# Online Supplement to "The Effort-Maximizing Move Order in Tullock Contests"

In this Online Supplement, we first present the existing results of the equilibrium characterization of all three contest formats—simultaneous, strong-lead and weak-lead sequential. We then provide the illustrative figures for the other cases of our findings of the optimal move order. The rest of the Supplement contains all the proofs of our findings in the main article.

## S.1 Review of Equilibrium Characterization

### S.1.1 Model Setup

The following two subsections review the equilibrium characterizations of the three contest formats, which serve as building blocks for our analysis of the effort-maximizing move order in two-player contests in the accompanying paper. Since the two-player simultaneous-move Tullock contest has been well studied in the literature, Section S.1.2 first summarizes existing equilibrium results for this model, where the contestants are denoted by players $A$ and $B$. Section S.1.3 then reviews the main findings of Gao, Lu, and Wang (2025), who provide a complete equilibrium analysis of a general two-player sequential Tullock contest with a leader and a follower. In this model, the contestants are labeled as $L$ (leader) and $F$ (follower).

### S.1.2 Equilibria in Simultaneous Contests

We consolidate the results of Nti (1999), Wang (2010), and Alcalde and Dahm (2010) into the following lemma, using our notation for the Tullock contest with players $A$ and $B$.[20]

**Lemma S.1.** *Consider a simultaneous Tullock contest in which players $A$ and $B$ compete for a unit prize. The two players have marginal costs $c_A$ and $c_B$, respectively,*

[20]For simplicity, the expectation operator is omitted throughout this paper.

*and* $c_A/c_B \in (0,1]$. *For any* $r \in (0,\infty)$, *the expected total effort in equilibrium is given by*

$$TE_{simu} = \begin{cases} \dfrac{(c_A+c_B)c_A^{r-1}c_B^{r-1}}{(c_A^r+c_B^r)^2}r & \text{for } r \in (0,\bar{r}], \\ \left(\dfrac{c_A+c_B}{c_B^2}\right)\dfrac{(r-1)^{\frac{r-1}{r}}}{r} & \text{for } r \in [\bar{r},2], \\ \dfrac{c_A+c_B}{2c_B^2} & \text{for } r \in [2,\infty), \end{cases}$$

*where* $\bar{r}$ *is uniquely determined by* $c_A/c_B = (\bar{r}-1)^{\frac{1}{\bar{r}}}$, *and* $\bar{r} \in (1,2]$.

### S.1.3 Equilibria in Sequential Contests

This section briefly reviews the equilibrium characterization in the general sequential-move Tullock contest model of Gao, Lu, and Wang (2025), where the two contestants are labeled $L$ (leader) and $F$ (follower). The analysis and results below will be used in Section 4, which provides proofs of the results in the accompanying paper.

In our sequential-move contest model, for any $c = \frac{c_L}{c_F} \in (0,\infty)$ and $r \in (0,\infty)$, the leader first exerts effort $x_L$ in stage 1. Upon observing $x_L$, the follower chooses effort $x_F$ in stage 2 to maximize their expected payoff. The follower's optimal choice $x_F$ either satisfies the following necessary first-order condition (FOC)

$$r x_F^{r-1} x_L^r = c_F \left(x_L^r + x_F^r\right)^2, \tag{1}$$

where $x_F > 0$ if $x_L$ is sufficiently small, or satisfies a corner condition $x_F = 0$ if $x_L$ is sufficiently large. Expecting the follower's best response in stage 2, denoted by $x_F(x_L)$, the leader chooses $x_L$ in stage 1 to maximize their expected payoff.

Let $\hat{x}_L$ denote the leader's *preemptive effort*, defined as the minimum effort required to preempt the follower.

When $r < 1$, the leader's optimal effort choices are always non-preemptive. With a slight abuse of notation, we write $\hat{x}_L = \infty$ in this case. When $r > 1$, the leader's

preemptive effort $\hat{x}_L$ is given by

$$\hat{x}_L = \frac{1}{c_F}\left(\frac{1}{r}\right)(r-1)^{\frac{r-1}{r}}. \tag{2}$$

The follower's best response to any $x_L \in (0, \hat{x}_L)$, denoted by $x_F(x_L)$, is given by the greater of the two solutions to equation (1). When $r = 1$, for any $x_L \in (0, \hat{x}_L)$, we have $x_F(x_L) = \sqrt{\frac{x_L}{c_F}} - x_L$, and the leader's preemptive effort is $\hat{x}_L = \frac{1}{c_F}$.

Gao, Lu, and Wang (2025) introduce the quotient $t$ as the follower-leader effort ratio, $t = \frac{x_F}{x_L}$, which is a function of $x_L$, as $x_F$ is essentially a function of $x_L$.[21] They further derive the following *characteristic equation*, which must be satisfied by the equilibrium effort ratio $t$ for given $(c, r)$:

$$c - t = cr\left(\frac{2}{1+t^r} - 1\right). \tag{3}$$

A valid solution, $t^*$, must satisfy both (3) and the players' participation constraints. This characteristic equation plays a vital role in identifying and characterizing the non-preemptive equilibrium candidate, and assists us in determining the types of sequential contest equilibria when both the preemptive and non-preemptive optimal effort levels coexist, and in ranking the three contest formats in the present study.

In what follows, we briefly describe the equilibrium types across different regions of the $(r, c)$ space. In a sequential contest with $r \in (0, 1)$, for any given $c > 0$, there always exists a unique equilibrium, which is non-preemptive. Intuitively, when the contest is sufficiently noisy (i.e., when $r \in (0, 1)$), it is impossible for the leader to preempt the follower, because the follower always has an incentive to exert positive effort in response to any finite effort from the leader.

In strong-lead sequential contests where $r \in [1, \infty)$ and $c \in (0, \frac{1}{2}]$, all equilibria are preemptive. In contrast, for contests with $r \in [1, \infty)$ and $c \in (\frac{1}{2}, 1]$, there exists a crit-

[21] If $t = 0$ in an equilibrium, it must be the case that $x_F = 0$, which corresponds to a preemptive equilibrium candidate. If $t > 0$ occurs in equilibrium, we have $x_F > 0$, which corresponds to a non-preemptive equilibrium candidate where both players exert strictly positive effort.

ical value $\hat{r}_s$, defined as a unique solution to the equation $\pi_L^{\text{nonp}}(t^*; c, r) = \pi_L^{\text{pre}}(c, r)$,[22] such that equilibria are non-preemptive for $r < \hat{r}_s$ and preemptive for $r \geq \hat{r}_s$. When $c \leq 1$, the critical value $\hat{r}_s \in [1, \sqrt{2})$, and it increases weakly with $c$, indicating that as the difference in players' abilities grows, the strong leader becomes more likely to preempt the weak follower.

In weak-lead sequential contests with $r \in [1, \infty)$ and $c \in (1, \infty)$, there exists a threshold value of $c$, denoted by $\hat{c}^h$, where $\hat{c}^h \approx 1.983$. When $c \in (1, \hat{c}^h]$, there exist two values, $\hat{r}_w^1$ and $\hat{r}_w^2$, with $1 < \hat{r}_w^1 \leq \hat{r}_w^2$, which are the two solutions to the equation $\pi_L^{\text{nonp}}(t^*; c, r) = \pi_L^{\text{pre}}(c, r)$ for $c \in (1, \hat{c}^h)$.[23] In this region, the leader preempts the follower in equilibrium if and only if $r \in [\hat{r}_w^1, \hat{r}_w^2] \subset \mathbb{R}^+$.[24]

## S.2 The Illustrative Figures for Other Cases

In generating the figures, we normalize $c_B = 1$ to facilitate the numerical computation of the equilibrium total effort which is without loss of generality. In each figure, the blue dotted line depicts the total effort in strong-lead contests; the red dotted line shows the total effort in weak-lead contests; the black solid line represents the total effort in simultaneous contests.[25]

[22] They are the leader's expected payoffs from a non-preemptive equilibrium candidate and a preemptive one, respectively.

[23] In particular, when $c = \hat{c}^h$, we have $\hat{r}_w^1 = \hat{r}_w^2$, yielding a unique solution.

[24] This *preemptive interval* $[\hat{r}_w^1, \hat{r}_w^2]$ becomes smaller as $c$ increases, and it eventually shrinks to a single point as $c$ approaches $\hat{c}^h$. Intuitively, the weak leader becomes less likely to preempt the strong follower as their ability difference grows. When the ability difference is sufficiently large ($c > \hat{c}^h$), only non-preemptive equilibria exist for any $r$. In these cases, preemption becomes prohibitively costly for the weak leader when facing a strong follower.

[25] While Case (c) is presented in the accompanying paper, we provide figures here for the remaining four cases—Cases (a), (b), (d), and (e). For Case (d), Lemma 3 shows that $\hat{r}_{SL}^1 \geq \hat{r}_w^1$, with a threshold $\hat{r}_{sw} \in [\hat{r}_w^1, \hat{r}_{SL}^1]$ where $TE_{SL} \gtrless TE_{WL}$ if $r \lessgtr \hat{r}_{sw}$. In Case (e), the equality $Sequ = Simu$ holds for all $r \in (0, \hat{r}_s)$, indicating identical total effort between sequential and simultaneous contests.

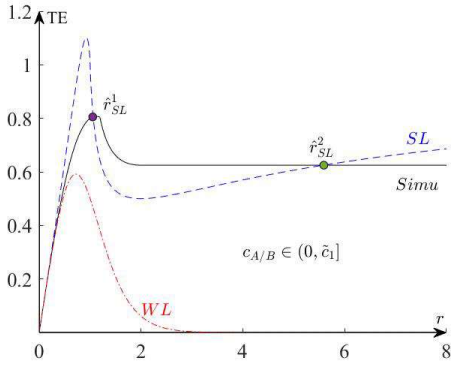
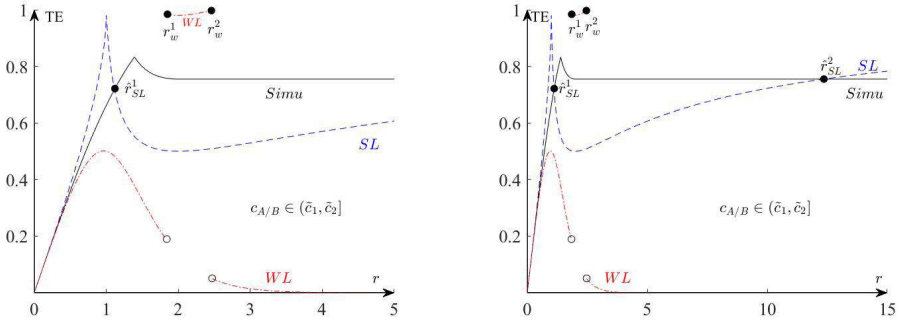



Figure 1: The total effort as a function of $r$ in Case (a), where $c_{A/B} \in (0, \tilde{c}_1]$. Illustrated for $c_{A/B} = 0.25$.




$r$ with smaller value; $r$ in a larger region.

Figure 2: The total effort as a function of $r$ in Case (b), where $c_{A/B} \in (\tilde{c}_1, \tilde{c}_2]$. Illustrated for $c_{A/B} = 0.51$.

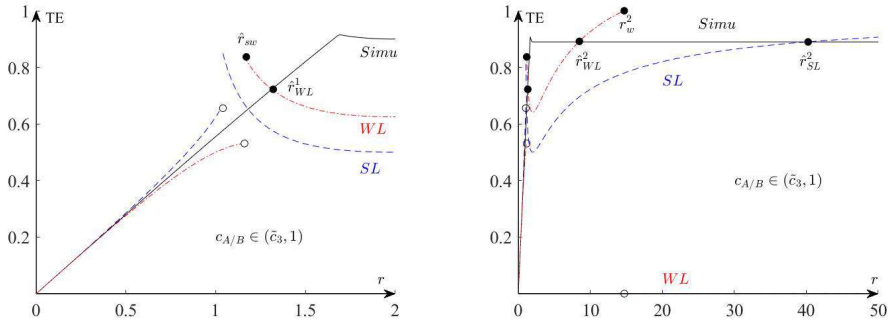
## S.3 Proofs

In this section, we provide proofs of the main results in the accompanying paper.

### S.3.1 Proof of Theorem 1

*Proof.* We first present an important continuity result for total effort at the threshold value $r = \hat{r}_s$—see Lemma S.2 and its proof below for details—which will be used in the subsequent proof of Theorem 1.

**Lemma S.2.** *Consider strong-lead sequential contests with* $c \in (0, 1]$.




$r$ near 1; $r$ in a larger region.

Figure 3: The total effort as a function of $r$ in Case (d), where $c_{A/B} \in (\tilde{c}_3, 1)$. Illustrated for $c_{A/B} = 0.8$ with $r$ in different scales.

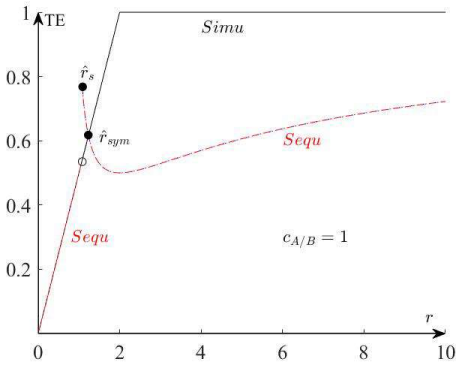



Figure 4: The total effort as a function of $r$ in Case (e), where $c_{A/B} = 1$.

*(i) When* $c \in (0, \frac{1}{2}]$, $\hat{r}_s = 1$, *it holds true that* $\lim_{r \to \hat{r}_s -} TE^{nonp}_{strong} = \lim_{r \to \hat{r}_s +} TE^{pre}_{strong}$.
*(ii) When* $c \in (\frac{1}{2}, 1]$, $\hat{r}_s > 1$, *it holds true that* $\lim_{r \to \hat{r}_s -} TE^{nonp}_{strong} < \lim_{r \to \hat{r}_s +} TE^{pre}_{strong}$.

**Proof of Lemma S.2 (i)**

The leader's expected payoff from preemption is given by $\pi^{pre}_L = 1 - c_L \hat{x}_L = 1 - \frac{c}{r}(r-1)^{\frac{r-1}{r}}$ for $r > 1$ and $1 - c$ for $r = 1$. From (2) and $\pi^{pre}_L$, the leader's preemptive effort converges to $\hat{x}_L = \frac{1}{c_F}$ and their expected payoff converges to $1 - c$, as $r$ approaches $1^+$. If we define $h(r) = \frac{1}{r}(r-1)^{\frac{r-1}{r}}$ for $r > 1$ and 1 for $r = 1$, $\pi^{pre}_L = 1 - c \cdot h(r)$.

When the preemptive equilibrium candidate exists, $\pi^{pre}_L$ strictly increases with $r$

when $r \in [1,2)$ and strictly decreases with $r$ when $r \in (2,+\infty)$, implying that $\pi_L^{pre}$ reaches its global maximum at $r=2$, for $r \in [1,\infty)$.

When $c \in (0,\frac{1}{2}]$, we have shown that $\hat{r}_s = 1$.

Using the following expressions of the two players' equilibrium effort levels obtained in Gao, Lu, and Wang (2025),

$$x_L = \frac{rt^{r-1}}{c_F\left(1+t^r\right)^2} \text{ and } x_F = \frac{rt^r}{c_F\left(1+t^r\right)^2}, \tag{4}$$

we obtain that $TE_{strong}^{nonp} = \frac{r(t^*)^{r-1}(1+t^*)}{c_F(1+(t^*)^r)^2}$ and $TE_{strong}^{pre} = \frac{1}{c_F}\left(\frac{1}{r}(r-1)^{\frac{r-1}{r}}\right)$. Using the fact that $t^* \to 0$ when $r \to 1$, we obtain that $\lim_{r\to\hat{r}_s-} TE_{strong}^{nonp} = \frac{1}{c_F} = \lim_{r\to\hat{r}_s+} TE_{strong}^{pre}$. For $c \in (0,\frac{1}{2}]$ and $r \leq 1$, there exists a unique solution of $t^*$.

As $r$ increases to 1, $t^*$ continuously decreases to 0, based on our observations of the functions $z = L(t)$ and $z = R(t)$ on the $tz$-plane,[26] where the two functions are defined as follows: $L(t) = c - t$ and $R(t) = cr(\frac{2}{1+t^r} - 1)$, which represent the left-hand side (LHS) and right-hand side (RHS), respectively, of equation (3). The effort ratio in a non-preemptive solution $t^*$, when it exists, must be a crossing point of $z = L(t)$ and $z = R(t)$ on the $tz$-plane. Because $t^* = 0$ implies that $x_F = 0$, the strong leader starts to preempt at $r = 1$ and the equilibrium effort varies continuously with $r$ in sequential contests. This completes the proof of Part (i) of Lemma S.2.

**Proof of Lemma S.2 (ii)**

In the proof of Theorem 3 in Gao, Lu, and Wang (2025), we establish the following result: when $c \in (\frac{1}{2},1]$, at $r = \hat{r}_s$, where $\hat{r}_s > 1$, the strong leader is indifferent between choosing the two equilibrium candidates—non-preemptive and preemptive. Let the leader's effort levels under the two equilibrium candidates be denoted by $x_L^{nonp}$ and $x_L^{pre}$, respectively. Let $\pi$ be the leader's expected payoff level at $r = \hat{r}_s$. Thus, we have $\pi_L^{nonp} = \pi_L^{pre} = \pi > 0$. Since $c = c_L/c_F$, it follows that $c_L x_L = c c_F x_L$, and we

[26] See Claim 3 in Appendix A.5 of Gao, Lu, and Wang (2025) for four key observations derived from analyzing these functions.

can further derive that

$$\pi_L^{nonp} = \frac{1}{1+t^r} - cc_F x_L^{nonp} = \pi, \tag{5}$$

$$\pi_L^{pre} = 1 - cc_F x_L^{pre} = \pi. \tag{6}$$

Since $t = x_F^{nonp}/x_L^{nonp}$, we have $x_F^{nonp} = tx_L^{nonp}$. Using (5) and (6), we derive the total effort levels under the two cases:

$$TE_{strong}^{nonp} = (1+t)x_L^{nonp} = \frac{1+t}{cc_F}\left(\frac{1}{1+t^r} - \pi\right),$$

$$TE_{strong}^{pre} = x_L^{pre} = \frac{1}{cc_F}(1-\pi).$$

Therefore, at $r = \hat{r}_s$, $TE_{strong}^{pre} > TE_{strong}^{nonp}$ if $\pi > \frac{1-t^{r-1}}{1+t^r}$. Using (5), the above inequality holds true if $cc_F x_L^{nonp} < \frac{t^{r-1}}{1+t^r}$. Using (4), we see that the above inequality holds true if $1+t^r > cr$. By (3), we derive that $1+t^r = 2cr/(cr+c-t)$, which implies that $1+t^r > cr$ holds true if $t > c+cr-2$. Since it has been shown that $t > cr-c$—see the second inequality of Proposition 4 in Gao, Lu, and Wang (2025)—the condition $t > c+cr-2$ holds if $cr-c \geq c+cr-2$, which clearly holds for $c \leq 1$. Therefore, when $c \in (\frac{1}{2}, 1]$, we always have $TE_{strong}^{nonp} < TE_{strong}^{pre}$ at $r = \hat{r}_s$. This completes the proof of Part (ii) of Lemma S.2.

Using the results in the proofs of Theorems 2 and 3 of Gao, Lu, and Wang (2025), we can show that $\hat{r}_s < \bar{r}$ for all $c \in (0, 1]$.

Next, we first derive the expected equilibrium effort levels of simultaneous contests for $c \in (0, 1)$. According to Nti (1999), Wang (2010), and Alcalde and Dahm (2010),

the total effort of a simultaneous contest is given by[27]

$$TE_{simu} = \begin{cases} \frac{(1+c)rc^{r-1}}{c_F(1+c^r)^2} & \text{for } r \in (0, \bar{r}], \\ \frac{(1+c)(r-1)^{\frac{r-1}{r}}}{c_F r} & \text{for } r \in [\bar{r}, 2], \\ \frac{1+c}{2c_F} & \text{for } r \in [2, \infty), \end{cases}$$

where $\bar{r}$ is determined by equation $c^{\bar{r}} = \bar{r} - 1$.

Given $c \in (0,1)$, when $r \in (0, \hat{r}_s)$, the total effort in a non-preemptive equilibrium of a sequential contest is given by $TE^{nonp}_{strong} = \frac{r(t^*)^{r-1}(1+t^*)}{c_F(1+(t^*)^r)^2}$. In this range of parameter values, one can see that the total effort levels of a sequential contest and its corresponding simultaneous contest are the function $(rt^{r-1}(1+t))/(c_F(1+t^r)^2)$ evaluated at $t = t^*$ and $t = c$, respectively. This is because when $t = c = c_L/c_F$, the total effort in a corresponding simultaneous contest is given by $TE_{simu} = \frac{rc_L^{r-1}c_F^{r-1}}{(c_L^r+c_F^r)^2}(c_L + c_F)$.

Recall that given $c \in (0,1)$, when $r \in (0, \hat{r}_s)$, we always have $t^* < c < 1$. To compare the two effort levels, it suffices to examine the monotonicity of this function. We need to show that this function is strictly decreasing in $t$ on $[t^*, c]$ for any given $r \in (0, \hat{r}_s)$.

It can be derived that

$$d\left\{\frac{t^{r-1}(1+t)}{(1+t^r)^2}\right\}/dt = \frac{t^{r-2}}{(1+t^r)^3}\left[r - 1 + rt - (r+1)t^r - rt^{r+1}\right]. \tag{7}$$

The term $[r - 1 + rt - (r+1)t^r - rt^{r+1}]$ is equal to $r - 1$ if $t = 0$ and $-2$ if $t = 1$. It is clear that this term is negative for any $r \in (0, 1]$. When $c \in (0, \frac{1}{2}]$, $\hat{r}_s = 1$, this term is negative for any $r \in (0, \hat{r}_s)$. The situation is subtle when $c \in (\frac{1}{2}, 1)$ and $r \in (1, \hat{r}_s)$ as $\hat{r}_s > 1$. Taking derivative of this term, we get $r - r(r+1)t^{r-1}(1+t)$, which is positive at $t = 0$. Thus, when $t$ increases, this term increases from $r - 1 > 0$ at $t = 0$, and then decreases to $-2$ at $t = 1$. Next, we show that this term is negative at $t = t^*$ for any given $r \in (1, \hat{r}_s)$.

[27] For notational simplicity, we omit the expectation operator throughout this Supplement.

Rewriting equation (3), we have that $t^r = \frac{cr-(c-t)}{cr+(c-t)}$, and the term in the square bracket of (7) can be rewritten as $2rt\left(\frac{c-t-1}{cr+c-t}\right)$, which is strictly negative at $t = t^*$, because for $r \in (1, \hat{r}_s)$, where $\hat{r}_s < \bar{r} \leq 2$, we get $t^* < c < 1$, and thus $c - t^* - 1 < 0$ and $cr + c - t > 0$. Therefore, we have shown that $(rt^{r-1}(1+t))/(c_F(1+t^r)^2)$ is strictly decreasing in $t$ on $[t^*, c]$ for any given $r \in (0, \hat{r}_s)$.

Now consider the case in which $c > 1$. Equation (7) can be rewritten as

$$d\left\{\frac{t^{r-1}(1+t)}{(1+t^r)^2}\right\}/dt = \frac{t^{r-2}}{(1+t^r)^3}\left[(r-1) - (r+1)t^r + rt(1-t^r)\right]. \tag{8}$$

When $c > 1$, that is, when $c_L > c_F$, we have $t = t^* > c > 1$, and it can be shown that (8) is negative, since $rt(1-t^r) < 0$ and $(r-1) - (r+1)t^r < 0$. This indicates that the total effort is larger in a simultaneous contest than that in a corresponding weak-lead sequential contest.

Hence, we have $TE^{nonp}_{strong} > TE_{simu}$ for $r \in (0, \hat{r}_s)$.

We now examine $TE^{pre}_{strong}$ and $TE_{simu}$ for $r \in [\hat{r}_s, 2]$. Recall that $\hat{r}_s < \bar{r}$. When $r \in [\bar{r}, 2]$, we derive that $TE^{pre}_{strong} = \left(\frac{1}{c_F}\right) h(r) < \left(\frac{1+c}{c_F}\right) h(r) = TE_{simu}$.

We seek to show that $TE^{pre}_{strong} > TE_{simu}$ at $r = \hat{r}_s$. When $c \in (0, \frac{1}{2}]$, $\hat{r}_s = 1$ and $t^* = 0$, then at $r = \hat{r}_s = 1$, $TE^{pre}_{strong} = \frac{1}{c_F} > \frac{1}{c_F}\left(\frac{1}{1+c}\right) = TE_{simu}$. When $c \in (\frac{1}{2}, 1)$, using Lemma S.2 $(ii)$, we obtain that $TE^{pre}_{strong} > TE_{simu}$, since $TE_{simu}(r = \hat{r}_s) < \lim_{r \to \hat{r}_s -} TE^{nonp}_{strong} < \lim_{r \to \hat{r}_s +} TE^{pre}_{strong}$.

So far, we have shown that $TE^{pre}_{strong} > TE_{simu}$ at $r = \hat{r}_s$ and $TE^{pre}_{strong} < TE_{simu}$ for all $r \in [\bar{r}, 2]$. Also, notice that when $r \in [\hat{r}_s, \bar{r}]$, both $TE^{pre}_{strong}$ and $TE_{simu}$ are continuous. Moreover, it can be shown that $TE^{pre}_{strong}$ is monotonically decreasing in $r$ and $TE_{simu}$ is monotonically increasing in $r$ when $r \in [\hat{r}_s, \bar{r}]$, for any given $c \in (\frac{1}{2}, 1]$. Based on the above results, there must exist an $\hat{r}^1_{SL}$, where $\hat{r}^1_{SL} \in (\hat{r}_s, \bar{r})$, such that $TE^{pre}_{strong} > TE_{simu}$ for $r \in [\hat{r}_s, \hat{r}^1_{SL})$, $TE^{pre}_{strong} < TE_{simu}$ for $r \in (\hat{r}^1_{SL}, 2]$, and $TE^{pre}_{strong} = TE_{simu}$ at $r = \hat{r}^1_{SL}$.

Recall that $TE_{simu} = \frac{1+c}{2c_F}$ for $r \in [2, \infty)$. At $r = 2$, $TE^{pre}_{strong} < TE_{simu}$, while $\lim_{r \to \infty} TE^{pre}_{strong} = \frac{1}{c_F} > \frac{1+c}{2c_F} = \lim_{r \to \infty} TE_{simu}$, given $c < 1$. Also, notice that

$TE_{simu}$ stays unchanged and $TE^{pre}_{strong}$ increases in $r$ on $[2,\infty)$. Combining the above results, we can conclude that there exists a unique $\hat{r}^2_{SL}$, where $\hat{r}^2_{SL} \in (2,\infty)$, such that $TE^{pre}_{strong} < TE_{simu}$ for $r \in [2,\hat{r}^2_{SL})$, while $TE^{pre}_{strong} > TE_{simu}$ for $r \in (\hat{r}^2_{SL},\infty)$, and $TE^{pre}_{strong} = TE_{simu}$ at $r = \hat{r}^2_{SL}$. □

### S.3.2 Proof of Theorem 2

*Proof.* When $c = 1$, it can be shown that $TE_{simu} = r/2$ for $r \in (0,2)$ and $TE_{simu} = 1$ for $r \in [2,\infty)$. When $c = 1$, we have $TE^{nonp}_{sym} = TE_{simu}$ when the leader chooses a non-preemptive equilibrium candidate over a preemptive equilibrium candidate, because in equilibrium $t^* = c = 1$. Also, we derive that $TE_{simu} = r/2 = TE^{nonp}_{sym}$, since

$$TE^{nonp}_{sym} = \frac{rc^{r-1}(1+c)}{c_F(1+c^r)^2} = \frac{r}{2}. \tag{9}$$

In addition, in such a non-preemptive equilibrium, each player may win with probability 1/2 by exerting effort $r/4$, and thus each player's expected payoff is $\pi^{nonp}_{sym} = \frac{1}{2} - \frac{r}{4}$. In contrast, in the case of preemption, $\pi^{pre}_{sym} = 1 - \frac{1}{r}(r-1)^{\frac{r-1}{r}}$. Comparing $\pi^{pre}_{sym}$ and $\pi^{nonp}_{sym}$, we derive that when $c = 1$, there exists a unique $\hat{r}_s$, where $\hat{r}_s \approx 1.0789$, such that $\pi^{nonp}_{sym} > \pi^{pre}_{sym}$ when $r < \hat{r}_s$, $\pi^{nonp}_{sym} = \pi^{pre}_{sym}$ when $r = \hat{r}_s$, and $\pi^{nonp}_{sym} < \pi^{pre}_{sym}$ when $r > \hat{r}_s$.

Also, in the case of preemption, total effort just equals the leader's effort. We derive that $TE^{pre}_{sym} \approx 0.7697$ at $r = \hat{r}_s$, using

$$TE^{pre}_{sym} = \frac{1}{r}(r-1)^{\frac{r-1}{r}}. \tag{10}$$

Meanwhile, when $r < \hat{r}_s$, we go back to the case with a non-preemptive equilibrium candidate. It can be derived that $TE^{nonp}_{sym}|_{r\to\hat{r}_s^-} \approx 0.5395 < 0.7697 \approx TE^{pre}_{sym}|_{r=\hat{r}_s}$, and thus there is a jump in the total effort function at the threshold value $\hat{r}_s$, which separates the non-preemptive and preemptive equilibria. This also confirms our result of Lemma S.2 (ii).

At last, using (9) and (10), we derive that when $c = 1$, there exists a unique $\hat{r}_{sym}$, where $\hat{r}_{sym} \approx 1.2324$, such that $TE^{pre}_{sym} > TE_{simu}$ for $r \in [\hat{r}_s, \hat{r}_{sym})$, $TE^{pre}_{sym} = TE_{simu}$ at $r = \hat{r}_{sym}$, and $TE^{pre}_{sym} < TE_{simu}$ for $r \in (\hat{r}_{sym}, \infty)$. □

### S.3.3 Proof of Lemma 1

*Proof.* Different from the equilibrium total effort expression obtained in the proof of Theorem 1 where $c \in (0, 1]$, the equilibrium total effort with $c \in (1, +\infty)$ is given by

$$TE_{simu} = \begin{cases} \frac{rc^r(1+c)}{c_L(1+c^r)^2} & \text{for } r \in (0, \bar{r}], \\ \frac{(1+c)(r-1)^{\frac{r-1}{r}}}{cc_L r} & \text{for } r \in [\bar{r}, 2], \\ \frac{1+c}{2cc_L} & \text{for } r \in [2, \infty). \end{cases} \tag{11}$$

When $c \in (1, +\infty)$, using the results of Wang (2010), it is easy to derive that $\bar{r}$ is the value of $r$ such that $c^{\bar{r}} = 1/(\bar{r} - 1)$ in the case with $c > 1$.

Using $\pi^{pre}_L = 1 - c \cdot h(r)$ and $1/2 \leq h(r) < 1$ obtained in the proof of Proposition 1 in Gao, Lu, and Wang (2025), we derive that when $c > 2$, the weak leader would never choose a preemptive equilibrium candidate, because it always yields a negative payoff. Suppose that when $c \leq 2$, the weak leader chooses to preempt the strong follower for some $r > 1$.In the proof of Proposition 1 in Gao, Lu, and Wang (2025), it is also shown that $TE^{pre}_{weak} = \frac{1}{c_F} h(r)$, and $TE^{pre}_{weak}$ reaches its minimum at $r = 2$. To obtain $\hat{c}^l$, we set $TE^{pre}_{weak} = TE_{simu}$ for $r \in [\bar{r}, 2]$, that is, $\left(\frac{1+c}{cc_L}\right) h(r) = \left(\frac{1}{c_F}\right) h(r)$, which leads to $c^2 - c - 1 = 0$. Solving this equation, we have $\hat{c}^l = (\sqrt{5} + 1)/2$, and $\hat{c}^l < \hat{c}^h$ because $\pi^{pre}_L > \pi^{nonp}_L$ at $c = \hat{c}^l$ and $r = 2$. □

### S.3.4 Proof of Theorem 3

*Proof.* **Proof of Part (i)**

It is straightforward to show that $\hat{r}^1_w < \bar{r} < 2 < \hat{r}^2_w$ for $c \in (1, \hat{c}^l)$. We next show that the expected total effort from a non-preemptive equilibrium of a weak-lead

sequential contest is lower than that of its corresponding simultaneous contest for $c \in (1, \hat{c}^l)$.

Different from the equilibrium total effort expression obtained in the proof of Theorem 1 where $c \in (0, 1]$, the equilibrium total effort with $c \in (1, +\infty)$ is given by

$$TE_{simu} = \begin{cases} \frac{rc^r(1+c)}{c_L(1+c^r)^2} & \text{for } r \in (0, \bar{r}], \\ \frac{(1+c)(r-1)^{\frac{r-1}{r}}}{cc_L r} & \text{for } r \in [\bar{r}, 2], \\ \frac{1+c}{2cc_L} & \text{for } r \in [2, \infty). \end{cases} \tag{12}$$

Different from the previous case where $c \in (0, 1]$, when $c \in (1, +\infty)$, using the results of Wang (2010), it is easy to derive that $\bar{r}$ is the value of $r$ such that $c^{\bar{r}} = 1/(\bar{r} - 1)$ in the case with $c > 1$.

When $c > 1$, the solution $t^*$ of equation (3) always exists, where $t^* > c$; and $t^*$ goes to $c$ as $r$ goes to 0 and goes to $c(r + 1)$ as $r$ goes to infinity. Similar to the case of $c \in (0, 1)$, when $c \in (1, \hat{c}^l)$ and $r \in (0, \hat{r}_w^1) \cup (\hat{r}_w^2, \infty)$, the equilibrium total effort level of a non-preemptive equilibrium candidate of the sequential contests is $TE_{weak}^{nonp} = \frac{r(t^*)^{r-1}(1+t^*)}{c_F(1+(t^*)^r)^2}$. For $r \in (0, \hat{r}_w^1)$, it is still true that the difference of total effort levels between a sequential contest and its corresponding simultaneous contest is the difference between the values of the function $(rt^{r-1}(1+t))/(c_F(1+t^r)^2)$ at $t = t^*$ and $t = c$. To compare the effort levels, it is sufficient to look at its monotonicity, and we show that this function is strictly decreasing in $t$ on $[c, t^*]$ for any given $r \in (0, \hat{r}_w^1)$. Note that

$$\left(\frac{t^{r-1}(1+t)}{(1+t^r)^2}\right)' = \frac{t^{r-2}}{(1+t^r)^3}\Big[(r-1) + rt - (r+1)t^r - rt^{r+1}\Big].$$

The expression in the square bracket is negative when $t > 1$ and $r > 0$, because $r - 1 < (r + 1)t^r$ and $rt < rt^{r+1}$. Therefore, $TE_{weak}^{nonp} < TE_{simu}$ for $r \in (0, \hat{r}_w^1)$.

As to $TE_{weak}^{nonp} < TE_{simu}$ for $r \in (\hat{r}_w^2, \infty)$, we need to verify that $\frac{r(t^*)^{r-1}(1+t^*)}{c_F(1+(t^*)^r)^2} < \frac{1+c}{2cc_L}$,

which is equivalent to

$$\frac{cr(t^*)^{r-1}(1+t^*)}{(1+(t^*)^r)^2} < \frac{1+c}{2c}. \tag{13}$$

We now verify the above inequality. We have shown that $t^* < c(r+1)$, and it is also true that $t^* > cr$ when $r \geq 2$. To see this, recall that $t^*$ solves (3)—i.e., $L(t) = R(t)$. If it were the case that $t = cr$, we would have $t^r = (cr)^r > 2r - 1$ for $c > 1$ and $r \geq 2$, which implies that $L(t) > R(t)$.

Thus, $t = cr$ must lie to the left of $t = t^*$ on the $t$-axis, which implies that $t^* > cr$.

Using $t^* > cr$, we derive that

$$\frac{cr(t^*)^{r-1}(1+t^*)}{(1+(t^*)^r)^2} < \frac{(1+cr)(t^*)^r}{(1+(t^*)^r)^2} < \frac{1+cr}{1+(cr)^r} < \frac{1+c}{2c},$$

implying that (13) holds, which in turn implies that $TE^{nonp}_{weak} < TE_{simu}$ when $c > 1$ and $r \in (\hat{r}^2_w, \infty)$, where $\hat{r}^2_w > 2$.

This completes the proof that $TE^{nonp}_{weak} < TE_{simu}$ when $c \in (1, \hat{c}^l)$ and $r \in (0, \hat{r}^1_w) \cup (\hat{r}^2_w, \infty)$. Therefore, we have shown the first half of Part (i).

We now turn to the second half of Part (i). Since $c \in (1, \hat{c}^l)$, $TE^{pre}_{weak} < TE_{simu}$ for $r \in (\bar{r}, 2)$. The key step in the following proof is to show that $TE^{pre}_{weak} > TE_{simu}$ at $r = \hat{r}^1_w$ and $r = \hat{r}^2_w$. The result then follows because both $TE^{pre}_{weak}$ and $TE_{simu}$ are piecewise monotone on $(\hat{r}^1_w, \bar{r})$ and $(2, \hat{r}^2_w)$.

We first verify that $TE^{pre}_{weak} > TE_{simu}$ at $r = \hat{r}^1_w$. This is equivalent to show

$$\frac{(r-1)^{\frac{r-1}{r}}}{r}\Bigg|_{r=\hat{r}^1_w} > \frac{rc^{r-1}(1+c)}{(1+c^r)^2}\Bigg|_{r=\hat{r}^1_w},$$

where $c \in (1, \hat{c}^l)$, and $\hat{r}^1_w$ $(< \bar{r})$ solves the following equation

$$\frac{c^2 - (cr - t^*)^2}{4crt^*} = 1 - \frac{c}{r}(r-1)^{\frac{r-1}{r}}. \tag{14}$$

Both $\hat{r}^1_w$ and $t^*$ are functions of $c$, which is the only independent variable here.

It can be derived that

$$\frac{rc^{r-1}(1+c)}{(1+c^r)^2} = \frac{r}{1+c^r}\left(1-\frac{1}{1+c^r}\right)\left(1+\frac{1}{c}\right) < \frac{r}{4}\left(1+\frac{1}{c}\right),$$

thus, a sufficient condition for $TE^{pre}_{weak} > TE_{simu}$ is

$$\left.\frac{(r-1)^{\frac{r-1}{r}}}{r}\right|_{r=\hat{r}^1_w} > \left.\frac{r}{4}\left(1+\frac{1}{c}\right)\right|_{r=\hat{r}^1_w}. \tag{15}$$

Let $\hat{r}^1_w|_{c=1}$ and $\hat{r}^1_w|_{c=\hat{c}^l}$ denote the values of $\hat{r}^1_w$ when $c=1$ and $c=\hat{c}^l$, respectively. Calculation shows that $\hat{r}^1_w|_{c=1} \approx 1.0789$ and $\hat{r}^1_w|_{c=\hat{c}^l} \approx 1.3336$. Our previous analysis shows that $\hat{r}^1_w$ is increasing with $c$ in the range of $c \in (1, \hat{c}^l)$, which implies that $\hat{r}^1_w \in (\hat{r}^1_w|_{c=1},\ \hat{r}^1_w|_{c=\hat{c}^l})$ when $c \in (1, \hat{c}^l)$. In the second half of the proof of Theorem 4 in Gao, Lu, and Wang (2025), we show that $\pi^{pre}_L$ decreases faster than $\pi^{nonp}_L$ in $c$ and the new crossing point occurs at the right of the original $\hat{r}^1_w$.

We rewrite inequality (15) as

$$\left.\frac{(r-1)^{\frac{r-1}{r}}}{r^2}\right|_{r=\hat{r}^1_w} > \left.\frac{1}{4}\left(1+\frac{1}{c}\right)\right|_{r=\hat{r}^1_w}.$$

It can be verified that this inequality holds true when $c \in (1, \hat{c}^l)$.

To verify that $TE^{pre}_{weak} > TE_{simu}$ when $r = \hat{r}^2_w$, we need to show that

$$\left.\frac{(r-1)^{\frac{r-1}{r}}}{c_F r}\right|_{r=\hat{r}^2_w} > \frac{1+c}{2cc_L}, \tag{16}$$

where $c \in (1, \hat{c}^l)$, and $\hat{r}^2_w > 2$ solves the following equation

$$\frac{c^2-(cr-t^*)^2}{4crt^*} = 1 - c\frac{(r-1)^{\frac{r-1}{r}}}{r}, \tag{17}$$

where $t^*$ solves equation (3). Using (17), we derive that (16) is equivalent to the inequality $1 - \frac{c^2-(cr-t^*)^2}{4crt^*} > \frac{1+c}{2c}$. After rearrangement of terms, the above inequality is

equivalent to

$$(c^2-1)(r^2-1)+(t^*-r)^2>1 \tag{18}$$

where $r=\hat{r}_w^2$. Next, we show that the above inequality is true due to $t^*>r+1$. When $t=r+1$ (rather than $t=t^*$), we obtain that $L(t)-R(t)=c-(r+1)-cr\left(\frac{2}{1+(r+1)^r}-1\right)$. It can be shown that given $c\in(1,\hat{c}^l)$, for any $r>2$, we always have $L(t)-R(t)>0$ at $t=r+1$. This implies that the solution $t^*$ to $L(t)=R(t)$ must satisfy $t^*>r+1$. Consequently, inequality (18) holds, demonstrating that $TE_{weak}^{pre}>TE_{simu}$ at $r=\hat{r}_w^2$.

**Proof of Part (ii)**

Most of the proof for this part is similar to that of Part (i) except that $TE_{weak}^{pre}=TE_{simu}$ for $r\in[\hat{r}_{WL}^1,\hat{r}_{WL}^2]$ when $c=\hat{c}^l$, which is the result of Lemma 1.

**Proof of Part (iii)**

When $c>\hat{c}^l$, using (2) and (12), we derive that $TE_{weak}^{pre}>TE_{simu}$ for $r\in[\bar{r},2]$. Recall that $TE_{weak}^{pre}=h(r)/c_F$ is increasing with $r$ when $r>2$, and $r=2$ is its global minimum; while $TE_{simu}=\frac{1+c}{2cc_L}$ stays unchanged when $r$ increases beyond 2. We thus have $TE_{weak}^{pre}>TE_{simu}$ for $r\in[2,\hat{r}_w^2]$ as long as this set is nonempty. Note that no preemptive equilibrium exists when $c>2$. Given $c\le 2$, it can be shown that $TE_{simu}$ reaches its global maximum at $r=\bar{r}$. This implies that $TE_{weak}^{pre}>TE_{simu}$ for $r\in[\hat{r}_w^1,\bar{r}]$ if this set is nonempty, because $TE_{weak}^{pre}$ decreases and $TE_{simu}$ increases in $r$ in this interval.

To show $TE_{weak}^{nonp}<TE_{simu}$ whenever the equilibrium is non-preemptive, the only thing left to show is that $TE_{weak}^{nonp}<TE_{simu}$ if $r\in[\bar{r},2]$. $TE_{weak}^{nonp}<TE_{simu}$ is equivalent to $\frac{r(t^*)^{r-1}(1+t^*)}{c_F(1+(t^*)^r)^2}<\frac{(1+c)(r-1)^{\frac{r-1}{r}}}{cc_Lr}$, given any $c>\hat{c}^l$. We rewrite the above inequality as

$$\left(\frac{c}{1+c}\right)\frac{cr(t^*)^{r-1}(1+t^*)}{(1+(t^*)^r)^2}<\frac{(r-1)^{\frac{r-1}{r}}}{r}. \tag{19}$$

We can show that $t^*>cr$ and $\frac{cr(t^*)^{r-1}(1+t^*)}{(1+(t^*)^r)^2}<\frac{c}{1+c^r}$. It can be verified that $c/(1+c^r)$

decreases in $c$ as long as $c^r \geq 1/(r-1)$, which is always true in this case. Thus, for any $c \geq \hat{c}^l = \frac{\sqrt{5}+1}{2}$ and $r \in [\bar{r}, 2]$, we always have $\frac{c}{1+c^r} \leq \frac{\hat{c}^l}{1+(\hat{c}^l)^r}$. Finally, we conclude that for any $c \geq \hat{c}^l = \frac{\sqrt{5}+1}{2}$ and $r \in [\bar{r}, 2]$, $TE^{nonp}_{weak} < TE_{simu}$ if

$$\left(\frac{c}{1+c^r}\right)|_{c=\hat{c}^l=\frac{\sqrt{5}+1}{2}} - \frac{1}{r}(r-1)^{\frac{r-1}{r}} < 0. \tag{20}$$

It can be shown that (20), which has a single variable $r$, always holds true for $r \in [1, 2]$. The original inequality follows for $r \in [\bar{r}, 2]$. □

### S.3.5 Proof of Theorem 4

*Proof.* For $r \in [\hat{r}^1_w, \hat{r}^2_w]$ whenever this interval exists, both types of the sequential contests have preemptive equilibria and the expected total effort shares the same functional form except for the value of $1/c_F$ by equation (2). Since $c_A < c_B$, $TE_{SL} < TE_{WL}$.

For those values of $r$ and $c$ such that the weak-lead sequential contests only have non-preemptive equilibria, we have $TE^{nonp}_{weak} < TE_{simu}$ by Theorem 3. The claim follows if we can show that $TE^{pre}_{strong} > TE^{nonp}_{weak}$ for $r \in (\hat{r}^1_{SL}, \hat{r}^2_{SL})$ as defined in Theorem 1, because both $TE^{nonp}_{strong}$ and $TE^{pre}_{strong}$ are greater than $TE_{simu}$ for $r$ outside this interval. Recall from the proof of Theorem 1 that $TE^{pre}_{strong} = \frac{1}{c_B r}(r-1)^{\frac{r-1}{r}}$ and from the proof of Theorem 3 that $TE^{nonp}_{weak} = \frac{1}{c_A}\frac{r(t^*)^{r-1}(1+t^*)}{(1+(t^*)^r)^2}$, where $t^*$ is the equilibrium value of $t$ solving equation (3) with $c = c_B/c_A > 1$. We need to show the following inequality for $r \in (\hat{r}^1_{SL}, \hat{r}^2_{SL})$,

$$\frac{cr(t^*)^{r-1}(1+t^*)}{(1+(t^*)^r)^2} < \frac{(r-1)^{\frac{r-1}{r}}}{r}. \tag{21}$$

We can show that this inequality holds for $r \in (1, 2]$, which is sufficient because it is a wider region on the left half of the interval with $1 < \hat{r}_s < \hat{r}^1_{SL}$ and $TE^{pre}_{strong}$ is increasing while $TE^{nonp}_{weak}$ is decreasing for $r > 2$ and on $r \in (2, \hat{r}^2_{SL})$ in particular.

□

### S.3.6 Proof of Lemma 2

*Proof.* In the current contest model with players $A$ and $B$, in which the designer is allowed to choose either player $A$ or $B$ to be the leader, we first show that for $c_{A/B} \in (\tilde{c}_1, 1)$, we always have $\hat{r}_s < \hat{r}_w^1$. At $r = \hat{r}_s$, in a strong-lead sequential contest, which is equivalent to the sequential contest model with $L = A$, $F = B$, and $c = c_{A/B} \in (0, 1)$, define $\Delta\pi_{strong} = \pi_{strong}^{pre} - \pi_{strong}^{nonp}$. It can be derived that

$$\Delta\pi_{strong} = 1 - \frac{c_{A/B}}{r}(r-1)^{\frac{r-1}{r}} - \frac{c_{A/B}^2 - (c_{A/B}r - t^*)^2}{4c_{A/B}rt^*} = 0,$$

where $t^*$ is determined by (3) with $c = c_{A/B}$.

Recall that the weak-lead sequential contest in the current model with $c_{A/B} \in (0, 1)$ is equivalent to the weak-lead sequential contest with $L = B$, $F = A$ and $c = c_{B/A} = 1/c_{A/B} > 1$. Thus, in a weak-lead sequential contest, at $r = \hat{r}_s$, define $\Delta\pi_{weak} = \pi_{weak}^{pre} - \pi_{weak}^{nonp}$ and it can be derived that

$$\Delta\pi_{weak} = 1 - \frac{c_{B/A}}{r}(r-1)^{\frac{r-1}{r}} - \frac{c_{B/A}^2 - (c_{B/A}r - t^*)^2}{4c_{B/A}rt^*},$$

where $c_{B/A} = c_B/c_A$ and $t^*$ is determined by

$$c_{B/A} - t^* = c_{B/A}r\left(\frac{2}{1 + (t^*)^r} - 1\right). \tag{22}$$

It is clear that $c_{A/B} < 1 < c_{B/A}$, given $c_{A/B} = c_A/c_B = 1/c_{B/A}$.

Recall that in the proof of Theorem 4 in Gao, Lu, and Wang (2025), we have defined a function $\Delta\pi$, where

$$\Delta\pi = 1 - \frac{c}{r}(r-1)^{\frac{r-1}{r}} - \frac{c^2 - (cr - t^*)^2}{4crt^*}.$$

In the current model, when setting $c = c_{A/B} < 1$ and $c = c_{B/A} > 1$, $\Delta\pi$ becomes $\Delta\pi_{strong}$ and $\Delta\pi_{weak}$, respectively. We emphasize two points regarding the above

function $\Delta\pi$. First, $\Delta\pi$ is a function of two variables, $r$ and $c$, because $t^*$ is determined once $r$ and $c$ are determined. Second, $\Delta\pi$ is a general function in the sense that it covers $\Delta\pi_{strong}$ and $\Delta\pi_{weak}$ as two special cases—i.e., $\Delta\pi = \Delta\pi_{strong}$ when $c = c_{A/B} < 1$, and $\Delta\pi = \Delta\pi_{weak}$ when $c = c_{B/A} = 1/c_{A/B} > 1$.

In the proof of Theorem 4 in Gao, Lu, and Wang (2025), we have shown that $\frac{d\Delta\pi}{dc} < 0$ for $c > 1$. Next, we show that $\frac{d\Delta\pi}{dc} < 0$ for $c \in (\frac{1}{2}, 1]$. It can be derived that $\frac{d\Delta\pi}{dc} < 0$ if and only if

$$\left|\frac{d\pi_L^{nonp}}{dc}\right| = \frac{(cr + c - t^*)(cr - c + t^*)}{4c^2 r t^*} < \frac{1}{r}(r-1)^{\frac{r-1}{r}}. \tag{23}$$

Recall that $R(t)$ and $L(t)$ are tangent at $r = \tilde{r}$ for $c \in (\frac{1}{2}, 1]$. It can be shown that when $r$ increases from 1 to $\tilde{r}$, $t^*$ decreases from $2c - 1$ to $c\sqrt{\tilde{r}^2 - 1}$; when $c$ increases from 1/2 to 1, $\tilde{r}$ increases from 1 to $\sqrt{2}$.[28] Thus, given $c$ fixed, when $\hat{r}_w^1 > \tilde{r}$, we must have $\hat{r}_w^1 > \hat{r}_s$ because $\hat{r}_s < \tilde{r}$.

We have known that $\hat{r}_w^1 \leq \tilde{r}$. Now we seek to show that in the current model, when $c \in (\tilde{c}_1, 1)$, $\hat{r}_w^2 < \hat{r}_{SL}^2$. At $r = \hat{r}_w^2$, $\pi_{weak}^{pre} = \pi_{weak}^{nonp}$, we have $1 - \frac{c_{B/A}}{r}(r-1)^{\frac{r-1}{r}} = \frac{c_{B/A}^2 - (c_{B/A} r - t^*)^2}{4 c_{B/A} r t^*}$, where $t^*$ is determined by (22). Substituting $c_{B/A} = 1/c_{A/B}$ into the above equation, we derive that $\frac{(r-1)^{\frac{r-1}{r}}}{r} = c_{A/B} - \frac{1 - (r - c_{A/B} t^*)^2}{4 r t^*}$, where $t^*$ is determined by (22).

Next, we show $\hat{r}_w^2 < \hat{r}_{SL}^2$ by proving that $TE_{strong}^{pre} < TE_{simu}$ at $r = \hat{r}_w^2$. As $\hat{r}_w^2 > 2$, $TE_{strong}^{pre} < TE_{simu}$ holds at $r = \hat{r}_w^2$ if and only if

$$\frac{(r-1)^{\frac{r-1}{r}}}{r} = c_{A/B} - \frac{1 - (r - c_{A/B} t^*)^2}{4 r t^*} < \frac{1 + c_{A/B}}{2},$$

which holds if and only if $\frac{1 - (r - c_{A/B} t^*)^2}{4 r t^*} > \frac{c_{A/B} - 1}{2}$. At $r = \hat{r}_w^2$, we have $\pi_{weak}^{nonp} = \frac{c_{B/A}^2 - (c_{B/A} r - t^*)^2}{4 c_{B/A} r t^*} = \frac{1 - (r - c_{A/B} t^*)^2}{4 c_{A/B} r t^*} > 0$, which implies that $\frac{1 - (r - c_{A/B} t^*)^2}{4 r t^*} > 0 > \frac{c_{A/B} - 1}{2}$ as $c_{A/B} < 1$. Thus, $TE_{strong}^{pre} < TE_{simu}$ at $r = \hat{r}_w^2$, implying that $\hat{r}_w^2 < \hat{r}_{SL}^2$. □

[28] This follows from the fact that $\tilde{r}$ and $c$ have a one-to-one relationship, as shown in the proof of Theorem 3 in Gao, Lu, and Wang (2025). Note that $\tilde{r}$ here corresponds to $r_s^1$ in Gao, Lu, and Wang (2025).

### S.3.7 Proof of Lemma 3

*Proof.* Recall that $TE^{pre}_{strong} = TE_{simu}$ when $r = \hat{r}^1_{SL}$. As we have shown that $\hat{r}^1_{SL} \in (\hat{r}_s, \overline{r})$, we derive that at $r = \hat{r}^1_{SL}$,

$$\frac{1}{r}(r-1)^{\frac{r-1}{r}} = \frac{(1+c_{A/B})c_{A/B}^{r-1}r}{(1+c_{A/B}^r)^2}. \tag{24}$$

Also recall that when the leader is the weak player—i.e., when $L = B$ and $F = A$, we have $\pi^{pre}_{weak} = \pi^{nonp}_{weak}$ at $r = \hat{r}^1_w$, which implies that $1 - \frac{c_{B/A}}{r}(r-1)^{\frac{r-1}{r}} = \frac{c^2_{B/A} - (c_{B/A}r - t^*)^2}{4c_{B/A}rt^*}$, where $t^*$ is determined by (22). From the above equation, we derive that at $r = \hat{r}^1_w$,

$$\frac{1}{r}(r-1)^{\frac{r-1}{r}} = \frac{(t^* + c_{B/A}r)^2 - c^2_{B/A}}{4c^2_{B/A}rt^*}. \tag{25}$$

We start from the case where $c = 1$. Using (24), we can derive that at $r = \hat{r}^1_{SL}$, $\frac{1}{r}(r-1)^{\frac{r-1}{r}} = \frac{r}{2}$, which implies that $\hat{r}^1_{SL} \approx 1.2324$. Using $c_{B/A} = 1/c_{A/B} = 1$ and (22), we obtain $t^* = 1$. Using (25), we further derive that at $r = \hat{r}^1_w$, $\frac{1}{r}(r-1)^{\frac{r-1}{r}} = \frac{(1+r)^2-1}{4r}$, which implies that $\hat{r}^1_w \approx 1.0789$. Clearly, $\hat{r}^1_{SL} > \hat{r}^1_w$ when $c = 1$. When $c_{A/B}$ decreases from 1, or equivalently, when $c_{B/A}$ increases from 1, $\hat{r}^1_w$ increases in $c_{B/A}$, shown in the proof of Theorem 4 in Gao, Lu, and Wang (2025). From (24), we show that $\hat{r}^1_{SL}$ decreases when $c_{A/B}$ decreases. Therefore, when $c_{A/B}$ decreases from 1, $\hat{r}^1_w$ increases and $\hat{r}^1_{SL}$ decreases, implying that there exists a unique $\tilde{c}_3$, where $\tilde{c}_3 \approx 0.7708$, such that $\hat{r}^1_{SL} < \hat{r}^1_w$ when $c \in (\tilde{c}_2, \tilde{c}_3)$, $\hat{r}^1_w = \hat{r}^1_{SL}$ when $c = \tilde{c}_3$, and $\hat{r}^1_w < \hat{r}^1_{SL}$ when $c \in (\tilde{c}_3, 1)$. □