underline{r}^{2}-1} \approx 1.5724$ and $\pi_{L}^{i n t}=\left[c^{2}-\left(c \underline{r}-t^{*}\right)^{2}\right] /\left(4 c \underline{r} t^{*}\right) \approx 0.0235$.

For $c \leq 1 / 2$, let the minimal value of $r$ that ensures the existence of $t^{*}$ be $r_{\min }$. Clearly, $r_{\text {min }} \geq \underline{r}$. Using $t=c \sqrt{r_{\text {min }}^{2}-1}$, we obtain that

$$
\pi_{L}^{i n t}\left(r_{\min }\right)=\left[1-\left(r_{\min }-\sqrt{r_{\min }^{2}-1}\right)^{2}\right] / 4 r_{\min } \sqrt{r_{\min }^{2}-1}
$$

We further derive that

$$
\frac{\partial \pi_{L}^{i n t}}{\partial r_{\min }}=-\frac{1}{2 r_{\min }^{2} \sqrt{r_{\min }^{2}-1}}<0
$$

which implies that $\pi_{L}^{i n t}\left(t^{*} ; c, r_{\min }\right)$ decreases with $r_{\text {min }}$. Thus, for any $c \in\left(0, \frac{1}{2}\right], \pi_{L}^{\text {int }}\left(t^{*} ; c, \underline{r}\right) \geq$ $\pi_{L}^{i n t}\left(t^{*} ; c, r_{\text {min }}\right)$, which means that $\pi_{L}^{i n t}\left(t^{*} ; c, \underline{r}\right) \approx 0.0235$ is the greatest expected payoff for the leader when choosing an interior solution-but it is strictly smaller than the minimal level of $\pi_{L}^{p r e}$, which equals $1-c \geq \frac{1}{2}$ by Proposition 5 and $c \leq \frac{1}{2}$. Thus, in these cases, an interior solution, when it exists, is always dominated by its corresponding preemptive solution. The above verifies our claim.

## A. 12 Proof of Lemma 3

Proof. We directly calculate $d \pi_{L}^{i n t} / d r$. In equation (11) and the following text of this proof, we treat $t^{*}$ as a function of $r$ for a given $c$ and denote $d t^{*} / d r$ by $t^{\prime}(r)$ for simplicity. Taking the derivative with respect to $r$ from both sides of equation (11), we have

$$
-t^{\prime}(r)=c\left(\frac{2}{1+\left(t^{*}\right)^{r}}-1\right)-2 c r \frac{\left(t^{*}\right)^{r}}{\left(1+\left(t^{*}\right)^{r}\right)^{2}}\left(\ln \left(t^{*}\right)+\frac{r}{t^{*}} t^{\prime}(r)\right),
$$

which implies that

$$
\begin{equation*}
-t^{\prime}(r)=c\left(\frac{2}{1+\left(t^{*}\right)^{r}}-1\right)-\frac{2 c r}{t^{*}}\left(\frac{1}{1+\left(t^{*}\right)^{r}}\right)\left(1-\frac{1}{1+\left(t^{*}\right)^{r}}\right)\left(t^{*} \ln \left(t^{*}\right)+r t^{\prime}(r)\right) . \tag{35}
\end{equation*}
$$

Rewriting equation (11), we have $\frac{1}{1+\left(t^{*}\right)^{r}}=\frac{c(r+1)-t^{*}}{2 c r}$. Replacing $\frac{1}{1+\left(t^{*}\right)^{r}}$ in equation (35) with the above expression, we obtain that

$$
-t^{\prime}(r)=\frac{c-t^{*}}{r}-\frac{c^{2} r^{2}-\left(c-t^{*}\right)^{2}}{2 c r t^{*}}\left(t^{*} \ln \left(t^{*}\right)+r t^{\prime}(r)\right)
$$

We further derive that

$$
\begin{equation*}
r\left(c^{2} r^{2}-c^{2}-\left(t^{*}\right)^{2}\right) t^{\prime}(r)=2 c t^{*}\left(c-t^{*}\right)-\left(c^{2} r^{2}-\left(c-t^{*}\right)^{2}\right) t^{*} \ln \left(t^{*}\right) \tag{36}
\end{equation*}
$$

The leader's expected payoff is given by (32). Taking the derivative with respect to $r$, we derive that

$$
\begin{aligned}
\frac{d \pi_{L}^{i n t}}{d r} & =\frac{1}{16 c^{2} r^{2}\left(t^{*}\right)^{2}}\left[-2\left(c r-t^{*}\right)\left(c-t^{\prime}(r)\right) 4 c r t^{*}-\left(c^{2}-\left(c r-t^{*}\right)^{2}\right)\left(4 c t^{*}+4 c r t^{\prime}(r)\right)\right] \\
& =-\frac{1}{4 c r^{2}\left(t^{*}\right)^{2}}\left[2\left(c r-t^{*}\right)\left(c-t^{\prime}(r)\right) r t^{*}+\left(c^{2}-\left(c r-t^{*}\right)^{2}\right)\left(t^{*}+r t^{\prime}(r)\right)\right] \\
& =-\frac{1}{4 c r^{2}\left(t^{*}\right)^{2}}\left[\left(c^{2}+c^{2} r^{2}-\left(t^{*}\right)^{2}\right) t^{*}+r\left(c^{2}-c^{2} r^{2}+\left(t^{*}\right)^{2}\right) t^{\prime}(r)\right]
\end{aligned}
$$

Substituting (36) into the above equation yields

$$
\begin{align*}
\frac{d \pi_{L}^{i n t}}{d r} & =-\frac{1}{4 c r^{2}\left(t^{*}\right)^{2}}\left[\left(c^{2}+c^{2} r^{2}-\left(t^{*}\right)^{2}\right) t^{*}-2 c t^{*}\left(c-t^{*}\right)+\left(c^{2} r^{2}-\left(c-t^{*}\right)^{2}\right) t^{*} \ln \left(t^{*}\right)\right] \\
& =-\frac{1}{4 c r^{2} t^{*}}\left[c^{2} r^{2}-\left(c-t^{*}\right)^{2}\right]\left(1+\ln \left(t^{*}\right)\right) \tag{37}
\end{align*}
$$

It can be shown that $c^{2} r^{2}-\left(c-t^{*}\right)^{2}>0$ for $r \geq 1$. To see this, if $t^{*} \leq 1$, we have $0 \leq t^{*} \leq c$ and $\left(c-t^{*}\right)^{2} \leq c^{2} \leq c^{2} r^{2}$; if $t^{*}>1$, we have $c \leq t^{*}<c(r+1)$ and $c^{2} r^{2}-\left(c-t^{*}\right)^{2}>0$.

We first analyze the simple case with $t^{*}>1$, which occurs when $c>1$; it also may occur when $c \in\left(\frac{1}{2}, 1\right]$ and $r$ is sufficiently large in the sense that $r \in\left[r_{s}^{1}, \infty\right)$. Given $t^{*}>1$, we have $1+\ln \left(t^{*}\right)>0$, which implies that $\pi_{L}^{i n t}$ is decreasing in $r$, using (37).

Next, we focus on the case in which $t^{*} \leq 1$, which occurs when $c \in\left(\frac{1}{2}, 1\right]$ and $r$ is sufficiently small in the sense that $r \in\left[1, r_{s}^{1}\right]$. In this case, because $t^{*}=2 c-1$ when $r=1$ and $t^{*}$ decreases with $r$, we obtain that $2 c-1$ is the upper bound of $t^{*}$; recall that we have shown that the lower bound of $t^{*}$ is the value of $t$ that solves the system of equations
involving (11) and $t=c \sqrt{r^{2}-1}$, where $z=L(t)$ and $z=R(t)$ are tangent to each other.
Let $\bar{c}_{1}$ be the value of $c$ such that $2 c-1=1 / e$, where $e$ is the natural number. Let $\bar{c}_{2}$ be the value of $c$ solving (11) and $t=c \sqrt{r^{2}-1}=1 / e$. Using (37), it can be shown that when $c \in\left(1 / 2, \bar{c}_{1}\right], d \pi_{L}^{i n t} / d r \geq 0$ for all $r$, which means that $\pi_{L}^{i n t}$ is increasing in $r$; when $c \in\left(\bar{c}_{1}, \bar{c}_{2}\right]$, $d \pi_{L}^{i n t} / d r \leq 0$ (resp. $d \pi_{L}^{i n t} / d r>0$ ) when $r$ is sufficiently small (resp. large), which means that $\pi_{L}^{i n t}$ is first decreasing and then increasing in $r$; when $c \in\left(\bar{c}_{2}, 1\right], d \pi_{L}^{i n t} / d r<0$ for all $r$, which means that $\pi_{L}^{i n t}$ is decreasing in $r$. It can be shown that when $r$ goes to infinity, $L(t)=c-t$ approaches $-c r$, and thus $t$ goes to $c+c r$, which further implies that $\pi_{L}^{i n t}$ goes to zero, using the expression of $\pi_{L}^{i n t}$ which is given in Proposition 4.

## A. 13 Proof of Theorem 3

Proof. In a sequential contest, the leader compares the expected payoffs between an interior solution and a preemptive solution and chooses effort $x_{L}$, which yields a higher expected payoff. The follower's best response $x_{F}$ is determined once $x_{L}$ is chosen. We focus on the case in which $c \in\left(\frac{1}{2}, 1\right]$.

We now compare the leader's expected payoffs $\pi_{L}^{i n t}\left(t^{*} ; c, r\right)$ and $\pi_{L}^{p r e}(c, r)$ directly. Recall that $r_{s}^{1}$ is the value of $r$ such that $R(t)$ and $L(t)$ are tangent at some $t \in(0,1]$, given $c \in\left(\frac{1}{2}, 1\right]$. We claim that $\pi_{L}^{i n t}\left(t^{*} ; c, r=1\right)>\pi_{L}^{p r e}(c, r=1)$ and $\pi_{L}^{i n t}\left(t^{*} ; c, r=r_{s}^{1}\right)<\pi_{L}^{p r e}\left(c, r=r_{s}^{1}\right)$.

To prove that the above claim is true, we first show that the inequality holds for $r=1$. It is easy to derive that $t^{*}=2 c-1$ when $r=1$. From (14) and (32), we obtain that

$$
\pi_{L}^{i n t}\left(t^{*} ; c, r=1\right)=\frac{c^{2}-(1-c)^{2}}{4 c(2 c-1)}=\frac{1}{4 c}
$$

and $\pi_{L}^{p r e}(c, r=1)=1-c$. The inequality $1-c<\frac{1}{4 c}$ holds true for $c>\frac{1}{2}$, which implies that $\pi_{L}^{i n t}\left(t^{*} ; c, r=1\right)>\pi_{L}^{p r e}(c, r=1)$.

Next, we show the inequality for $r=r_{s}^{1}$. Recall that $r_{s}^{1}$ is the solution of (11) and $t=c \sqrt{r^{2}-1}$, and we have

$$
1-\sqrt{r^{2}-1}=r\left(\frac{2}{1+c^{r}\left(\sqrt{r^{2}-1}\right)^{r}}-1\right)
$$

The solution $r_{s}^{1}$ should be a function of $c$, but there is no closed-form expression. Fortunately, this function is one-to-one. We have the following inverse function:

$$
\begin{equation*}
c=\frac{1}{\sqrt{\left(r_{s}^{1}\right)^{2}-1}}\left(\frac{r_{s}^{1}-\left(1-\sqrt{\left(r_{s}^{1}\right)^{2}-1}\right)}{r_{s}^{1}+\left(1-\sqrt{\left(r_{s}^{1}\right)^{2}-1}\right)}\right)^{\frac{1}{r_{s}^{1}}} \tag{38}
\end{equation*}
$$

Using (38), we can show that $c$ increases with $r_{s}^{1}$ when $r_{s}^{1} \in(1, \sqrt{2}]$, which further implies that $r_{s}^{1}$ increases from 1 to $\sqrt{2}$ when $c$ increases from $1 / 2$ to 1 .


Figure A.3: The numerical illustration of $c$ as a function of $r_{s}^{1}$.

The original form of the expected payoff from an interior solution is given by

$$
\begin{equation*}
\pi_{L}^{i n t}\left(t^{*} ; c, r_{s}^{1}\right)=\frac{1}{1+\left(t^{*}\right)^{r_{s}^{1}}}-\frac{c r_{s}^{1}\left(t^{*}\right)^{r_{s}^{1}-1}}{\left(1+\left(t^{*}\right)^{r_{s}^{1}}\right)^{2}} \tag{39}
\end{equation*}
$$

Recall that $L(t)$ and $R(t)$ are tangent at $t=t^{*}$ when $r=r_{s}^{1}$, we derive that $R^{\prime}\left(t^{*}\right)=$ $-\frac{2 c\left(r_{s}^{1}\right)^{2}\left(t^{*}\right)^{r_{s}^{1}-1}}{\left(1+\left(t^{*}\right)^{r_{s}^{1}}\right)^{2}}=-1=L^{\prime}\left(t^{*}\right)$, which implies that

$$
\begin{equation*}
\frac{c r_{s}^{1}\left(t^{*}\right)^{r_{s}^{1}-1}}{\left(1+\left(t^{*}\right)^{r_{s}^{1}}\right)^{2}}=\frac{1}{2 r_{s}^{1}} \tag{40}
\end{equation*}
$$

Using (11) and $t^{*}=c \sqrt{\left(r_{s}^{1}\right)^{2}-1}$, we get

$$
\begin{equation*}
\frac{1}{1+\left(t^{*}\right)^{r_{s}^{1}}}=\frac{r_{s}^{1}+1-\sqrt{\left(r_{s}^{1}\right)^{2}-1}}{2 r_{s}^{1}} . \tag{41}
\end{equation*}
$$

Substituting (40) and (41) into (39), we obtain the expression of $\pi_{L}^{i n t}\left(t^{*} ; c, r_{s}^{1}\right)$. We derive
that

$$
\begin{aligned}
& \pi_{L}^{p r e}\left(c, r_{s}^{1}\right)-\pi_{L}^{i n t}\left(t^{*} ; c, r_{s}^{1}\right) \\
= & {\left[1-\frac{c}{r_{s}^{1}}\left(r_{s}^{1}-1\right)^{\frac{r_{s}^{1}-1}{r_{s}^{1}}}\right]-\left[\frac{r_{s}^{1}+1-\sqrt{\left(r_{s}^{1}\right)^{2}-1}}{2 r_{s}^{1}}-\frac{1}{2 r_{s}^{1}}\right] } \\
= & \frac{r_{s}^{1}+\sqrt{\left(r_{s}^{1}\right)^{2}-1}}{2 r_{s}^{1}}-\frac{c}{r_{s}^{1}}\left(r_{s}^{1}-1\right)^{\frac{r_{s}^{1}-1}{r_{s}^{1}}}
\end{aligned}
$$

Substituting (38) into the above equation, we get

$$
\begin{aligned}
& \pi_{L}^{p r e}\left(c, r_{s}^{1}\right)-\pi_{L}^{i n t}\left(t^{*} ; c, r_{s}^{1}\right) \\
= & \frac{1}{2 r_{s}^{1}}\left[\left(r_{s}^{1}+\sqrt{\left(r_{s}^{1}\right)^{2}-1}\right)-\frac{2}{\sqrt{\left(r_{s}^{1}\right)^{2}-1}}\left(\frac{r_{s}^{1}-\left(1-\sqrt{\left(r_{s}^{1}\right)^{2}-1}\right)}{r_{s}^{1}+\left(1-\sqrt{\left(r_{s}^{1}\right)^{2}-1}\right)}\right)^{\frac{1}{r_{s}^{1}}}\left(r_{s}^{1}-1\right)^{\frac{r_{s}^{1}-1}{r_{s}^{1}}}\right] .
\end{aligned}
$$

Using the above equation, which is a function of a single variable $r_{s}^{1}$, it can be shown that $\pi_{L}^{p r e}\left(c, r_{s}^{1}\right)>\pi_{L}^{i n t}\left(t^{*} ; c, r_{s}^{1}\right)$ for all $r_{s}^{1} \in[1, \sqrt{2}]$.


Figure A.4: The numerical illustration of the payoff difference between both solutions, $\pi_{L}^{p r e}\left(c, r_{s}^{1}\right)-\pi_{L}^{\text {int }}\left(t^{*} ; c, r_{s}^{1}\right)$, as a function of $r_{s}^{1}$, where $r_{s}^{1} \in[1, \sqrt{2}]$.

Next, we show the following claim: Given $c \in\left(\frac{1}{2}, 1\right]$, for any $r \in\left[1, r_{s}^{1}\right]$, we always have

$$
\begin{equation*}
\frac{d \pi_{L}^{i n t}}{d r}<\frac{d \pi_{L}^{p r e}}{d r} \tag{42}
\end{equation*}
$$

The above claim implies that when $r$ increases in the region $\left[1, r_{s}^{1}\right], \pi_{L}^{p r e}$ increases at a faster
rate than $\pi_{L}^{i n t}$. Using (14) and (37), we obtain that

$$
\begin{align*}
\frac{d \pi_{L}^{p r e}}{d r} & =-\frac{c(r-1)^{\frac{r-1}{r}}}{r^{3}} \ln (r-1) \\
\frac{d \pi_{L}^{i n t}}{d r} & =-\frac{1}{4 c r^{2} t^{*}}\left[c^{2} r^{2}-\left(c-t^{*}\right)^{2}\right]\left(1+\ln \left(t^{*}\right)\right) \tag{43}
\end{align*}
$$

Recall that using $1 \geq-R^{\prime}\left(t^{*}\right)$, we have shown (30), which implies that

$$
\begin{equation*}
2 c t^{*} \geq\left[c^{2} r^{2}-\left(c-t^{*}\right)^{2}\right] \tag{44}
\end{equation*}
$$

Thus, using (43) and (44), we derive that in the case with $d \pi_{L}^{i n t} / d r>0$, we must have $d \pi_{L}^{i n t} / d r \leq-\frac{1}{2 r^{2}}\left(1+\ln \left(t^{*}\right)\right)$. Therefore, we can derive that (42) holds if

$$
\begin{equation*}
-\left(1+\ln \left(t^{*}\right)\right) \leq-\frac{2 c}{r}(r-1)^{\frac{r-1}{r}} \ln (r-1) \tag{45}
\end{equation*}
$$

Recall that we have shown in Proposition 3 that given $c \in\left(\frac{1}{2}, 1\right]$ and $r \in\left[1, r_{s}^{1}\right], t^{*}<c \leq 1$ and $t^{*}$ is decreasing in $r$-i.e., $t^{* \prime}(r)<0$. Using (36) and $t^{* \prime}(r)<0$, we derive that $\left(t^{*}\right)^{2}>$ $c^{2}\left(r^{2}-1\right)$, which implies that $t^{*}>c \sqrt{r^{2}-1}$, as $c^{2} r^{2}-\left(c-t^{*}\right)^{2}>0$ and $\ln \left(t^{*}\right)<0$. Given $t^{*}>c \sqrt{r^{2}-1}$, we derive that (45) holds if $-\ln \left(c \sqrt{r^{2}-1}\right)<-\frac{2 c}{r}(r-1)^{\frac{r-1}{r}} \ln (r-1)+1$. For $r \in\left[1, r_{s}^{1}\right]$, the above inequality holds for all $c \in\left(\frac{1}{2}, 1\right]$ if it holds for $c=1 / 2$-i.e.,

$$
-\ln \left(\frac{1}{2} \sqrt{r^{2}-1}\right)<-\frac{1}{r}(r-1)^{\frac{r-1}{r}} \ln (r-1)+1 .
$$

It can be shown that the above inequality, which has a single variable $r$, holds for any $r \in[1, \sqrt{2}]$. This implies that (42) holds for any $r \in\left[1, r_{s}^{1}\right]$ given $c \in\left(\frac{1}{2}, 1\right]$-i.e., when $r$ increases in this region, $\pi_{L}^{p r e}$ increases faster than $\pi_{L}^{i n t}$.

Finally, combining both results: (1) $\pi_{L}^{p r e}$ increasing faster than $\pi_{L}^{i n t}$ with $r$, and (2) $\pi_{L}^{i n t}\left(t^{*} ; c, r=1\right)>\pi_{L}^{p r e}(c, r=1)$ and $\pi_{L}^{i n t}\left(t^{*} ; c, r=r_{s}^{1}\right)<\pi_{L}^{p r e}\left(c, r=r_{s}^{1}\right)$, we conclude that there exists a unique value of $\hat{r}_{s}$, where $\hat{r}_{s} \in\left(1, r_{s}^{1}\right)$, such that $\pi_{L}^{\text {int }}\left(t^{*} ; c, r\right) \geq \pi_{L}^{p r e}(c, r)$ when $r \leq \hat{r}_{s}$, and $\pi_{L}^{i n t}\left(t^{*} ; c, r\right)<\pi_{L}^{p r e}(c, r)$ when $r>\hat{r}_{s}$.

When $r>r_{s}^{1}$, there is no interior solution for $t^{*} \leq 1$, and we have $t^{*}>1$ if an interior solution exists. Next, we show that when there exists a $t^{*}(>1)$, we must have $\pi_{L}^{p r e}(c, r)>$ $\pi_{L}^{i n t}\left(t^{*} ; c, r\right)$.

First, we show that when there exists a $t^{*}(>1), \pi_{L}^{i n t}\left(t^{*} ; c, r\right)=\left(c^{2}-\left(c r-t^{*}\right)^{2}\right) / 4 c r t^{*}<$ $1 / r^{2}$. We prove the above inequality by contradiction. Assume that $\pi_{L}^{i n t}$ and $1 / r^{2}$ have at least one crossing point, denoted by $r=r_{c}$. At this crossing point, $\left(c^{2}-\left(c r-t^{*}\right)^{2}\right) / 4 c r t^{*}=1 / r^{2}$.

Given $c$ and $r$, the above equation yields two possible solutions of $t^{*}$, denoted by

$$
t_{1}^{*}=\frac{c r^{2}-2 c-\sqrt{4 c^{2}-3 c^{2} r^{2}}}{r}, t_{2}^{*}=\frac{c r^{2}-2 c+\sqrt{4 c^{2}-3 c^{2} r^{2}}}{r} .
$$

We show that both $t_{1}^{*}$ and $t_{2}^{*}$ cannot exist. This is because a valid $t^{*}>1$ at least requires that $\max \left\{t_{1}^{*}, t_{2}^{*}\right\}=t_{2}^{*}>1$, and it can be shown that $r>\sqrt{2}$ is a necessary condition for $t_{2}^{*}>1$; however, $r>\sqrt{2}$ implies that $4 c^{2}-3 c^{2} r^{2}<0$, which rules out the existence of both $t_{1}^{*}$ and $t_{2}^{*}$. The above result indicates that $\pi_{L}^{i n t}(r)$ and $1 / r^{2}$ can never cross when $t^{*}>1$-i.e., for all $r \in\left[r_{\min }(c), \infty\right)$, it is either the case that $\pi_{L}^{i n t}(r)>1 / r^{2}$ or $\pi_{L}^{i n t}(r)<1 / r^{2}$, where $r_{\text {min }}(c)$ is the minimal value of $r$ that ensures the existence of $t^{*}>1$. It can be shown that $r_{\text {min }}(c)>\sqrt{2}$ for any $c \in\left(\frac{1}{2}, 1\right]$.

Moreover, when $r=r_{\text {min }}(c), L(t)$ and $R(t)$ are tangent to each other, $t^{*}=c \sqrt{r^{2}-1}$, which implies that

$$
\begin{align*}
\pi_{L}^{i n t}\left(t^{*} ; c, r\right) & =\frac{c^{2}-\left(c r-t^{*}\right)^{2}}{4 c r t^{*}} \\
& =\frac{1-\left(r-\sqrt{r^{2}-1}\right)^{2}}{4 r \sqrt{r^{2}-1}} . \tag{46}
\end{align*}
$$

We can further show that for any $r>1$,

$$
\pi_{L}^{i n t}\left(t^{*} ; c, r\right)-\frac{1}{r^{2}}=-\frac{2-r^{2}+r \sqrt{r^{2}-1}}{2 r^{2}}<0
$$

Thus, we can conclude that $\pi_{L}^{\text {int }}(r)<1 / r^{2}$ for any $r \in\left[r_{\min }(c), \infty\right)$.
Next, we show that $\pi_{L}^{p r e}(c, r)>1 / r^{2}$ for any $r \in[2,+\infty)$, which further implies that $\pi_{L}^{\text {pre }}(c, r)>\pi_{L}^{\text {int }}\left(t^{*} ; c, r\right)$ for $r \in\left[r_{\min }(c), \infty\right)$ due to $r_{\text {min }}(c)>2$. We first show $r_{\min }(c)>2$ by contradiction. When $r_{\text {min }}(c) \leq 2$, a valid $t^{*}(>1)$ requires that there exists at least one $t(>1)$ that satisfies $L(t)=R(t)$ at $r=2$ i.e., $\frac{1}{2}\left(1-\frac{t}{c}\right)=\frac{1-t^{2}}{1+t^{2}}$. However, it can be shown that the unique solution of the above equation is strictly negative for any $c \in\left(\frac{1}{2}, 1\right]$, which rules out the possibility of $r_{\min }(c) \leq 2$ - thus, it must be the case that $r_{\text {min }}(c)>2$.

For $r \in[2, \infty), \pi_{L}^{p r e}(c, r)>1 / r^{2}$ holds if $c<(1+1 / r)(r-1)^{\frac{1}{r}}$. The above inequality always holds for $r \in[2, \infty)$, because $c \leq 1$ and the RHS of the above inequality, which first increases and then decreases with $r$, is strictly greater than 1 for any $r \in[2, \infty) .{ }^{43}$ Thus, we have $\pi_{L}^{p r e}(c, r)>1 / r^{2}>\pi_{L}^{i n t}\left(t^{*} ; c, r\right)$ for $r \in[2,+\infty)$. Finally, we can conclude that $\pi_{L}^{p r e}(c, r)>\pi_{L}^{i n t}\left(t^{*} ; c, r\right)$ for all $r \in\left[r_{\text {min }}(c), \infty\right)$.

The above verifies our claim and $\hat{r}_{s} \in\left[1, r_{s}^{1}\right)$.

[^26]
## A. 14 Proofs of Lemma 4 and Theorem 4

Proof. When $c>1$, the contest becomes a weak-lead sequential contest, in which the leader is the weak player. We compare expected payoffs $\pi_{L}^{\text {int }}\left(t^{*} ; c, r\right)$ and $\pi_{L}^{p r e}(c, r)$ directly.

Because the curve defined by $z=R(t)$ on the $t z$-plane has a fixed point at $(t=1, z=0)$, and $L(1)=c-1>0=R(1)$ when $c>1, \lim _{t \rightarrow \infty} R(t)=-c r>-\infty=\lim _{t \rightarrow \infty} L(t)$, the solution of (11) exists for all $r>0$, and $t^{*}>1$. It can be derived that given $c>1$ and $r>1, t^{*}$ increases with $r$, and $t^{*} \rightarrow c$ when $r \rightarrow 0$, while $t^{*} \rightarrow c(1+r)$ when $r \rightarrow \infty$ because $z=R(t)$ has an asymptotic line $z=-c r$. It can further be shown that $\pi_{L}^{\text {int }}\left(t^{*} ; c, r\right)$, as given by Corollary 4 , decreases with $r$ when $r \geq 1$.

As to the expected payoff in a preemptive case, $\pi_{L}^{p r e}(c, r)=1-c \cdot h(r)$, where $h(r)$ is given by (34). Proposition 5 indicates that $\pi_{L}^{p r e} \rightarrow 1-c$ when either $r \rightarrow 1$ or $r \rightarrow \infty$; $\pi_{L}^{p r e}(c, r)$ monotonically increases to $1-c / 2$ at $r=2$ and monotonically decreases after that until infinity.

We have the ranking of expected payoffs with $r=1$ and $r \rightarrow \infty$, respectively, as follows. At $r=1$, we have

$$
\begin{equation*}
\pi_{L}^{i n t}\left(t^{*} ; c, r\right)=\frac{1}{4 c}>1-c=\pi_{L}^{p r e}(c, r) \tag{47}
\end{equation*}
$$

when $r \rightarrow \infty$, we have $\lim _{r \rightarrow \infty} \pi_{L}^{i n t}\left(t^{*} ; c, r\right)=0>1-c=\lim _{r \rightarrow \infty} \pi_{L}^{p r e}(c, r)$.
When $r=2$ and $c$ is sufficiently close to 1 , we have $\pi_{L}^{p r e}(c, r)>\pi_{L}^{i n t}\left(t^{*} ; c, r\right)$. To see this, for example, when $r=2$ and $c=1.2$, we have $\pi_{L}^{p r e}(c=1.2, r=2)=0.4$, while the numerical solution of (11) shows that $t^{*}=3.164$, and $\pi_{L}^{i n t}\left(t^{*}=3.164 ; c=1.2, r=2\right)=0.0282$, which is smaller than $\pi_{L}^{p r e}(c=1.2, r=2)=0.4$. However, when $c$ increases to a sufficiently high value, for instance, $c>2$, it results in $\pi_{L}^{p r e}(c, r)<0$ for all $r \geq 1$.

Both $\pi_{L}^{\text {int }}$ and $\pi_{L}^{p r e}$ are smooth functions of $r$ with any higher order of derivative. We claim that there exists a critical value $\hat{c}^{h}$ such that these two functions have a unique crossing point when $c=\hat{c}^{h}$; when $c$ increases from 1 to $\hat{c}^{h}$ and to infinity, the crossing points between $\pi_{L}^{i n t}$ and $\pi_{L}^{p r e}$ reduce from two to one and to zero. Next, we show that the above claim is true.

For $r>1$, let's define $\Delta \pi=\pi_{L}^{p r e}-\pi_{L}^{i n t}$, which can be further expressed as

$$
\begin{equation*}
\Delta \pi=1-\frac{c}{r}(r-1)^{\frac{r-1}{r}}-\frac{c^{2}-\left(c r-t^{*}\right)^{2}}{4 c r t^{*}} \tag{48}
\end{equation*}
$$

where $t^{*}$ is determined by

$$
\begin{equation*}
c-t^{*}=c r\left(\frac{2}{1+t^{* r}}-1\right) \tag{49}
\end{equation*}
$$

We further derive that

$$
\begin{equation*}
\frac{d \Delta \pi}{d c}=-\frac{1}{r}(r-1)^{\frac{r-1}{r}}-\frac{d \pi_{L}^{i n t}}{d c} . \tag{50}
\end{equation*}
$$

From (32), we obtain that

$$
\begin{equation*}
\frac{d \pi_{L}^{i n t}}{d c}=\frac{8 c r t^{*}\left[c-\left(c r-t^{*}\right)\left(r-t^{* \prime}(c)\right)\right]-4 r\left(t^{*}+c t^{* \prime}(c)\right)\left[c^{2}-\left(c r-t^{*}\right)^{2}\right]}{16 c^{2} r^{2} t^{* 2}}, \tag{51}
\end{equation*}
$$

where $t^{* \prime}(c)=\frac{d t^{*}}{d c}$ and it can be shown that

$$
t^{* \prime}(c)=\frac{2\left(t^{*}\right)^{2}}{c^{2}+\left(t^{*}\right)^{2}-c^{2} r^{2}}>0
$$

obtained from (31) and (49). Substituting the above expression of $t^{* \prime}(c)$ into (51) yields

$$
\frac{d \pi_{L}^{i n t}}{d c}=-\frac{\left(c r+c-t^{*}\right)\left(c r-c+t^{*}\right)}{4 c^{2} r t^{*}}<0 .
$$

Notice that $\frac{d \Delta \pi}{d c}<0$ if and only if

$$
\begin{equation*}
\left|\frac{d \pi_{L}^{i n t}}{d c}\right|=\frac{\left(c r+c-t^{*}\right)\left(c r-c+t^{*}\right)}{4 c^{2} r t^{*}}<\frac{1}{r}(r-1)^{\frac{r-1}{r}} . \tag{52}
\end{equation*}
$$

As we have shown that $\frac{1}{2} \leq \frac{1}{r}(r-1)^{\frac{r-1}{r}}<1$ in Proposition 5 , a necessary condition for $\frac{d \Delta \pi}{d c}<0$ is that $\frac{\left(c r+c-t^{*}\right)\left(c r-c+t^{*}\right)}{4 c^{2} r t^{*}}<\frac{1}{2}$, which holds if $c^{2} r\left(r-2 t^{*}\right)<\left(t^{*}-c\right)^{2}$. When $1<r<2$, we have $t^{*}>c>1>\frac{r}{2}$, then $r-2 t^{*}<0$, and the above inequality holds; when $r \geq 2$, the above inequality holds because $t^{*}>c(r-1)>\frac{r}{2}$ given $c>1$, and we still have $r-2 t^{*}<0$.

In sum, we have shown that $\frac{d \Delta \pi}{d c}<0$, which implies that when $c$ increases from one to infinity, $\pi_{L}^{\text {pre }}$ becomes relatively smaller compared with $\pi_{L}^{i n t}$ for any $r \geq 1$.

Next, we show that $\pi_{L}^{i n t}$ is a convex function for $r \geq 1$. Recall that (36) yields

$$
\begin{equation*}
t^{* \prime}(r)=\frac{d t^{*}}{d r}=\frac{2 c t^{*}\left(t^{*}-c\right)+\left(c^{2} r^{2}-\left(c-t^{*}\right)^{2}\right) t^{*} \ln \left(t^{*}\right)}{r\left(\left(t^{*}\right)^{2}-c^{2}\left(r^{2}-1\right)\right)}>0 . \tag{53}
\end{equation*}
$$

Using (37), we derive that

$$
\frac{d^{2} \pi_{L}^{i n t}}{d r^{2}}=A+B+C
$$

where

$$
\begin{aligned}
A & =\left(\frac{1}{2 c r^{3} t^{*}}+\frac{t^{* \prime}(r)}{4 c r^{2}\left(t^{*}\right)^{2}}\right)\left(c^{2} r^{2}-\left(c-t^{*}\right)^{2}\right)\left(1+\ln \left(t^{*}\right)\right) \\
B & =-\frac{1}{2 c r^{2} t^{*}}\left(c^{2} r+\left(c-t^{*}\right) t^{* \prime}(r)\right)\left(1+\ln \left(t^{*}\right)\right) \\
C & =-\frac{1}{4 c r^{2} t^{*}}\left(c^{2} r^{2}-\left(c-t^{*}\right)^{2}\right) \frac{t^{* \prime}(r)}{t^{*}}
\end{aligned}
$$

Notice that $\frac{d^{2} \pi_{L}^{i n t}}{d r^{2}}>0$ if and only if

$$
\begin{aligned}
& \left(\frac{2}{r}+\frac{t^{* \prime}(r)}{t^{*}}\right)\left(c^{2} r^{2}-\left(c-t^{*}\right)^{2}\right)\left(1+\ln \left(t^{*}\right)\right) \\
> & 2\left(c^{2} r+\left(c-t^{*}\right) t^{* \prime}(r)\right)\left(1+\ln \left(t^{*}\right)\right)+\left(c^{2} r^{2}-\left(c-t^{*}\right)^{2}\right)\left(\frac{t^{* \prime}(r)}{t^{*}}\right),
\end{aligned}
$$

which is equivalent to

$$
\left[\frac{t^{* \prime}(r)}{t^{*}}+2\left(\frac{1}{r}-\frac{c^{2} r+\left(c-t^{*}\right) t^{* \prime}(r)}{c^{2} r^{2}-\left(c-t^{*}\right)^{2}}\right)\right]\left(1+\ln \left(t^{*}\right)\right)>\frac{t^{* \prime}(r)}{t^{*}} .
$$

Thus, a necessary condition which ensures $\frac{d^{2} \pi_{L}^{i n t}}{d r^{2}}>0$ is that $\frac{1}{r}>\frac{c^{2} r+\left(c-t^{*}\right) t^{* \prime}(r)}{c^{2} r^{2}-\left(c-t^{*}\right)^{2}}$, which holds true if $t^{* \prime}(r)>\frac{t^{*}-c}{r}$. Using (53), $t^{* \prime}(r)>\frac{t^{*}-c}{r}$ is equivalent to

$$
\frac{2 c t^{*}\left(t^{*}-c\right)+\left(c^{2} r^{2}-\left(t^{*}-c\right)^{2}\right) t^{*} \ln \left(t^{*}\right)}{c^{2}+\left(t^{*}\right)^{2}-c^{2} r^{2}}>t^{*}-c
$$

which holds if $\left(c^{2} r^{2}-\left(t^{*}-c\right)^{2}\right)\left(t^{*} \ln \left(t^{*}\right)+t^{*}-c\right)>0$. The above inequality holds true as $c<t^{*}<c(r+1)$. Thus, $\pi_{L}^{\text {int }}$ is convex when $r \geq 1$.

Besides the convexity of $\pi_{L}^{i n t}$, Proposition 5 shows that $\pi_{L}^{p r e}$ increases with $r$ when $r \in(1,2)$ then reaches its maximum at $r=2$, and decreases with $r$ when $r \in(2,+\infty)$. Meanwhile, $\pi_{L}^{\text {int }}$ decreases with $r$ when $r \in(1,+\infty)$. Also, we show that $\pi_{L}^{\text {pre }}(r=1)=$ $\lim _{r \rightarrow \infty} \pi_{L}^{\text {pre }}=1-c<0$, while $\pi_{L}^{\text {int }}(r=1)=\frac{1}{4 c}>0$ and $\lim _{r \rightarrow \infty} \pi_{L}^{i n t}=0$, and $\pi_{L}^{\text {pre }}(r)$ is concave when $r<\hat{r}_{c}$ and convex when $r \geq \hat{r}_{c}$, where $\hat{r}_{c} \approx 3,2771$.

Therefore, using $\pi_{L}^{p r e}(r=2)=1-c / 2$, we conclude that for $c(>1)$ sufficiently close to $1, \pi_{L}^{p r e}$ and $\pi_{L}^{\text {int }}$ will have exactly two crossing points, denoted by $\hat{r}_{w(c=1)}^{1}$ and $\hat{r}_{w(c=1)}^{2}$, respectively, where $1<\hat{r}_{w(c=1)}^{1}<\hat{r}_{w(c=1)}^{2}$. Moreover, $\hat{r}_{w}^{1}$ and $\hat{r}_{w}^{2}$ represent the two zero points of $\Delta \pi(r):=\pi_{L}^{p r e}(r)-\pi_{L}^{i n t}(r)$. This relationship further implies that $\Delta \pi(r) \geq 0$ if and only if $r \in\left[\hat{r}_{w(c=1)}^{1}, \hat{r}_{w(c=1)}^{2}\right]$. We have just shown that $\frac{d \Delta \pi}{d c}<0$ when $c>1$ and $r>1$. Hence, as $c$ increases from 1 , the curve of $\Delta \pi(r)$ will move strictly downward at each point on the interval $(1, \infty)$. Also, when $c \geq 2$, we have $\pi_{L}^{\text {pre }}(r) \leq \pi_{L}^{p r e}(r=2) \leq 0$ and thus $\Delta \pi<0$. Combining these results, we conclude that as $c$ increases, the interval $\left[\hat{r}_{w(c=1)}^{1}, \hat{r}_{w(c=1)}^{2}\right]$ will shrink to a point at some $\hat{c}^{h}<2$. When $c>\hat{c}^{h}, \pi_{L}^{p r e}$ is strictly smaller than $\pi_{L}^{i n t}$ for all $r>1$. The specific value of $\hat{c}^{h}$ is determined by the unique solution of the system of equations involving $\pi_{L}^{\text {int }}\left(t^{*} ; c, r\right)=\pi_{L}^{p r e}(c, r)$ and $\frac{d}{d r} \pi_{L}^{p r e}=\frac{d}{d r} \pi_{L}^{i n t}$. Thus, using (48) and (43), we can derive the unique solution of the system of equations involving $\Delta \pi=0$ and $\frac{d}{d r} \pi_{L}^{p r e}=\frac{d}{d r} \pi_{L}^{i n t}$, which yields that $\hat{c}^{h} \approx 1.9831$.

## A. 15 Proof of Proposition 6

Proof. We first establish the fact that, when $c \in(0,1)$ and $r \in\left(0, \hat{r}_{s}\right)$, we have $p_{S L}>p_{W L}>$ $p_{\text {Simu }}>1 / 2$ which are the winning probabilities of player $A$ in the strong-lead sequential, weak-lead sequential and simultaneous contests, respectively. Here, $\hat{r}_{s}$ can be determined by Theorem 3. For values of $r$ in $\left(0, \hat{r}_{s}\right)$, strong-lead sequential contests have interior equilibria, as well as weak-lead sequential contests according to the proofs of Lemma 4 and Theorem 4.

We seek to show that

$$
p_{S L}=\frac{1}{1+t_{A}^{r}}>p_{W L}=\frac{t_{B}^{r}}{1+t_{B}^{r}}
$$

for these values of $r$. We rewrite equation (17) as

$$
\tilde{c} r\left(\frac{2}{1+t_{B}^{r}}-1\right)=\tilde{c} r\left(\frac{2 / t_{B}^{r}}{1+1 / t_{B}^{r}}-1\right)=-\tilde{c} r\left(\frac{2}{1+1 / t_{B}^{r}}-1\right)=\tilde{c}-\frac{\tilde{c}^{2}}{1 / t_{B}^{r}}
$$

Therefore, $t=\frac{1}{t_{B}}$ must be a solution to the following equation with a single variable $t$ :

$$
\frac{\tilde{c}^{2}}{t}-\tilde{c}=\tilde{c} r\left(\frac{2}{1+t^{r}}-1\right)
$$

Also, notice that $t_{B}$ is the unique solution of equation (17), which is greater than $1 / \tilde{c}$, by the previous analysis in the general sequential contest model with players $L$ and $F$. In the meantime, $t_{A}$ is the solution of the following characteristic equation

$$
\tilde{c}-t=\tilde{c} r\left(\frac{2}{1+t^{r}}-1\right)
$$

We keep the notation of the two functions (12):

$$
\begin{aligned}
& \tilde{L}(t)=\tilde{c}-t \\
& \tilde{R}(t)=\tilde{c} r\left(\frac{2}{1+t^{r}}-1\right),
\end{aligned}
$$

which are the $L H S$ and $R H S$ of the above characteristic equation, respectively. We further define

$$
\begin{equation*}
\tilde{L}_{B}(t)=\frac{\tilde{c}^{2}}{t}-\tilde{c} \tag{54}
\end{equation*}
$$

Based on our previous analysis of the characteristic equation in the general sequential contest model, it can be shown that when $r \in\left(0, \hat{r}_{s}\right), t_{A}<c$ is either the unique solution of equation (15), or the greater solution if equation (15) has two solutions; the values of $t_{A}$ and $1 / t_{B}$ are uniquely identified by the cross points between $z=\tilde{L}(t)$ and $z=\tilde{R}(t)$ and between
$z=\tilde{L}_{B}(t)$ and $z=\tilde{R}(t)$, respectively, on the $t z$-plane. We seek to show that

$$
t_{A}<\frac{1}{t_{B}}
$$

which immediately indicates that

$$
p_{S L}=\frac{1}{1+t_{A}^{r}}>p_{W L}=\frac{t_{B}^{r}}{1+t_{B}^{r}} .
$$

Taking partial derivative of $\tilde{L}_{B}(t)$ with respect to $t$ evaluating at $t=\tilde{c}$, we have

$$
\left.\frac{\partial \tilde{L}_{B}(t)}{\partial t}\right|_{t=\tilde{c}}=-\left.\frac{\tilde{c}^{2}}{t^{2}}\right|_{t=\tilde{c}}=-1 .
$$

This implies that $z=\tilde{L}_{B}(t)$ is a convex function of $t$ on $t \in(0,1)$, and $z=\tilde{L}(t)$ is its supporting straight line at $t=\tilde{c}$, where $\tilde{c} \in(0,1)$. As $t$ decreases from 1 to 0 , the curve defined by $z=\tilde{R}(t)$ must cross $z=\tilde{L}_{B}(t)$ first, and then $z=\tilde{L}(t)$, which leads to $t_{A}<1 / t_{B}$.

For $r \geq \hat{r}_{s}$, the winning probability of player $A$ is one in strong-lead sequential contests, i.e., $p_{S L}=1$, as the strong leader preempts the weak follower in equilibrium. Therefore, $p_{S L}>p_{W L}$ holds for all the values of $c \in(0,1)$ and $r \in(0, \infty)$ including the case that, for certain values of $c$ and $r$, weak-lead sequential contests have preemptive equilibria, where $p_{W L}=0$.

We next compare $p_{W L}$ and $p_{\text {Simu }}$. We seek to show that the strong player's winning probability in an interior solution (whenever it exists) in a weak-lead sequential contest, denoted by $p_{W L}$, is greater than $p_{\text {Simu }}$ for any $r \in(0, \infty)$. Because $p_{W L}=t_{B}^{r} /\left(1+t_{B}^{r}\right)$ in interior equilibria and $p_{W L}=0$ in preemptive equilibria, our claim follows easily.

According to Theorem 1, in interior equilibria of weak-lead sequential contests, we always have $t_{B}>1 / \tilde{c}$, i.e., $1 / t_{B}<\tilde{c}$, which further implies that

$$
\frac{t_{B}^{r}}{1+t_{B}^{r}}>\frac{1}{1+\tilde{c}^{r}}
$$

holds true for all values of $r, r \in(0, \infty)$. To prove $t_{B}^{r} /\left(1+t_{B}^{r}\right)>p_{\text {Simu }}$, we show a stronger result that $1 /\left(1+\tilde{c}^{r}\right)>p_{\text {Simu }}$ for $r \in(\bar{r}, \infty)$.

To facilitate the analysis, we first examine two functions $p_{1}$ and $p_{2}$, defined as

$$
\begin{gathered}
p_{1}=\frac{1}{1+\tilde{c}^{r}} \\
p_{2}=1-\frac{\tilde{c}}{r}(r-1)^{\frac{r-1}{r}},
\end{gathered}
$$

for $r \in[1,2]$. Using Proposition 5, we have $1 /(1+\tilde{c})>1-\tilde{c}$ and $1 /\left(1+\tilde{c}^{2}\right)>1-\tilde{c} / 2$, that is, $\left.p_{1}\right|_{r=1}>\left.p_{2}\right|_{r=1}$ and $\left.p_{1}\right|_{r=2}>\left.p_{2}\right|_{r=2}$. We obtain that

$$
\begin{gathered}
\frac{d p_{1}}{d r}=-\frac{\tilde{c}^{r} \ln \tilde{c}}{\left(1+\tilde{c}^{r}\right)^{2}} \\
\frac{d p_{2}}{d r}=-\frac{\tilde{c}}{r^{3}}(r-1)^{\frac{r-1}{r}} \ln (r-1)
\end{gathered}
$$

Using the above results and equation $\tilde{c}^{\bar{r}}=\bar{r}-1$, which determines the unique value of $\bar{r}$, we further derive that

$$
\begin{gathered}
\left.\frac{d p_{1}}{d r}\right|_{r=\bar{r}}=-\frac{(\bar{r}-1) l n \tilde{c}}{\bar{r}^{2}}=\left.\frac{d p_{2}}{d r}\right|_{r=\bar{r}}, \\
\left.p_{1}\right|_{r=\bar{r}}=\frac{1}{\bar{r}}=\left.p_{2}\right|_{r=\bar{r}} .
\end{gathered}
$$

Combining both results: (a) $p_{1}>p_{2}$ at both points $r=1$ and $r=2$, and (b) $p_{1}=p_{2}$ and $d p_{1} / d r=d p_{2} / d r$ at point $r=\bar{r}$, it can be shown that $p_{1}(r) \geq p_{2}(r)$ on $r \in[1,2]$, with equality only holding when $r=\bar{r} .{ }^{44}$

Because $p_{\text {Simu }}$ is a constant, $1-\tilde{c} / 2$, for $r \in[2, \infty)$, and $p_{2}(r=2)=1-\tilde{c} / 2$, it is straightforward to show that $1 /\left(1+\tilde{c}^{r}\right) \geq 1 /\left(1+\tilde{c}^{2}\right)>p_{\text {Simu }}=1-\tilde{c} / 2$ for $r \in[2, \infty)$. Combining the above two results, we have shown that $1 /\left(1+\tilde{c}^{r}\right)>p_{\text {Simu }}$ for all $\tilde{c} \in(0,1)$ and $r \in(\bar{r}, \infty)$.

This finishes the proof that the strong player's winning probability in an interior solution, $p_{W L}=t_{B}^{r} /\left(1+t_{B}^{r}\right)$, is greater than $p_{\text {Simu }}$ for all $r \in(0, \infty)$ whenever the interior solution exists. Given the result that $p_{\text {Simu }}>1 / 2$ and $p_{W L}=0$ in any preemptive equilibrium of a weak-lead sequential contest, the claims of (i), (ii), and (iii) follow.

When $c=1$, direct calculation shows that $p_{S L}=p_{W L}=p_{\text {Simu }}=1 / 2$ for $r \in\left(0, \hat{r}_{s}\right)$. When $r \geq \hat{r}_{s}, p_{S L}=1$ in a sequential contest led by player $A, p_{W L}=0$ in a sequential contest led by $B$, and $p_{\text {Simu }}=1 / 2$. This completes the proof for claim (iv).

## A. 16 Proof of Theorem 5

Proof. (i) When $\tilde{c} \in\left(0,1 / \hat{c}^{h}\right)$, the winning probabilities are ranked as: $p_{S L}>p_{W L}>$ $p_{\text {Simu }}>1 / 2$, which are the winning chances of the strong player (player $A$ ) in the three contest formats. We can write down the normal form game between the two coaches as shown in Table A.1.

This is a zero-sum-type normal form stage game and it is easy to see that there is no pure-strategy equilibrium. There is a unique mixed-strategy equilibrium in which player $A$ randomizes between two actions, Lead and Follow, with probabilities $\left(p_{W L}-p_{\text {Simu }}\right) /\left(p_{S L}+\right.$

[^27]Table A.1: The coaches-decision stage game that determines the move order.

\[

\]

$\left.p_{W L}-2 p_{\text {Simu }}\right)$ and $\left(p_{S L}-p_{\text {Simu }}\right) /\left(p_{S L}+p_{W L}-2 p_{\text {Simu }}\right)$, while player $B$ randomizes between Lead and Follow with probabilities $\left(p_{S L}-p_{\text {Simu }}\right) /\left(p_{S L}+p_{W L}-2 p_{\text {Simu }}\right)$ and $\left(p_{W L}-\right.$ $\left.p_{\text {Simu }}\right) /\left(p_{S L}+p_{W L}-2 p_{\text {Simu }}\right)$. Next, we show that given the opponent choosing the equilibrium strategy, each player is indifferent between choosing Lead and Follow.

We present the solution process for the mixed-strategy equilibrium, which also serves as a proof of its uniqueness. Let $(x, 1-x)$ and $(y, 1-y)$ be the strategy profiles of the two players' coaches; they are the probabilities that coaches $A$ and $B$ place on strategies Lead and Follow, respectively. In a mixed-strategy equilibrium, if it exists, the randomization $(x, 1-x)$ by coach $A$ makes coach $B$ indifferent between choosing Lead and Follow. From Table A.1, we obtain the equation

$$
x\left(1-p_{\text {Simu }}\right)+(1-x)\left(1-p_{W L}\right)=x\left(1-p_{S L}\right)+(1-x)\left(1-p_{\text {Simu }}\right),
$$

which is

$$
x\left(p_{S L}-p_{\text {Simu }}\right)=(1-x)\left(p_{W L}-p_{\text {Simu }}\right) .
$$

This implies that

$$
x=\frac{p_{W L}-p_{\text {Simu }}}{p_{S L}+p_{W L}-2 p_{\text {Simu }}} .
$$

Likewise, the randomization $(y, 1-y)$ by coach $B$ makes coach $A$ indifferent between choosing Lead and Follow. We have

$$
y p_{\text {Simu }}+(1-y) p_{S L}=y p_{W L}+(1-y) p_{\text {Simu }}
$$

which is

$$
y\left(p_{W L}-p_{\text {Simu }}\right)=(1-y)\left(p_{S L}-p_{\text {Simu }}\right),
$$

and therefore

$$
y=\frac{p_{S L}-p_{S i m u}}{p_{S L}+p_{W L}-2 p_{\text {Simu }}} .
$$

It is straightforward to verify that $0<x<y<1$, as $p_{S L}>p_{W L}>p_{\text {Simu }}>1 / 2$. Thus, our result follows. Notice that in this mixed-strategy equilibrium, $x<y$ implies that the strong contestant chooses Lead with a greater probability, while the weak contestant chooses Follow
with a greater probability.
(ii) When $\tilde{c} \in\left[1 / \hat{c}^{h}, 1\right)$, all weak-lead sequential contests have interior equilibria for $r \in\left(0, \hat{r}_{w}^{1}\right) \cup\left(\hat{r}_{w}^{2}, \infty\right)$, and the ranking of the winning probabilities remains the same, $p_{S L}>$ $p_{W L}>p_{\text {Simu }}>1 / 2$. We can obtain the same result for the mixed-strategy equilibrium as in case (i) following the same procedure.
(iii) When $\tilde{c} \in\left[1 / \hat{c}^{h}, 1\right)$, weak-lead sequential contests have preemptive equilibria for $r \in\left[\hat{r}_{w}^{1}, \hat{r}_{w}^{2}\right]$, in which the strong follower $A$ is preempted. The winning probabilities for player $A$ in the three contest formats are ranked as $p_{S L}>p_{\text {Simu }}>1 / 2>p_{W L}=0$ by Proposition 6. Choosing Lead is a dominant strategy for both coaches. In this situation, both coaches choose Lead and the endogenous equilibrium move order results in a simultaneous contest.
(iv) When $\tilde{c}=1$, players $A$ and $B$ are symmetric in ability. The winning probabilities are equal to $1 / 2$ for both coaches in any contest format for $r<\hat{r}_{s}$; the equilibrium of the coach-decision game is a collection of continuum mixed strategies in which both contestants randomize arbitrarily between choosing Lead and Follow. However, they all prefer to choose Lead and preempt the opponent for $r \geq \hat{r}_{s}$, as the equilibria are preemptive in sequential contests. Therefore, both contestants choosing Lead results in a simultaneous contest.


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[^1]:    ${ }^{1}$ Simultaneous-move contests have been thoroughly studied in the literature, and the players' behaviors are well understood.
    ${ }^{2}$ This type of equilibrium is referred to as an interior equilibrium in this paper.

[^2]:    ${ }^{3}$ For instance, in aircraft procurement processes, the rules for determining the winning contender are standard for all participants, resulting in a relatively accurate winning technology with a large value for $r$. Conversely, when the government is developing a next-generation weapon system, winning a contract from the Department of Defense can be influenced by numerous random factors, resulting in a noisy winning technology and a small value for $r$.
    ${ }^{4}$ For $r$ close to zero, winning is primarily by luck, resulting in a low accuracy level of the contest. As $r$ increases, the player with higher effort enjoys a greater probability of winning, leading to a more accurate contest. This trend continues until it converges to the case of an all-pay auction as $r \rightarrow \infty$.

[^3]:    ${ }^{5}$ The marginal cost ratio $c$ indicates a strong-lead contest if $c \leq 1$ (in particular, $c=1$ corresponds to the case with two symmetric players) and a weak-lead contest when $c>1$.

[^4]:    ${ }^{6}$ These scenarios closely mirror certain real-world practices where the objectives of coaches and contestants on the same team are not perfectly aligned. In sports, for example, while a coach endeavors to maximize the team's winning chances, the athlete, in maximizing their expected payoff, must balance their expected revenue (winning chance times prize value) with the incurred cost (marginal cost times effort). Similarly, this scenario is prevalent in other economic or political events. For instance, senior managers overseeing R\&D scientists in an innovation contest, or political advisors guiding election candidates in their campaigns for political office, face similar dynamics.
    ${ }^{7}$ We show that this interval disappears when the two players are sufficiently asymmetric in ability.

[^5]:    ${ }^{8}$ Intuitively, when the contestants' asymmetry level is sufficiently large, we always have a mixed-strategy equilibrium. In these cases, it is optimal for the strong player to choose a different strategy as his opponent does; while it is optimal for the weak player to choose the same strategy as his opponent does. Given the above best responses of the two players, a pure-strategy equilibrium is never possible. Conversely, when the contestants' asymmetry level is sufficiently small, we have a pure-strategy equilibrium for $r$ in a preemptive interval. In these cases, the weak player is strong enough to preempt the strong player in a weak-lead sequential contest, so choosing Lead is a dominant strategy for either player, leading to a simultaneous contest with certainty.
    ${ }^{9}$ For example, we show that, for any $r \in(0, \infty)$, a strong-lead sequential contest maximizes the strong player's winning chance among the three contest formats.
    ${ }^{10}$ The coach-contestant model can be seen as an extension of the original model with players $A$ and $B$, with the addition of coaches $A$ and $B$, who aim to maximize their players' winning chances, respectively.

[^6]:    ${ }^{11}$ See Konrad (2009) and Fu and Wu (2019) for surveys.
    ${ }^{12}$ For $r$ in the middle range, a mixed-strategy equilibrium of Wang (2010) exists, in which the strong player adopts a pure strategy and the weak player adopts a mixed strategy. For $r$ in the high range, the all-pay auction equilibrium (in mixed strategies) of Alcalde and Dahm (2010) exists.
    ${ }^{13}$ His main result holds for two CSF forms: the logit form, which is more general than the Tullock CSF, and the probit form, which is similar to the tournament model of Lazear and Rosen (1981).
    ${ }^{14}$ In a similar model with endogenous timing of moves, Protopappas (2023) demonstrates that the designer's payoff can be enhanced by offering a lower price to the player(s) who move(s) early.
    ${ }^{15}$ In the literature on multi-battle contests (e.g., Klumpp and Polborn, 2006; Fu, Ke, and Tan, 2015; Barbieri and Serena, 2021), the battles in a best-of-three contest can be played sequentially, where each

[^7]:    battle is modeled as a Tullock contest with players exerting effort simultaneously. Some papers refer to these contests as "sequential contests." However, in this paper, the term "sequential contests" is defined differently, wherein players make an effort sequentially in a single-battle contest.
    ${ }^{16}$ Klunover (2018) derives narrow bounds for the rent-dissipation rate using the same method.
    ${ }^{17}$ Notably, to the best of our knowledge, existing literature on sequential-move contests, including Dixit (1987), which focuses on cases equivalent to Tullock contests with $r<1$, does not cover the case with $r>1$.
    ${ }^{18}$ The detailed analysis is provided in a separate paper (Gao, Lu, and Wang, 2024) due to space constraints.

[^8]:    ${ }^{19}$ See Kovenock and Roberson (2012) for an overview of this literature.
    ${ }^{20} \mathrm{Fu}$, Wu, and Zhu (2022) also establish equilibrium existence in a generalized multi-prize nested lottery contest model without assuming symmetric players.

[^9]:    ${ }^{21}$ The case where $c=1$ (i.e., $c_{L}=c_{F}$ ) is also referred to as a symmetric sequential contest.
    ${ }^{22}$ There is no loss of generality in assuming that the follower would choose zero effort when he is indifferent because the leader can always increase his effort by an arbitrarily small amount so that the follower stays inactive as his best response.

[^10]:    ${ }^{23}$ The preemptive effort level of the leader, $x_{L}$, is always strictly positive in a preemptive solution by definition.
    ${ }^{24}$ Note that $x_{F} \leq 1 / c_{F}$ can be obtained by the participation constraint of $\pi_{F} \geq 0$.

[^11]:    ${ }^{25}$ As mentioned earlier, $x_{L}>0$ in equilibrium, which ensures that $t$ is well-defined.

[^12]:    ${ }^{26}$ When $x_{i}\left(t^{*}\right)$ is a global maximizer, his expected payoff at $x_{i}\left(t^{*}\right)$ is greater than that at $x_{i}=0$, which implies that player $i$ 's participation constraint holds automatically.

[^13]:    ${ }^{27}$ Also, the strong-weak effort ratio is greater in a sequential contest than in a simultaneous contest.

[^14]:    ${ }^{28}$ This system of equations has two unknowns $r$ and $t$. Recall that functions $L$ and $R$ are given by (12).
    ${ }^{29}$ When two solutions exist, the larger value is chosen for $t^{*}$ since the smaller solution would lead to a strictly negative expected payoff for the follower.

[^15]:    ${ }^{30}$ Note that the leader's effort is greater in a preemptive solution than in its corresponding interior solution when both types of solutions exist.

[^16]:    ${ }^{31} \mathrm{An}$ interpretation for the absence of interior solutions is provided in Remark 2 following Proposition 3.
    ${ }^{32}$ The value of $\hat{c}^{h}$ is determined by the unique solution of the following system of equations with two unknowns $c$ and $r: \pi_{L}^{i n t}\left(t^{*} ; c, r\right)=\pi_{L}^{p r e}(c, r)$ and $\frac{d}{d r} \pi_{L}^{p r e}=\frac{d}{d r} \pi_{L}^{i n t}$.

[^17]:    ${ }^{33} \mathrm{~A}$ simultaneous move contest prevails if both coaches choose the same action.

[^18]:    ${ }^{34}$ Because either player ( $A$ or $B$ ) can be the leader in a sequential contest, it suffices to consider $c_{A} \leq c_{B}$ in analyzing the three contest formats: (i) a strong-lead sequential contest, where player $A$ exerts effort first; (ii) a weak-lead sequential contest, where player $B$ exerts effort first; (iii) a simultaneous contest, where both players exert effort simultaneously.

[^19]:    ${ }^{35}$ Once the ranking order of player $A$ winning probabilities (in the three contest formats) is given, the ranking order of player $B$ 's winning probabilities follow straightforwardly.

[^20]:    ${ }^{36}$ With $\tilde{c}=1$, player $A$ is as strong as player $B$. For consistency in notation, we continue to use $p_{S L}$ and $p_{W L}$ to denote the winning probabilities of player $A$ in sequential contests where player $A$ and $B$ are the leaders, respectively.

[^21]:    ${ }^{37}$ The logic of having a mixed-strategy equilibrium is the same as that given in the previous paragraph.
    ${ }^{38}$ Characterizing interior solutions and comparing them to their corresponding preemptive solutions is highly challenging due to the lack of explicit expressions for interior solutions.

[^22]:    ${ }^{39}$ In Gao, Lu, and Wang (2024), which is a follow-up work to this paper, we demonstrate that a stronglead sequential contest is effort-maximizing when the contest's accuracy level is in the low or high range, but either a simultaneous or a weak-lead sequential contest can be effort-maximizing when the contest's accuracy level falls within the middle range. These results contrast with the existing literature focusing on $r=1$ (Linster, 1993), which suggests that a strong-lead sequential contest is always optimal for maximizing total effort. Gao, Lu, and Wang (2024) reveal that Linster's results hold for $r<1$ but vary significantly for $r>1$. In an ongoing research project, we further explore how payoff-maximizing contestants would choose their move order independently, which introduces an additional dimension in the decision-making of the contestants, closely resembling many economic practices.

[^23]:    ${ }^{40}$ It is straightforward to see that when $\hat{x}_{L}$ is weakly smaller than $\frac{1}{2 r c_{L}}$, there is at most a single solution to the leader's FOC.

[^24]:    ${ }^{41}$ Following a similar procedure in the previous case with $r \in[1, \infty)$ and $c \in(1, \infty)$, it can be shown that $t^{*}$ is the larger solution, since the smaller one leads to a strictly negative payoff for the follower.

[^25]:    ${ }^{42}$ Note that here $r=\sqrt{2}$ is the maximal value of $r$ such that $t^{*} \leq 1$, which occurs when $c=1, t^{*}=1$, and $R^{\prime}(t=1)=-1$.

[^26]:    ${ }^{43}$ It can be shown that this term goes to 1 when $r$ goes to infinity.

[^27]:    ${ }^{44}$ The proof is omitted for brevity. It can be provided upon request from the authors.

