Performance Bundling across Multiple Competitions

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Abstract

In a multi-project contract environment with a single agent who is subject to limited liability, the optimality of performance bundling across all projects is well established: The agent should be rewarded only if all projects are successful. In this paper, we study performance bundling across multiple competitions between two (possibly asymmetric) players. We find that in addition to a beneficial cost-saving effect that diminishes with asymmetry across players, performance bundling causes a counterproductive unbalancing effect, which intensifies with this asymmetry. Thus, performance bundling is desirable only if the players are sufficiently symmetric. Otherwise, a set of independent contests is optimal.

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Keywords: Multidimensional contest, Performance bundling, Cost-saving effect, Unbalancing effect.

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1 Introduction

In contests, multiple agents are incentivized by the chance of winning prizes to supply productive effort. Very often, the same set of players compete with each other in multiple competitive activities. For example, in research universities, faculty members compete in both teaching and research. Students in the same cohort in a school compete in multiple subjects. Internal promotions within organizations typically require that the candidates prove their competence in multiple dimensions; for example, professional expertise and managerial skills. Many sporting events require that participants compete on multiple fronts in one race. For instance, in an Ironman Triathlon, all athletes compete in three dimensions: swimming, bicycling, and running.

In these situations, there are different ways to organize competitions. First, the organizer can run a set of independent single-dimensional (henceforth: SD) contests: Each dimensional competition is held separately, and dimensional winners are awarded independently. Second, the organizer can introduce performance bundling, in which players’ bundled performances across dimensional competitions determine their one-shot overall rewards. At the extreme, for instance, the organizer can run a grand multidimensional (henceforth: MD) contest in which a player is awarded if and only if he wins all dimensions of the competition. Throughout this paper, we assume that the organizer aims to maximize her expected payoff, which equals her benefit obtained from players’ effort less her expected expenditure of prizes paid to the players.

The central issue we investigate in this paper is the optimal way to organize such multiple competitions. Can a set of independent SD contests, a grand MD contest, or a mix of the two be optimal in certain scenarios? What is the key driving force that determines the optimal design?

We adopt the following stylized setting to illustrate the main findings of the paper. Suppose a department seeks to use contests to motivate two of its members, players A and B, to exert productive effort in two dimensions/activities: teaching and research. The department rewards players based on their two-dimensional winning status, and targets optimally balancing effort elicitation and prize costs. The players get different prizes, depending on whether they win in both activities, a single teaching activity, a single research activity, or no activity.\footnote{Without loss of generality, we normalize the prize for zero wins to be zero. Our prize structure covers as special cases a set of independent SD contests, a grand MD contest, and a mix of the two.} In each dimension/activity, players’ winning chances are determined by a Tullock
contest technology given their effort levels, which are continuous.

For heuristic purpose, we start with an environment in which players are symmetric in the sense that they are equally capable in each dimension/activity—i.e., players’ marginal effort costs are symmetric across the two dimensions/activities. For this setting, we find that the grand contest always dominates, i.e., it is optimal to only reward a player who wins in both dimensions. The above result holds even when the two symmetric players are not equally capable across the two activities and the contest technologies are asymmetric across the two activities. The dominance of the grand contest can be explained by a cost-saving effect of performance bundling: A single grand prize can be used to incentivize effort making in both dimensions at the same time, but it is only allocated when a player wins in both dimensions. Note that with symmetric players, for any eligible prize structure, at equilibrium each player wins each SD competition with half a chance. This means that if a grand MD contest is adopted, creating 2 units of (grand) prize spread would cost the organizer 1 unit in expectation, which generates a 1 unit prize spread in each dimension. In contrast, if a set of independent SD contests is adopted, creating a 1 unit prize spread in each dimension would cost the organizer 2 units, as the two SD prizes are allocated with probability one. As a result of the cost-saving effect, the two prize spreads in both dimensions that are generated by any eligible prize structure can be replicated (in the sense that the same effort levels are induced) by increasing the grand MD prize and lowering the two SD prizes appropriately, which is less costly for the organizer.

Laux (2001) and Zhao (2008) find that in a single-agent multi-project contract environment with discrete effort (low or high), performance bundling that rewards a player if and only if he succeeds in all projects is optimal, as it relaxes the limited liability constraint of the agent. The economics behind this, however, are similar. Any alternative contract that induces high effort in all activities can be replaced by such a performance bundling contract that costs the same on average—but strictly relaxes the incentive compatibility conditions. As a result, there exists such a performance bundling contract that costs less on average and induces high effort in all projects.\(^2\)

For a more complete picture of the effects of performance bundling in contests, we further allow the two players to be asymmetric in their marginal effort costs, while assuming each player has the same marginal effort cost across two dimensions. With asymmetric players, the cost-saving effect is still present: A single grand prize can be employed simultaneously

\(^2\)It is worth pointing out that while Laux (2001) focuses on homogeneous projects, his insight does extend to an environment with heterogenous projects. Details are available from the authors.
to incentivize effort making in both dimensions, but is allocated if and only if a player wins in both dimensions. However, there is a complication: The players’ winning chances in each dimension are no longer fixed, as in the scenario with symmetric players. In particular, the stronger player definitely wins in each dimension with a higher chance, which is endogenously determined by the equilibrium effort induced by the prevailing prize structure. An increase in the grand prize tends to further tilt the already unbalanced competition between the two asymmetric players, since a player’s incremental prize spread in one dimension, which is generated by the increase of the grand prize, is proportional to his own winning chance in the other dimension. We call this effect the \textit{unbalancing effect}. As is well understood in the contest literature, the unbalancing effect would tend to dampen players’ effort supply.

Moreover, due to the unbalancing effect, including a grand MD prize increases the winning chance of the stronger player in each dimension; this weakens the cost-saving effect, since the chance of the grand prize being allocated gets higher. We find that when the grand prize gets sufficiently large, the unbalancing effect can even entail a \textit{semi-pure} strategy equilibrium rather than a pure strategy equilibrium, in which the weaker player does not play a pure strategy and his actions in the two dimensions/activities are perfectly correlated: He either exerts zero effort or a fixed positive effort in both dimensions, while the stronger player plays a pure strategy. In this equilibrium, the winning statuses are correlated; this further weakens the cost-saving effect, as the correlated winning statuses in the two dimensions/activities increase the chance of the grand prize being allocated.

While the cost-saving effect favors the introduction of a grand MD prize, the unbalancing effect tends to go against it. Whether introducing a grand prize is optimal depends on the trade-off of the two effects. One can expect that when players become more asymmetric, the cost-saving effect of introducing a grand prize gets relatively weaker (compared with the unbalancing effect), as the chance of one player winning in both dimensions (and thus the grand prize is allocated) gets higher. Therefore, introducing a grand MD prize would eventually become unpalatable. This is what we have discovered for the optimal prize structure. We find that when the players are sufficiently asymmetric, a set of independent SD contests is optimal; when the players are sufficiently symmetric, a grand MD contest is optimal. If the asymmetry of the players lies in between, a coexistence of SD and grand MD prizes is optimal.

Our analysis shows that the optimal evaluation scheme (i.e., the optimal prize structure) always involves a bundled-performance feature (i.e., an MD feature), provided that the players are not too asymmetric. This is consistent with many situations we observe in
the real world. For instance, a junior professor in a department may get a teaching (resp. research) reward if he excels in the single dimension of teaching (resp. research), which can be seen as an SD prize. He will get tenure or promotion if he excels in both dimensions, which can be seen as the grand MD prize. Moreover, our analysis provides an explanation of prize retention, which can also be observed in reality—in some scenarios, when no one stands out without controversy, no one gets the grand prize (for instance, a promotion or a highly esteemed reward), and thus the prize is retained.\footnote{In this regard, our paper is related to the literature on the applications of contests/tournaments in the field of labor economics. Chan (1996) shows that adding a new external player to the competition reduces internal players’ chances of promotion, and thus reduces their incentive to work, so an external player should be recruited only if she is significantly superior to the internal players. In contrast, we show that when players are sufficiently symmetric, the grand prize should be retained when no one wins in all dimensions, so the possibility of prize retention actually enhances internal competition rather than reducing it.}

Our paper belongs to the well-established literature on multi-battle contests. Strategic behavior in multi-battle contests has been extensively studied in different environments by Snyder (1989), Laslier and Picard (2002), Szentes and Rosenthal (2003), Roberson (2006), Klumpp and Polborn (2006), Kvasov (2007), Konrad and Kovenock (2009, 2010), Malheug and Yates (2010), Roberson and Kvasov (2012), Gelder (2014), Fu, Lu, and Pan (2015), Boyer, Konrad, and Roberson (2017), Häfner (2017), and Fu and Wu (2018), among many others. Feng and Lu (2018) consider optimal prize allocation in dynamic multi-battle contests in which the central issue is how to optimally mitigate the momentum effect. In this paper, we further this line of research by considering a setting with multiple simultaneous competitions and study the optimal prize structure that optimally balances effort elicitation and prize costs.

Our study is closely related to Laux (2001) and Zhao (2008), who find that in a single-agent multi-project model with limited liability, it is optimal for the principal to reward the agent if and only if all projects are successful. In other words, a pure bundled-performance-evaluation (BPE) scheme is optimal. Chen (2010, 2012) confirms this insight in more general environments. In this paper, we extend their insights on the advantage of performance bundling to a multi-agent contest setting when players are not significantly asymmetric, and thus the cost-saving effect dominates. More importantly, when players are significantly asymmetric, our analysis shows that the dominance of the pure BPE scheme, which is always true in a contract setting, no longer holds in the contest setting due to a counterproductive unbalancing effect of performance bundling. Our analysis illustrates the subtlety of performance bundling in multi-agent environments in which the interactions among asymmetric agents create additional important complications. It should be emphasized that the setting
of this study differs from the contract literature related to Laux (2001) and Zhao (2008) mainly in the following aspects: We consider a multiple-agent contest model with continuous effort levels, in which agents can be asymmetric, while they focus on a single-agent contract model with discrete effort levels.\(^4\)

### 2 Multiple competitions between symmetric players

We consider a contest model in which two players, indexed by \( N \in \{A, B\} \), compete by exerting effort in two activities/dimensions, indexed by \( K = \{1, 2\} \). One can consider the dimensions to be teaching and research. Player \( A \)'s effort profile is denoted by \( x_A = (x_{A1}, x_{A2}) \), where \( x_{Ak} \) is his effort in dimension \( k \in \{1, 2\} \). Similarly, player \( B \)'s effort profile is denoted by \( x_B = (x_{B1}, x_{B2}) \). The two players are symmetric in the sense that the cost function of player \( i \) is \( C(x_i) = c_1 x_{i1} + c_2 x_{i2} \), where \( c_1 > 0 \) and \( c_2 > 0 \), \( \forall i \in \{A, B\} \).

We assume that only the rank information on the competitions—i.e., the winning outcomes in the two dimensions—is available. Therefore, the organizer considers the following prize structure \( V = (V_1, V_2, V_{12}) \): If a player only wins in dimension \( k \) and loses in the other dimension, he gets the SD prize \( V_k \), where \( k \in \{1, 2\} \); if a player wins in both dimensions, he gets the prize \( V_{12} \); if the player does not win in any dimension, he gets zero. We impose limited liability on the players and focus on a prize structure \( (V_1, V_2, V_{12}) \) such that \( V_1, V_2 \geq 0 \) and \( V_{12} \geq V_1 + V_2 \). The above prize structure is general, as it not only covers the following two extreme cases: (i) a set of two independent SD contests with \( V_1 > 0 \), \( V_2 > 0 \), \( V_{12} = V_1 + V_2 \), and (ii) a grand MD contest based on pure bundled-performance-evaluation (BPE) with \( V_1 = 0 \), \( V_2 = 0 \), \( V_{12} > 0 \), but also covers the cases in between, which have features of the above two extreme schemes, e.g., \( V_1 > 0 \), \( V_2 > 0 \) and \( V_{12} - V_1 - V_2 > 0 \).

To facilitate our analysis, the above prize structure \( (V_1, V_2, V_{12}) \) can be reinterpreted as the following equivalent prize structure \( (V_1, V_2, W) \): a player gets \( V_k \) for certain if he wins in dimension \( k \); if he wins in both dimensions, in addition to the two SD prizes \( V_1 \) and \( V_2 \), he gets an *extra* prize \( W \), i.e., he gets the prize \( V_{12} = V_1 + V_2 + W \), where \( W \geq 0 \). We call \( W \) the grand MD prize, as it is rewarded if and only if a player wins in both dimensions.

The competition in each dimension is modeled as a generalized Tullock contest. That is,

\(^4\)Chen (2012) considers a contract model with multiple agents, but his focus is on verifiability (and observability) issues of agents' effort rather than optimal design.
the probability that player $A$ wins in dimension $k$ is given by

$$p^k_A = \frac{x^{r_k} A_k}{x^{r_k} A_k + x^{r_k} B_k},$$

whenever $x_{Ak} + x_{Bk} > 0$, and $p^k_A = 1/2$ otherwise, where $r_k > 0$, $\forall k \in \{1, 2\}$. The parameter $r_k$, which is often referred to as the discriminatory power, can also be interpreted as the accuracy level of the competition in dimension $k$.\footnote{A lower (resp. higher) $r_k$ means the competition in dimension $k$ is more (resp. less) noisy.} The probability that player $B$ wins in dimension $k$ is $p^k_B = 1 - p^k_A$. In this paper, we focus on the case in which $0 < r_1 + r_2 \leq 2$, which, as will be shown later, ensures the existence and uniqueness of a pure-strategy equilibrium for any prize structure.\footnote{In the Appendix, we show that with symmetric players, when $0 < r_1 + r_2 \leq 2$, for any prize structure there exists a unique pure-strategy equilibrium, which is a symmetric equilibrium in which each player plays the same pure strategy. Otherwise, when $r_1 + r_2 > 2$, a pure-strategy equilibrium may not exist.}

The contest organizer benefits from high effort elicited in both activities/dimensions, and she suffers from higher expected expenditure on prizes paid to the players. Let $\psi(\overline{x}_1, \overline{x}_2)$ denote the organizer’s benefit function, where $\overline{x}_k = E(x_{Ak} + x_{Bk})$ denotes the expected total effort elicited in dimension $k$. We assume $\psi(\overline{x}_1, \overline{x}_2)$ is increasing in each dimension and concave in both dimensions. The organizer’s payoff, which is denoted by $\Psi$, is determined by her benefit less the expected cost of prizes allocated. The organizer aims to find the optimal prize structure that maximizes her expected payoff.

2.1 An example

We first present a simple example with $r_1 = r_2 = 1$ to illustrate the main idea of the model with symmetric players. In this example, we aim to compare two extreme prize structures: (i) a set of two independent SD contests, in which $V_1 = V_2 = V$ and $W = 0$, and (ii) a grand MD contest, in which $V_1 = V_2 = 0$ and $W = 2V$.

For prize structure (i), player $i$’s expected payoff is

$$\pi_i^{SD} = \left( \frac{x_{i1}}{x_{i1} + x_{j1}} \right) V + \left( \frac{x_{i2}}{x_{i2} + x_{j2}} \right) V - c_1 x_{i1} - c_2 x_{i2}.$$  

It is standard to obtain that in the unique pure-strategy equilibrium,

$$x_{A1}^{SD} = x_{B1}^{SD} = \frac{V}{4c_1}, \text{ and } x_{A2}^{SD} = x_{B2}^{SD} = \frac{V}{4c_2}.$$  

The organizer’s expected payoff is

$$\Psi^{SD} = \psi\left( \frac{V}{2c_1}, \frac{V}{2c_2} \right) - 2V. \quad (1)$$
For prize structure (\(ii\)), player \(i\)'s expected payoff is

\[
\pi_i^G = \left( \frac{x_{i1}}{x_{i1} + x_{j1}} \right) \left( \frac{x_{i2}}{x_{i2} + x_{j2}} \right) 2V - c_1x_{i1} - c_2x_{i2}.
\]

It can be shown that in the unique pure-strategy equilibrium,

\[
x_{A1}^G = x_{B1}^G = \frac{V}{4c_1}, \quad \text{and} \quad x_{A2}^G = x_{B2}^G = \frac{V}{4c_2}.
\]

The organizer’s expected payoff is

\[
\Psi^G = \psi \left( \frac{V}{2c_1}, \frac{V}{2c_2} \right) - V. \tag{2}
\]

By comparing (1) and (2), it is clear that \(\Psi^G > \Psi^{SD}\), i.e., the grand contest strictly dominates the two independent SD contests for the organizer. The logic is simple: Both prize structures induce the same effort from each player in each dimension/activity—which provides the same benefit for the organizer—while her expected expenditure for prize allocation is \(V_1 + V_2 = 2V\) in (i) and only \(V\) in (ii), as the grand MD prize \(W = 2V\) is only allocated with probability 1/2.

2.2 Dominance of the grand contest

We now consider a general model with a general prize structure \((V_1, V_2, W)\) and general discriminatory powers \(r_1\) and \(r_2\), where \(V_1, V_2, W \geq 0\) and \(r_1, r_2 > 0\). In this model, player \(A\)'s expected payoff is

\[
\pi_A = \left( \frac{x_{A1}^r}{x_{A1}^r + x_{B1}^r} \right) \left( \frac{x_{A2}^r}{x_{A2}^r + x_{B2}^r} \right) W + \frac{x_{A1}^r}{x_{A1}^r + x_{B1}^r} V_1 + \frac{x_{A2}^r}{x_{A2}^r + x_{B2}^r} V_2 - c_1x_{A1} - c_2x_{A2}, \tag{3}
\]

while player \(B\)'s expected payoff is

\[
\pi_B = \left( \frac{x_{B1}^r}{x_{A1}^r + x_{B1}^r} \right) \left( \frac{x_{B2}^r}{x_{A2}^r + x_{B2}^r} \right) W + \frac{x_{B1}^r}{x_{A1}^r + x_{B1}^r} V_1 + \frac{x_{B2}^r}{x_{A2}^r + x_{B2}^r} V_2 - c_1x_{B1} - c_2x_{B2}. \tag{4}
\]

Consider a symmetric equilibrium in which player \(A\) exerts effort \((x_{A1}, x_{A2})\) and player \(B\) exerts effort \((x_{B1}, x_{B2})\), where \(x_{A1} = x_{B1} = x_{A1}^*\) and \(x_{A2} = x_{B2} = x_{A2}^*\). The existence and uniqueness of a symmetric pure-strategy equilibrium is established formally in Lemma 1.

**Lemma 1** For any prize structure \((V_1, V_2, W)\) considered in this paper, if \(r_1 + r_2 \leq 2\), it can be shown that: (i) existence: there always exists a symmetric equilibrium under which
each player exerts the same level of effort in each dimension; (ii) uniqueness: the symmetric equilibrium is the unique pure-strategy equilibrium.\textsuperscript{7} Equilibrium efforts are given by

\begin{align*}
x_1^* &= \frac{r_1}{8c_1} (W + 2V_1), \\
x_2^* &= \frac{r_2}{8c_2} (W + 2V_2).
\end{align*}

\textbf{Proof.} See the Appendix. \hfill \square

In the symmetric pure-strategy equilibrium, using (5) and (6), the expected payoff of player $i \in \{A, B\}$ is

\[
\pi_i^* = \frac{1}{4} W + \frac{1}{2} V_1 + \frac{1}{2} V_2 - c_1 x_1^* - c_2 x_2^*
\]

\[
= \left( 2 - \frac{r_1 - r_2}{8} \right) W + \left( 2 - \frac{r_1}{4} \right) V_1 + \left( 2 - \frac{r_2}{4} \right) V_2 \geq 0.
\]

It can be shown that $\pi_i^* = 0$ if and only if $r_1 + r_2 = 2$ and $V_1 = V_2 = 0$. Otherwise, $\pi_i^* > 0$.

We are now ready to consider the optimal design problem. For this purpose, we first solve for the constrained optimal prize structure given a fixed expected expenditure for prize allocation, say $\Gamma > 0$. We will show that the constrained optimal prize structure must be $(\hat{V}_1, \hat{V}_2, \hat{W}) = (0, 0, 2\Gamma)$.

In the symmetric equilibrium where two players exert the same level of effort in each dimension, $V_1$ and $V_2$ are allocated with probability $1$ and $W$ is allocated with probability $1/2$. Then the expected expenditure constraint requires:

\[
V_1 + V_2 + \frac{1}{2} W = \Gamma. \tag{7}
\]

The organizer’s problem is to maximize $\psi(x_1^*, x_2^*)$ by choosing $(V_1, V_2, W)$, subject to (5), (6), (7), and $V_1, V_2, W \geq 0$.

Using (5), (6), and (7), we derive that

\begin{align*}
x_1^* &= \frac{r_1}{4c_1} (\Gamma - V_2), \\
x_2^* &= \frac{r_2}{4c_2} (\Gamma - V_1).
\end{align*}

From (8) and (9), it is clear to see that $x_k^*$ increases in $V_l$, $\forall k, l \in \{1, 2\}$ and $k \neq l$. As a result, it is optimal to choose $V_1 = V_2 = 0$, which also implies $W = 2\Gamma$. We thus have the following proposition.

\textsuperscript{7}It can be shown that a pure-strategy equilibrium may not exist if $r_1 + r_2 > 2$. 

9
Proposition 1 When players are symmetric, for any given expected prize expenditure \( \Gamma > 0 \), the optimal prize structure is \( \tilde{V} = (\tilde{V}_1, \tilde{V}_2, \tilde{W}) = (0, 0, 2\Gamma) \).

Intuitions

To illustrate the intuitions behind the result, consider the following two changes from an original prize structure \((V_1, V_2, W)\): (a) let both \(V_1\) and \(V_2\) increase by \(\Delta > 0\); (b) let \(W\) increase by \(2\Delta\). From (5) and (6), we can see that the equilibrium effort levels after changes (a) and (b) are exactly the same since in either case, \(W + 2V_k\) increases by \(2\Delta, \forall k\). However, we can see that (b) costs less than (a) for the organizer in expectation, as \(\Gamma_a > \Gamma_b\), where

\[
\Gamma_a = (V_1 + \Delta) + (V_2 + \Delta) + \frac{1}{2}W = \left(V_1 + V_2 + \frac{1}{2}W\right) + 2\Delta,
\]

\[
\Gamma_b = V_1 + V_2 + \frac{1}{2}(W + 2\Delta) = \left(V_1 + V_2 + \frac{1}{2}W\right) + \Delta.
\]

To induce the same effort increase, increasing grand MD prize \(W\) costs less than increasing the SD prizes. We call this the cost-saving effect of performance bundling, as \(W\) is based on bundled performance across two dimensions. With performance bundling, a single grand MD prize is employed simultaneously to incentivize effort making in both dimensions, but it is only allocated when a player wins in both dimensions. With symmetric players, at equilibrium each player wins each dimension with half a chance. If a grand contest is adopted, creating 2 units of (grand) prize spread costs the organizer a 1 unit in expectation, and it generates a 1 unit prize spread in each dimension. In contrast, if a set of independent SD contests is adopted, creating a 1 unit prize spread in each dimension costs the organizer 2 units, as two SD prizes are allocated for certain. As a result of the cost-saving effect, the prize spreads of two dimensions that are generated by any eligible prize structure can be replicated (in the sense that the same effort levels are induced) by increasing the grand MD prize and lowering the two SD prizes appropriately, which costs the organizer less.

To further understand the cost-saving effect, consider the following thought experiment. Starting from an original prize allocation \((V_1, V_2, W)\), consider a transfer from an SD prize, say \(V_1\), to \(W\), while keeping \(\Gamma\) unchanged, i.e., let \(V_1\) decrease by \(\Delta\) and \(W\) increase by \(2\Delta\). One can check that the expected prize expenditure continues to be \(\Gamma\), as

\[
(V_1 - \Delta) + V_2 + \frac{1}{2}(W + 2\Delta) = V_1 + V_2 + \frac{1}{2}W.
\]

Using (5) and (6), we conclude that after this “T-fixed” transfer, \(x_1^*\) stays the same and \(x_2^*\)
is strictly higher, as
\[(W + 2\Delta) + 2(V_1 - \Delta) = W + 2V_1,\]
\[(W + 2\Delta) + 2V_2 > W + 2V_2.\]
So any “T-fixed” transfer from \(V_k\) to \(W\) will induce the same effort in dimension \(k\), but strictly higher effort in the other dimension \(l\). Thus, the organizer always finds it beneficial to make such a transfer from \(V_k\) to \(W\), which explains the dominance of the grand contest.

For any given expected prize expenditure \(\Gamma\), we have shown that it is optimal to choose \(V_1 = V_2 = 0\) and \(W = 2\Gamma\). Then, for any given \(\Gamma\), the organizer’s expected payoff \(\Psi\) can be written as
\[\Psi(\Gamma) = \psi \left( \frac{r_1}{2c_1}, \frac{r_2}{2\Gamma} \right) - \Gamma.\]

**Assumption 1** \(\Psi(\Gamma)\) is maximized uniquely at some \(\Gamma^* \in (0, +\infty)\).

Clearly, the above assumption holds if \(\lim_{z \to 0^+} \Psi(z) > 0\), \(\lim_{z \to +\infty} \Psi(z) < 0\), and \(\Psi''(\cdot) < 0\). The following theorem summarizes the optimal prize structure.

**Theorem 1** Under Assumption 1, there exists a unique \(\Gamma^* > 0\) such that \((V_1^*, V_2^*, W^*) = (0, 0, 2\Gamma^*)\) is the optimal prize structure that maximizes the organizer’s expected payoff.

### 3 Asymmetry across players

With asymmetric players, a general analysis is technically challenging. To make our analysis both insightful and tractable, in this section we consider the following stylized model with two asymmetric players. Assume that \(r_1 = r_2 = 1\), \(c_{A1} = c_{A2} = c_A\) and \(c_{B1} = c_{B2} = c_B\). Without loss of generality, we assume
\[\theta = \frac{c_A}{c_B} \leq 1.\]
This specification models a situation in which each player is equally able in two activities and player \(A\) is more able than player \(B\).

We continue to assume that the organizer’s benefit function is \(\psi(\pi_1, \pi_2)\), where \(\pi_k = E(x_{Ak} + x_{Bk})\) denotes expected total effort elicited in dimension \(k\), and \(\psi(\pi_1, \pi_2)\) is increasing.
in each dimension and concave in both dimensions. Moreover, let the organizer focus on a
prize structure \((V_1, V_2, W) = (V, V, W)\) where \(V \geq 0\) and \(W \geq 0\).

With symmetric players, we have shown that a pure-strategy equilibrium always exists
for any possible prize structure. However, in this model with asymmetry across players, for
a given prize structure \((V, V, W)\), we find that a pure-strategy equilibrium exists if and only
if \(W\) is sufficiently small relative to \(V\).

When a pure-strategy equilibrium exists, player \(A\) exerts effort \(x_{A1} = x_{A2} = x_{A}^{\text{pure}} > 0\),
while player \(B\) exerts effort \(x_{B1} = x_{B2} = x_{B}^{\text{pure}} > 0\). Define
\[
k_{A}^{\text{pure}} = \frac{x_{B}^{\text{pure}}}{x_{A}^{\text{pure}}},
\]
which is the equilibrium effort ratio of the two players. We have the following results. The
proof is relegated to the Appendix.

**Proposition 2** When \(W \leq \left(\sqrt{\frac{3-\theta}{1-\theta}} - 1\right)\), there exists a unique pure-strategy equilibrium, in
which players’ equilibrium efforts are given by
\[
x_{A}^{\text{pure}} = \frac{\theta V}{2c_A(1-\theta)^3W^2} \left\{ \frac{[(1-\theta)W - 2(1+\theta)V] \sqrt{(1-\theta)^2(W + V)^2 + 4\theta V^2}}{+ 2(1+\theta)^2V^2 + (1-\theta)^2W^2 + (1-\theta)(1-3\theta)WV} \right\};
\]
\[
x_{B}^{\text{pure}} = \frac{\theta V}{2c_A(1-\theta)^3W^2} \left\{ \frac{[(1-\theta)W + 2(1+\theta)V] \sqrt{(1-\theta)^2(W + V)^2 + 4\theta V^2}}{- 2(1+\theta)^2V^2 - (1-\theta)^2W^2 - (1-\theta)(3-\theta)WV} \right\},
\]
and the equilibrium effort ratio is
\[
k_{A}^{\text{pure}} = \frac{\sqrt{(1-\theta)^2(W + V)^2 + 4\theta V^2} - (1-\theta)(W + V)}{2V}.
\]

Following Proposition 2, we can further obtain the following results.

**Corollary 1** In the pure-strategy equilibrium of Proposition 2, the expected total prize
expenditure is
\[
TP_{E}^{\text{pure}} = \frac{(1-\theta)^2W^2 + (1+\theta)^2V^2 + 3(1-\theta)^2WV - (1+\theta)V \sqrt{(1-\theta)^2(W + V)^2 + 4\theta V^2}}{(1-\theta)^2W},
\]
the expected total effort is
\[
TE_{E}^{\text{pure}} = \frac{2\theta V \left[ \sqrt{(1-\theta)^2(W + V)^2 + 4\theta V^2} - (1+\theta)V \right]}{c_A(1-\theta)^2W},
\]

\(^8\)The prize structure \((V, V, W)\) is general enough for our purpose of illustrating the impact of asymmetry
across players on the optimal prize structure.
and the expected total effort in each dimension is

\[ TE_{1}^{\text{pure}} = TE_{2}^{\text{pure}} = \frac{1}{2} TE^{\text{pure}}. \]

When \( W \) is sufficiently large compared to \( V \), a pure-strategy equilibrium does not exist. We will show in the following proposition that there exists a “semi-pure” equilibrium, under which player \( A \) exerts effort \( x_{A1} = x_{A2} = x_{A}^{\text{semi}} > 0 \), while player \( B \) exerts effort \( x_{B1} = x_{B2} = x_{B}^{\text{semi}} > 0 \) with probability \( p^{\text{semi}} < 1 \), and remains inactive, i.e., he exerts zero effort in every dimension, with probability \( 1 - p^{\text{semi}} \). The proof is relegated to the Appendix.

**Proposition 3** When \( \frac{W}{V} > \left( \sqrt{\frac{9-\theta}{1-\theta}} - 1 \right) \), a semi-pure equilibrium exists, in which players’ equilibrium efforts are given by

\[
x_{A}^{\text{semi}} = \theta \frac{(W + 2V)^2}{8cAW},
\]

\[
x_{B}^{\text{semi}} = \theta \frac{(W - 2V)(W + 2V)}{8cAW},
\]

player \( B \)’s participation probability is

\[
p^{\text{semi}} = \theta \frac{W(W + 2V)}{(W - 2V)(W + 4V)} < 1,
\]

and the equilibrium effort ratio is

\[
k^{\text{semi}} = \frac{x_{B}^{\text{semi}}}{x_{A}^{\text{semi}}} = \frac{W - 2V}{W + 2V}.
\]

Following Proposition 3, we can further obtain the following results.

**Corollary 2** In the semi-pure equilibrium of Proposition 3, the expected total prize expenditure is

\[
TPE^{\text{semi}} = \frac{(W + 2V)[2(W + 4V) - \theta(W + 2V)]}{2(W + 4V)},
\]

the expected total effort is

\[
TE^{\text{semi}} = \frac{\theta[(1 + \theta)W + 4V](W + 2V)^2}{4cAW(W + 4V)},
\]

and the expected total effort in each dimension is

\[
TE_{1}^{\text{semi}} = TE_{2}^{\text{semi}} = \frac{1}{2} TE^{\text{semi}}.
\]
One can see that the above semi-pure equilibrium covers the pure-strategy equilibrium of Proposition 2 as a special case. In this sense, both the pure-strategy equilibrium (with $p^{\text{semi}} = 1$) and semi-pure equilibrium (with $p^{\text{semi}} < 1$) can be unified under the same umbrella.

Note that under semi-pure equilibria, for any given expected expenditure of prize allocation, the weaker player’s participation probability $p^{\text{semi}}$ decreases in $W$. Moreover, when $\frac{W}{V} = \sqrt{\frac{9 - \theta}{1 - \theta}} - 1$, i.e., when $\frac{W}{V}$ takes the threshold value separating pure and semi-pure strategy equilibria, we have

\[
TE^{\text{pure}} = TE^{\text{semi}} = \frac{\theta}{2c_A} \left( \sqrt{\frac{9 - \theta}{1 - \theta}} + 1 \right) V,
\]

\[
TPE^{\text{pure}} = TPE^{\text{semi}} = \frac{1}{4} \left[ (3 - \theta) \sqrt{\frac{9 - \theta}{1 - \theta}} + 7 - \theta \right] V.
\]

Therefore, both $TE$ and $TPE$ are continuous in $W$ for any given $\Gamma > 0$. This is also consistent with our previous argument that the pure-strategy equilibria and semi-pure equilibria belong to the more general semi-pure equilibria with $p^{\text{semi}} \leq 1$.

Given that we have characterized the equilibrium for any $W$ and $V$, next we seek to work out the optimal prize structure for a given expected expenditure $\Gamma$ of prize allocation. To solve for the optimal prize structure $(V, V, W)$, which maximizes the organizer’s expected payoff $\psi\left(\frac{1}{2}TE, \frac{1}{2}TE\right) - TPE$, we must identify the prize structure which maximizes $TE$ for any given $TPE = \Gamma > 0$.

Define the prize ratio

\[
t = \frac{W}{V} \quad \text{and} \quad \bar{t}(\theta) = \sqrt{\frac{9 - \theta}{1 - \theta}} - 1.
\]

It is clear that $\bar{t}(\theta)$ strictly increases in $\theta$. In the following lemma, we derive the expression for total effort $TE$, given expected budget $\Gamma$.

**Lemma 2** For any prize structure $(V, V, W)$ such that at equilibrium the expected total prize expenditure allocated to players equals a given $\Gamma > 0$, the expected total effort $TE$ can be written as the product of $\Gamma$ and a function of (prize ratio) $t$ and (ability ratio) $\theta$, i.e.,

\[
TE = \begin{cases} 
\frac{\theta}{c_A} \left( \frac{2\sqrt{(1-\theta)^2(t+1)^2+4\theta-2(1+\theta)} - 2(1+\theta)}{(1-\theta)^2(t+3)+2(1+\theta)^2 - (1+\theta)\sqrt{(1-\theta)^2(t+1)^2+4\theta}} \right) \Gamma, & t \leq \bar{t}(\theta), \\
\frac{\theta}{c_A} \left( \frac{(1+\theta)(t+4)(t+2)}{2(t+4)-(\theta(t+2))} \right) \Gamma, & t > \bar{t}(\theta).
\end{cases}
\]
Lemma 2 is very useful, as it illustrates that for given $\theta$ and total prize expenditure $\Gamma$, the expected total effort elicited is simply a function of the prize ratio $t$. Using Lemma 2, we can pin down the optimal effort ratio $t^*(\theta)$ that maximizes $TE$ subject to a binding fixed expected prize expenditure $\Gamma > 0$. If $t^*(\theta) = 0$, we thus have $W = 0$ and $V > 0$ at optimum; if $t^*(\theta) \in (0, +\infty)$, we have $W > 0$ and $V > 0$ at optimum; if $t^*(\theta) = +\infty$, we have $W > 0$ and $V = 0$ at optimum.

Let $(V^*, V^*, W^*)$ be the optimal prize structure that maximizes the expected total effort subject to a fixed total prize expenditure $\Gamma > 0$.

**Proposition 4** There exist cutoffs $\theta \in (0, \frac{1}{2})$ and $\tilde{\theta} \in (\frac{1}{2}, 1)$ such that for any fixed binding expected prize expenditure $\Gamma > 0$, (i) when $\theta \leq \tilde{\theta}$, $W^* = 0$ and $V^* > 0$; (ii) when $\theta \in (\theta, \tilde{\theta})$, $W^* > 0$ and $V^* > 0$; and (iii) when $\theta \geq \tilde{\theta}$, $W^* > 0$ and $V^* = 0$.

The result of Proposition 4 can be shown more explicitly in Figures 1 to 3. Figure 1 shows that when $\theta \leq \tilde{\theta}$, expected total effort strictly decreases in the prize ratio $t$ for any given expected prize expenditure $\Gamma$, which implies that $W^* = 0$ and $V^* > 0$. Figure 2 depicts the relationship between expected total effort and the prize ratio for moderate $\theta$, i.e., when $\theta \in (\tilde{\theta}, \tilde{\theta})$. Figure 2(a) shows that when $\theta \in (\tilde{\theta}, \frac{1}{2}]$, the expected total effort first increases and then decreases in $t$, while Figure 2(b) shows that the expected total effort first increases, then decreases, and finally increases in $t$. In both cases, however, the optimal grand MD prize and SD prizes are strictly positive, i.e., $W^* > 0$ and $V^* > 0$. Lastly, Figure 3 covers the case in which $\theta$ is high, i.e., when $\theta \geq \tilde{\theta}$. As shown in the proof of Proposition 4, there exists $\theta_1 \in (\tilde{\theta}, 1)$ such that two subcases should be considered separately: $\theta \in [\tilde{\theta}, \theta_1]$ and $\theta \in (\theta_1, 1]$, as depicted in Figure 3. For both subcases, it is evident that total effort is maximized when $t$ goes to infinity, i.e., $V^* = 0$. Thus, it is optimal to allocate the entire expected budget to the grand MD prize only.

From the above result, we can see that when the two players’ ability ratio is sufficiently large (when $\theta \geq \tilde{\theta}$), our previous result remains unchanged, i.e., the grand MD contest continues to dominate, under which $V^* = 0$ at optimum, and thus there is no SD feature; in contrast, when the two players’ ability ratio is sufficiently small (when $\theta \leq \tilde{\theta}$), the result is reversed—a set of two independent SD contests dominates, under which $W^* = 0$, and thus there is no MD feature; when the two players’ ability ratio is in the middle range (i.e., when

---

8As shown in the proof of the proposition, $\theta = 2 - \sqrt{3} \approx 0.2679$, and $\tilde{\theta} \approx 0.79$, which do not depend on $\Gamma$. 

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Figure 1: When $0 < \theta \leq \bar{\theta}$, the optimal prize allocation has both MD and SD features, under which $W^* > 0$ and $V^* > 0$.

**Intuitions**

With asymmetric players, a “T-fixed” transfer from $V$ to $W$ has two effects: First, as in the case with symmetric players, there is a *cost-saving effect*, which enhances competition efficiency, and thus tends to increase the total effort supply. Moreover, there is another effect called the *unbalancing effect*, which, in the asymmetric case we consider, may weaken the competitive balance of the battlefield and thus tends to decrease expected total effort. With asymmetric players, the players’ winning chances in each activity/dimension are no longer fixed, as in the scenario with symmetric players. The stronger player wins each dimension with a higher chance, which is endogenously determined by the equilibrium effort induced by the prevailing prize structure. A larger grand prize tends to further tilt the already unbalanced competition between the two asymmetric players, since a player’s incremental prize spread in one dimension, which is generated by the grand prize, is proportional to his own winning chance in the other dimension. The unbalancing effect tends to dampen the effort supply of the players. Moreover, due to the unbalancing effect, introducing a
Figure 2: When $\underline{\theta} < \theta \leq \frac{1}{2}$

(b) $\frac{1}{2} < \theta < \tilde{\theta}$

Figure 2: When $\underline{\theta} < \theta < \tilde{\theta}$
Figure 3: When $\tilde{\theta} \leq \theta < 1$
grand prize increases the chance of the stronger player winning it, which would weaken the cost-saving effect, since the chance of the grand prize being allocated gets higher.

More specifically, we further elaborate on the unbalancing effect by considering a Γ-fixed transfer from $V$ to $W$, i.e., an increase in the prize ratio $t = \frac{W}{V}$. When $W$ is sufficiently small, a pure-strategy equilibrium exists. A higher prize ratio $t$, which increases equilibrium effort ratio $k^{\text{pure}}$, always leads to a less balanced effort being made in equilibrium, and thus tends to lower the total effort. When $W$ is sufficiently large, a semi-pure equilibrium exists under which the weaker player only enters the contest actively with a probability $p^{\text{semi}} < 1$. As the prize ratio increases, the probability $p^{\text{semi}}$ decreases, which also tends to lower the expected total effort. To sum, in the asymmetric case we consider, a Γ-fixed transfer from $V$ to $W$—i.e., an increase in $t$—may lead to a less balanced competition, which can be shown by a lower equilibrium effort ratio $k^{\text{pure}}$ when a pure-strategy equilibrium exists, or a lower participation probability $p^{\text{semi}}$ when a semi-pure equilibrium exists. Therefore, the unbalancing effect can be negative in inducing total effort. This is in contrast to the cost-saving effect, which is always positive in inducing total effort.

Whether introducing a grand MD prize is optimal depends on the trade-off of the cost-saving effect and the unbalancing effect, both of which relate to the ability ratio of the two players $\theta$. The cost-saving effect diminishes when $\theta$ decreases (i.e., when the two players become more asymmetric), as the grand prize is allocated with a higher chance, while the unbalancing effect increases when $\theta$ decreases. When the ability ratio is sufficiently large, the unbalancing effect is small, and thus the total effect (i.e., the sum of the cost-saving effect and the unbalancing effect) is always positive. Therefore, a transfer from $V$ to $W$ is always beneficial, implying that $W^* > 0$ and $V^* = 0$, i.e., a grand MD contest is optimal. When the ability ratio is sufficiently small, the negative unbalancing effect always dominates the positive cost-saving effect, which implies that a transfer from $W$ to $V$ is always beneficial for the organizer. Thus, $W^* = 0$ and $V^* > 0$, i.e., the set of two independent SD contests is optimal. When the ability ratio is in the middle range, the cost-saving effect dominates when $W$ is relatively small and the unbalancing effect dominates otherwise. Hence, the optimal prize allocation requires $W^* > 0$ and $V^* > 0$.

Once we have derived the optimal prize structure with an expected budget $\Gamma > 0$, the optimal prize structure without any fixed expected budget can be derived by finding the optimal $\Gamma$ that maximizes the organizer’s expected payoff, which is written as

$$\Phi(\Gamma) = \psi \left( \frac{1}{2} TE^*(\Gamma), \frac{1}{2} TE^*(\Gamma) \right) - \Gamma,$$
where $TE^*(\Gamma)$ is the expected total effort when the optimal prize allocation is chosen given an expected budget $\Gamma$, which is a linear function of $\Gamma$ by Lemma 2.

**Assumption 2** $\Phi(\Gamma)$ is maximized uniquely at some $\Gamma^{**} \in (0, +\infty)$.

Clearly, the above assumption holds if $\lim_{z \to 0^+} \Phi(z) > 0$, $\lim_{z \to +\infty} \Phi(z) < 0$, and $\Phi''(\cdot) < 0$. Let $(V^{**}, V^{**}, W^{**})$ denote the optimal prize structure that maximizes the organizer’s expected payoff without any fixed expected budget of prize allocation. Under Assumption 2, Proposition 4 thus means the following result.

**Theorem 2** There exist cutoffs $\theta \in (0, \frac{1}{2})$ and $\tilde{\theta} \in (\frac{1}{2}, 1)$ as defined in Proposition 4, such that (i) when $\theta \leq \tilde{\theta}$, $W^{**} = 0$ and $V^{**} > 0$; (ii) when $\theta \in (\tilde{\theta}, \bar{\theta})$, $W^{**} > 0$ and $V^{**} > 0$; and (iii) when $\theta \geq \bar{\theta}$, $W^{**} > 0$ and $V^{**} = 0$.

## 4 Concluding remarks

In this paper, we study the optimal prize structure when two players compete across multiple activities/dimensions. We find that the asymmetry between players is a key factor that determines the optimal prize structure. Our study demonstrates that whether the optimal prize structure should feature performance bundling depends on the trade-off between two effects—the cost-saving effect and the unbalancing effect. The cost-saving effect favors performance bundling, while the unbalancing effect goes against it. Since the cost-saving effect decreases and the unbalancing effect increases with asymmetry between players, the cost-saving effect dominates if and only if the asymmetry between players is in the low range, and the unbalancing effect dominates if and only if the asymmetry is in the high range. As a result, a grand multi-dimensional (MD) contest that only awards the player winning all activities/dimensions is optimal if and only if the asymmetry is in the low range; a set of independent single single-dimensional (SD) contests is optimal if and only if the asymmetry is in the high range; and a mix of the two structures is optimal if the asymmetry is in the middle range.

Our findings provide useful guidance for choosing a prize structure to incentivize multiple competitions among employees in organizations, and predict that a wide range of prize structures can be observed in practice. In particular, in organizations with homogenous employees, performance bundling would be desirable to enhance organizational performance.
In organizations with rather heterogenous employees, in contrast, performance bundling can be detrimental.

On the one hand, our results confirm that the insight of Laux (2001) and Zhao (2008) for a single-agent contract setting still has bite in our multi-agent contest setting. On the other hand, we find that performance bundling in a contest setting entails an additional unbalancing effect, which is unique to a contest setting when players are asymmetric. When players are significantly asymmetric, the unbalancing effect could dominate, which leads to the optimality of a set of independent SD contests. This is in dramatic contrast to the optimality of the pure bundled-performance-evaluation (BPE) in the contract literature.

5 Appendix

5.1 Proof of Lemma 1

We first show the existence of the symmetric pure-strategy equilibrium, and prove that the symmetric equilibrium is indeed the unique pure-strategy equilibrium. We can restrict our attention to the case with $W > 0$.

It is easy to verify that any pure-strategy equilibrium must be interior, in which the following first-order conditions must be satisfied:

$$
\frac{r_1 x_{A1}^{r_1-1} x_{B1}}{(x_{A1}^{r_1} + x_{B1})^2} \left( \frac{x_{A2}^{r_2}}{x_{A2}^{r_1} + x_{B2}^{r_2}} W + V_1 \right) - c_1 = 0,
$$

$$
\frac{r_2 x_{A2}^{r_2-1} x_{B2}}{(x_{A2}^{r_2} + x_{B2})^2} \left( \frac{x_{A1}^{r_1}}{x_{A1}^{r_1} + x_{B1}^{r_1}} W + V_2 \right) - c_2 = 0,
$$

$$
\frac{r_1 x_{B1}^{r_1-1} x_{A1}}{(x_{A1}^{r_1} + x_{B1}^{r_1})^2} \left( \frac{x_{B2}^{r_2}}{x_{A2}^{r_2} + x_{B2}^{r_2}} W + V_1 \right) - c_1 = 0,
$$

$$
\frac{r_2 x_{B2}^{r_2-1} x_{A2}}{(x_{A2}^{r_2} + x_{B2}^{r_2})^2} \left( \frac{x_{B1}^{r_1}}{x_{A1}^{r_1} + x_{B1}^{r_1}} W + V_2 \right) - c_2 = 0.
$$

The solution to the above first-order conditions is given by $x_{A1} = x_{B1} = x_1^* > 0$ and $x_{A2} = x_{B2} = x_2^* > 0$, where $x_1^*$ and $x_2^*$ are defined as in Lemma 1. We next show this is indeed the unique equilibrium in pure strategy.
5.1.1 Existence

To show the existence of the symmetric equilibrium, by symmetry, we just need to show that given player $B$'s equilibrium strategy $(x_{B1}, x_{B2}) = (x_1^*, x_2^*)$, it is optimal for player $A$ to choose $(x_{A1}, x_{A2}) = (x_1^*, x_2^*)$.

Provided that player $B$ has chosen $(x_1^*, x_2^*)$, player $A$’s payoff can be written as

$$
\pi_A(x_{A1}, x_{A2}) = \left(\frac{x_{A1}^{r_1}}{x_{A1}^{r_1} + (x_1^*)^{r_1}}\right) \left(\frac{x_{A2}^{r_2}}{x_{A2}^{r_2} + (x_2^*)^{r_2}}\right) W + \frac{x_{A1}^{r_1}}{x_{A1}^{r_1} + (x_1^*)^{r_1}} V_1 + \frac{x_{A2}^{r_2}}{x_{A2}^{r_2} + (x_2^*)^{r_2}} V_2 - c_1 x_{A1} - c_2 x_{A2}.
$$

The first-order conditions are

$$
r_1 \frac{x_{A1}^{r_1-1}(x_1^*)^{r_1}}{(x_{A1}^{r_1} + (x_1^*)^{r_1})^2} \left[ - \frac{x_{A2}^{r_2}}{x_{A2}^{r_2} + (x_2^*)^{r_2}} W + V_1 \right] = c_1 = 0, \tag{10}
$$

$$
r_2 \frac{x_{A2}^{r_2-1}(x_2^*)^{r_2}}{(x_{A2}^{r_2} + (x_2^*)^{r_2})^2} \left[ - \frac{x_{A1}^{r_1}}{x_{A1}^{r_1} + (x_1^*)^{r_1}} W + V_2 \right] = c_2 = 0. \tag{11}
$$

The second derivatives of $\pi_A$ with respect to $x_{A1}$ and $x_{A2}$ can be derived as

$$
\frac{\partial^2 \pi_A}{\partial x_{A1}^2} = -r_1 \frac{(1 + r_1)x_{A1}^{r_1} + (1 - r_1)(x_1^*)^{r_1}}{(x_{A1}^{r_1} + (x_1^*)^{r_1})^3} \left[ - \frac{x_{A2}^{r_2}}{x_{A2}^{r_2} + (x_2^*)^{r_2}} W + V_1 \right],
$$

$$
\frac{\partial^2 \pi_A}{\partial x_{A2}^2} = -r_2 \frac{(1 + r_2)x_{A2}^{r_2} + (1 - r_2)(x_2^*)^{r_2}}{(x_{A2}^{r_2} + (x_2^*)^{r_2})^3} \left[ - \frac{x_{A1}^{r_1}}{x_{A1}^{r_1} + (x_1^*)^{r_1}} W + V_2 \right],
$$

$$
\frac{\partial^2 \pi_A}{\partial x_{A1} \partial x_{A2}} = -r_1 r_2 \frac{x_{A1}^{r_1-1}x_{A2}^{r_2-1}(x_1^*)^{r_1}(x_2^*)^{r_2}}{(x_{A1}^{r_1} + (x_1^*)^{r_1})^2(x_{A2}^{r_2} + (x_2^*)^{r_2})^2} W.
$$

First we show that $(x_1^*, x_2^*)$ is a local maximum for $\pi_A$. Note that first-order conditions (10) and (11) are clearly satisfied when $(x_{A1}, x_{A2}) = (x_1^*, x_2^*)$ by the definition of $x_1^*$ and $x_2^*$. Second-order conditions are also satisfied, given that

$$
\left. \frac{\partial^2 \pi_A}{\partial x_{A1}^2} \right|_{x_{A1}=x_1^*, x_{A2}=x_2^*} = -\frac{r_1}{8(x_1^*)^2} (W + 2V_1) < 0,
$$

$$
\left. \begin{vmatrix} \frac{\partial^2 \pi_A}{\partial x_{A1}^2} & \frac{\partial^2 \pi_A}{\partial x_{A1} \partial x_{A2}} \\ \frac{\partial^2 \pi_A}{\partial x_{A1} \partial x_{A2}} & \frac{\partial^2 \pi_A}{\partial x_{A2}^2} \end{vmatrix} \right|_{x_{A1}=x_1^*, x_{A2}=x_2^*} = \frac{r_1 r_2}{256(x_1^* x_2^*)^2} [4(W + 2V_1)(W + 2V_2) - r_1 r_2 W^2] = \frac{r_1 r_2}{256(x_1^* x_2^*)^2} (4 - r_1 r_2) W^2 > 0.
$$
It can be derived that player $A$’s payoff when he chooses $(x^*_1, x^*_2)$ is

$$\pi_A^* = \frac{1}{4} W + \frac{1}{2} V_1 + \frac{1}{2} V_2 - c_1 x^*_1 - c_2 x^*_2$$

$$= \left(\frac{2 - r_1 - r_2}{8}\right) W + \left(\frac{2 - r_1}{4}\right) V_1 + \left(\frac{2 - r_2}{4}\right) V_2 \geq 0,$$

where the inequality holds strictly unless $r_1 = r_2 = 1$ and $V_1 = V_2 = 0$.

To show that $(x^*_1, x^*_2)$ is also a global maximum for $\pi_A$, it is sufficient to show that there does not exist another local maximum, denoted by $(x'_{A1}, x'_{A2})$, that yields a strictly higher payoff for player $A$ than $\pi_A^*$. We first rule out the case in which player $A$’s effort in at least one dimension is zero. Without loss of generality, suppose $x'_{A1} = 0$ and $x'_{A2} \geq 0$. Denote $\hat{x}_2$ the effort of each player in the symmetric equilibrium when $W = V_1 = 0$ and $V_2 \geq 0$, where $\hat{x}_2 = \frac{r_2 V_2}{4c_2}$. It is clear that $x^*_2 \geq \hat{x}_2$. Therefore, we get that

$$\pi_A(0, x'_{A2}) = \frac{(x'_{A2})^{r_2}}{(x'_{A2})^{r_2} + (x^*_2)^{r_2}} V_2 - c_2 x'_{A2}$$

$$\leq \frac{(x'_{A2})^{r_2}}{(x'_{A2})^{r_2} + (\hat{x}_2)^{r_2}} V_2 - c_2 x'_{A2} \leq \left(\frac{2 - r_2}{4}\right) V_2 \leq \pi_A^*,$$

where the first inequality follows from $\hat{x}_2 \leq x^*_2$, and the second inequality follows from the definition of $\hat{x}_2$. Therefore, it remains to consider the case $x'_{Aj} > 0$ for $j = 1, 2$, in which the two first-order conditions (10) and (11) must both be satisfied for $(x_{A1}, x_{A2}) = (x'_{A1}, x'_{A2})$. From these two conditions, it is clear that we only need to consider the following two cases:

(i) $x'_{A1} \geq x^*_1$ and $x'_{A2} \geq x^*_2$ and (ii) $0 < x'_{A1} < x^*_1$ and $0 < x'_{A2} < x^*_2$.

In case (i), we claim that $(x^*_1, x^*_2)$ is the unique local maximum. To prove this, it is sufficient to show that $\pi_A$ is concave. This is true given that the second-order conditions are satisfied

$$\frac{\partial^2 \pi_A}{\partial x^2_{A1}}|_{x_{A1}=x'_{A1}, x_{A2}=x'_{A2}} \leq -\frac{2r_1(x'_{A1})^{r_1-2}(x^*_1)^{2r_1}}{((x'_{A1})^{r_1} + (x^*_1)^{r_1})^3} \left(\frac{1}{2} W + V_1\right) < 0,$$

where

$$\frac{\partial^2 \pi_A}{\partial x^2_{A1}} \frac{\partial^2 \pi_A}{\partial x^2_{A2}} \left|_{x_{A1}=x'_{A1}, x_{A2}=x'_{A2}} \right. \geq \frac{r_1 r_2 (x'_{A1})^{r_1-2}(x'_{A2})^{r_2-2}(x^*_1)^{2r_1}(x^*_2)^{2r_2}}{((x'_{A1})^{r_1} + (x^*_1)^{r_1})^3((x'_{A2})^{r_2} + (x^*_2)^{r_2})^3} \left[ 4 \left(\frac{(x'_{A2})^{r_2}}{(x'_{A2})^{r_2} + (x^*_2)^{r_2}} W + V_1\right) \left(\frac{(x'_{A1})^{r_1}}{(x'_{A1})^{r_1} + (x^*_1)^{r_1}} W + V_2\right) - r_1 r_2 \left(\frac{(x'_{A2})^{r_2}}{(x'_{A2})^{r_2} + (x^*_2)^{r_2}} W\right)^2 \right]$$

$$\geq \frac{r_1 r_2 (x'_{A1})^{2r_1-2}(x'_{A2})^{2r_2-2}(x^*_1)^{2r_1}(x^*_2)^{2r_2}}{((x'_{A1})^{r_1} + (x^*_1)^{r_1})^4((x'_{A2})^{r_2} + (x^*_2)^{r_2})^4} (4 - r_1 r_2) W^2 > 0.$$
In case \((ii)\), the two first-order conditions \((10)\) and \((11)\) must be satisfied for \((x_{A1}, x_{A2}) = (x'_{A1}, x'_{A2})\). Let

\[ k_1 = \left( \frac{x'_{A1}}{x_1^*} \right)^{r_1} < 1, \text{ and } k_2 = \left( \frac{x'_{A2}}{x_2^*} \right)^{r_2} < 1. \]

From the first-order conditions, we obtain

\[
\frac{r_1 k_1}{(k_1 + 1)^2} \left( \frac{k_2}{k_2 + 1} W + V_1 \right) = c_1 x'_{A1},
\]
\[
\frac{r_2 k_2}{(k_2 + 1)^2} \left( \frac{k_1}{k_1 + 1} W + V_2 \right) = c_2 x'_{A2}.
\]

Given player \(B\)'s equilibrium efforts \((x^*_1, x^*_2)\), player \(A\)'s expected payoff when he chooses efforts \((x'_{A1}, x'_{A2})\) is

\[
\pi_A(x'_{A1}, x'_{A2}) = \left( \frac{k_1}{k_1 + 1} \right) \left( \frac{k_2}{k_2 + 1} \right) W + \frac{k_1}{k_1 + 1} V_1 + \frac{k_2}{k_2 + 1} V_2 - c_1 x'_{A1} - c_2 x'_{A2}
\]
\[
= \left( \frac{k_1}{k_1 + 1} \right) \left( \frac{k_2}{k_2 + 1} \right) \left( 1 - \frac{r_1}{k_1 + 1} - \frac{r_2}{k_2 + 1} \right) W
\]
\[
+ \frac{k_1}{k_1 + 1} \left( 1 - \frac{r_1}{k_1 + 1} \right) V_1 + \frac{k_2}{k_2 + 1} \left( 1 - \frac{r_2}{k_2 + 1} \right) V_2.
\]

It is clear that \(\pi^*_A > \pi_A(x'_{A1}, x'_{A2})\) when \(k_1 < 1\) and \(k_2 < 1\).

In sum, it is optimal for player \(A\) to choose \((x^*_1, x^*_2)\) given player \(B\)'s equilibrium efforts \((x^*_1, x^*_2)\), which implies the existence of the symmetric equilibrium.

### 5.1.2 Uniqueness

To show that the above symmetric equilibrium is indeed the unique pure-strategy equilibrium, we just need to rule out any other interior equilibrium. In the following, we show it by contradiction. Suppose there is an equilibrium in which \(x_{ik} > 0\) for any \(i \in \{A, B\}\) and \(k \in \{1, 2\}\), and \(x_{B1} > x_{A1}\). By the first-order conditions, we get

\[
\frac{r_1 x_{A1}^{r_1} x_{B1}^{r_1}}{(x_{A1}^{r_1} + x_{B1}^{r_1})^2} \left( \frac{x_{A2}^{r_2}}{x_{A2}^{r_2} + x_{B2}^{r_2}} W + V_1 \right) = c_1 x_{A1},
\]
\[
\frac{r_1 x_{B1}^{r_1} x_{A1}^{r_1}}{(x_{A1}^{r_1} + x_{B1}^{r_1})^2} \left( \frac{x_{B2}^{r_2}}{x_{A2}^{r_2} + x_{B2}^{r_2}} W + V_1 \right) = c_1 x_{B1},
\]

which implies that

\[
x_{A1} = \frac{x_{A2}^{r_2}}{x_{A2}^{r_2} + x_{B2}^{r_2}} W + V_1,
\]
\[
x_{B1} = \frac{x_{B2}^{r_2}}{x_{A2}^{r_2} + x_{B2}^{r_2}} W + V_1.
\]

(12)
Given that \( x_{B1} > x_{A1} \), we must have \( x_{B2} > x_{A2} \), which further implies that

\[
\frac{x_{A1}}{x_{B1}} \geq \left( \frac{x_{A2}}{x_{B2}} \right)^{r_2},
\]

where the equality of (13) holds if and only if \( V_1 = 0 \). Following a similar logic, we get that

\[
\frac{x_{A2}}{x_{B2}} \geq \left( \frac{x_{A1}}{x_{B1}} \right)^{r_1},
\]

where the equality of (14) holds if and only if \( V_2 = 0 \). Thus, (13) and (14) together imply

\[
\frac{x_{A1}}{x_{B1}} \geq \left( \frac{x_{A1}}{x_{B1}} \right)^{r_1 r_2}.
\]

Given \( x_{B1} > x_{A1} \) and \( r_1 r_2 \leq 1 \), (15) holds only when \( r_1 = r_2 = 1 \) and \( V_1 = V_2 = 0 \). \(^{10}\)

It remains to check the case in which \( r_1 = r_2 = 1 \) and \( V_1 = V_2 = 0 \). Given that \( x_{B1} > x_{A1} \) and \( x_{B2} > x_{A2} \), player A’s equilibrium efforts are

\[
x_{A1} = \frac{k_1}{c_1(k_1+1)^2} \left( \frac{k_2}{k_2+1} \right) W,
\]

\[
x_{A2} = \frac{k_2}{c_2(k_2+1)^2} \left( \frac{k_1}{k_1+1} \right) W,
\]

where \( k_1 = \frac{x_{A1}}{x_{B1}} < 1 \) and \( k_2 = \frac{x_{A2}}{x_{B2}} < 1 \). Then player A’s equilibrium payoff is

\[
\pi_A = \frac{x_{A1}}{x_{A1} + x_{B1}} \left( \frac{x_{A2}}{x_{A2} + x_{B2}} \right) W - c_1 x_{A1} - c_2 x_{A2}
\]

\[
= \frac{k_1}{k_1+1} \left( \frac{k_2}{k_2+1} \right) W - \frac{k_1}{(k_1+1)^2} \left( \frac{k_2}{k_2+1} \right) W - \frac{k_2}{(k_2+1)^2} \left( \frac{k_1}{k_1+1} \right) W
\]

\[
= \frac{k_1}{k_1+1} \left( \frac{k_2}{k_2+1} \right) \left( 1 - \frac{1}{k_1+1} - \frac{1}{k_2+1} \right) W
\]

\[
< 0,
\]

which implies that such an equilibrium cannot exist.

As a result, the symmetric pure-strategy equilibrium is the unique pure-strategy equilibrium.

\(^{10}\) Notice that when \( r_1 + r_2 \leq 2 \), it is easy to obtain that \( r_1 r_2 \leq 1 \), where \( r_1 r_2 = 1 \) if and only if \( r_1 = r_2 = 1 \).
5.2 Proof of Proposition 2

To characterize the unique pure-strategy equilibrium, we just need to focus on the case with $W > 0$. Consider any pure-strategy equilibrium in which players’ efforts are $x_{ij}$ for any $i \in \{A, B\}$ and $j \in \{1, 2\}$. It is easy to check that the equilibrium must be interior (i.e., $x_{ij} > 0$). Otherwise, at least one player has an incentive to lower her effort in at least one dimension. Given that the equilibrium is interior, the following first-order conditions must hold:

\[
\frac{x_{B1}}{(x_{A1} + x_{B1})^2} \left( \frac{x_{A2}}{x_{A2} + x_{B2}} W + V \right) - c_A = 0, \\
\frac{x_{B2}}{(x_{A2} + x_{B2})^2} \left( \frac{x_{A1}}{x_{A1} + x_{B1}} W + V \right) - c_A = 0, \\
\frac{x_{A1}}{(x_{A1} + x_{B1})^2} \left( \frac{x_{B2}}{x_{A2} + x_{B2}} W + V \right) - c_B = 0, \\
\frac{x_{A2}}{(x_{A2} + x_{B2})^2} \left( \frac{x_{B1}}{x_{A1} + x_{B1}} W + V \right) - c_B = 0.
\]

Define $k_1 = \frac{x_{B1}}{x_{A1}} > 0$ and $k_2 = \frac{x_{B2}}{x_{A2}} > 0$. Then the first-order conditions can be written as

\[
k_1 = \frac{\theta(W k_2 + V k_2 + V)}{V k_2 + W + V}, \\
k_2 = \frac{\theta(W k_1 + V k_1 + V)}{V k_1 + W + V}, \\
x_{A1} = \frac{k_1}{(1 + k_1)^2 c_A} \left( \frac{1}{1 + k_2} W + V \right), \\
x_{A2} = \frac{k_2}{(1 + k_2)^2 c_A} \left( \frac{1}{1 + k_1} W + V \right).
\]

It can be further derived that there exists a unique solution to the above first-order conditions, such that $k_1 = k_2 = k_\text{pure}$, $x_{A1} = x_{A2} = x_{A}^{\text{pure}}$, and $x_{B1} = x_{B2} = x_{B}^{\text{pure}}$, where $k_\text{pure}$, $x_{A}^{\text{pure}}$, and $x_{B}^{\text{pure}}$ are given by Proposition 2. Therefore, the only possible pure-strategy equilibrium is when player A exerts effort $(x_{A}^{\text{pure}}, x_{A}^{\text{pure}})$ and player B exerts effort $(x_{B}^{\text{pure}}, x_{B}^{\text{pure}})$. Moreover, given the above efforts, players’ payoffs are

\[
\pi_A^{\text{pure}} = \frac{1}{(1 + k_\text{pure})^2} \left( \frac{1 - k_\text{pure}}{1 + k_\text{pure}} W + 2V \right), \\
\pi_B^{\text{pure}} = \frac{(k_\text{pure})^2}{(1 + k_\text{pure})^2} \left( \frac{k_\text{pure} - 1}{1 + k_\text{pure}} W + 2V \right).
\]


When \( W \geq \left( \sqrt{\frac{9-\theta}{1-\theta}} - 1 \right) \), we have \( \frac{W-2V}{W+2V} \leq k_{\text{pure}} \leq \theta \). Therefore, it is clear that \( \pi_A^{\text{pure}} > 0 \) and \( \pi_B^{\text{pure}} \geq 0 \). Note that \( \pi_B^{\text{pure}} = 0 \) if and only if \( \frac{W}{V} = \left( \sqrt{\frac{9-\theta}{1-\theta}} - 1 \right) \).

It remains to show that the strategy profile \((x_A^{\text{pure}}, x_A^{\text{pure}})\) and \((x_B^{\text{pure}}, x_B^{\text{pure}})\) indeed constitutes an equilibrium. We consider the following two steps.

Step 1: We show that it is optimal for player A to choose \((x_A^{\text{pure}}, x_A^{\text{pure}})\) given player B’s equilibrium strategy \((x_B^{\text{pure}}, x_B^{\text{pure}})\).

Provided that player B has chosen \((x_B^{\text{pure}}, x_B^{\text{pure}})\), player A’s payoff is written as

\[
\pi_A(x_A, x_A^2) = \frac{x_A}{x_A + x_A^2} \left( \frac{x_{A^2}}{x_A + x_A^2} W + \frac{x_A}{x_A + x_A^2} V \right) - c_A x_A^1 - c_A x_A^2.
\]

We first claim that \((x_A^{\text{pure}}, x_A^{\text{pure}})\) is a local maximum for \(\pi_A\). Note that the first-order conditions are written as

\[
\begin{align*}
\frac{x_B^{\text{pure}}}{(x_A + x_B^{\text{pure}})^2} & \left( \frac{x_{A^2}}{x_A + x_A^2} W + V \right) - c_A = 0, \\
\frac{x_B^{\text{pure}}}{(x_A + x_B^{\text{pure}})^2} & \left( \frac{x_A}{x_A + x_B^{\text{pure}}} W + V \right) - c_A = 0,
\end{align*}
\]

which clearly hold when \((x_A, x_A^2) = (x_A^{\text{pure}}, x_A^{\text{pure}})\) by the definition of \(x_A^{\text{pure}}\). The second derivatives of \(\pi_A\) with respect to \(x_A\) and \(x_A^2\) are

\[
\begin{align*}
\frac{\partial^2 \pi_A}{\partial x_A^2} & = - \frac{2x_B^{\text{pure}}}{(x_A + x_B^{\text{pure}})^3} \left( \frac{x_{A^2}}{x_A + x_B^{\text{pure}}} W + V \right), \\
\frac{\partial^2 \pi_A}{\partial x_A^2} & = - \frac{2x_B^{\text{pure}}}{(x_A + x_B^{\text{pure}})^3} \left( \frac{x_A}{x_A + x_B^{\text{pure}}} W + V \right), \\
\frac{\partial^2 \pi_A}{\partial x_A \partial x_A^2} & = \left( \frac{x_B^{\text{pure}}}{x_A + x_B^{\text{pure}}} \right)^2 \frac{1}{(x_A + x_B^{\text{pure}})^2 (x_A + x_B^{\text{pure}})^2} W.
\end{align*}
\]

The second-order conditions are also satisfied at \((x_A^1, x_A^2) = (x_A^{\text{pure}}, x_A^{\text{pure}})\), since

\[
\frac{\partial^2 \pi_A}{\partial x_A^2} \bigg|_{x_A = x_A^2 = x_A^{\text{pure}}} = - \frac{2x_B^{\text{pure}}}{(x_A^{\text{pure}} + x_B^{\text{pure}})^3} \left( \frac{x_A^{\text{pure}}}{x_A^{\text{pure}} + x_B^{\text{pure}}} W + V \right) < 0,
\]

\[
\begin{bmatrix}
\frac{\partial^2 \pi_A}{\partial x_A^2} & \frac{\partial^2 \pi_A}{\partial x_A \partial x_A^2} & \frac{\partial^2 \pi_A}{\partial x_A^2} \\
\frac{\partial^2 \pi_A}{\partial x_A \partial x_A^2} & \frac{\partial^2 \pi_A}{\partial x_A^2} & \frac{\partial^2 \pi_A}{\partial x_A^2} \\
\frac{\partial^2 \pi_A}{\partial x_A \partial x_A^2} & \frac{\partial^2 \pi_A}{\partial x_A^2} & \frac{\partial^2 \pi_A}{\partial x_A^2}
\end{bmatrix} \bigg|_{x_A = x_A^2 = x_A^{\text{pure}}}
\]

\[
= \left( x_B^{\text{pure}} \right)^2 \frac{2(W + V)x_A^{\text{pure}} + (W + 2V)x_B^{\text{pure}}}{(x_A^{\text{pure}} + x_B^{\text{pure}})^2} \frac{2(W + V)x_A^{\text{pure}} + (2V - W)x_B^{\text{pure}}}{(x_A^{\text{pure}} + x_B^{\text{pure}})^2} > 0,
\]

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given that $x^\text{pure}_A > x^\text{pure}_B$.

Then we just need to show there does not exist another local maximum, denoted by $(x'_{A1}, x'_{A2})$, that yields a strictly higher payoff for player $A$ than $\pi_A^\text{pure}$. We first rule out the case in which player $A$’s effort in at least one dimension is zero. Without loss of generality, suppose $x'_{A1} = 0$ and $x'_{A2} \geq 0$. It is easy to rule out the case in which $x'_{A1} = x'_{A2} = 0$, in which case player $A$ always obtains zero payoff. When $x'_{A1} = 0$ and $x'_{A2} > 0$, the first-order condition (19) must be satisfied, i.e.,

$$\frac{x^\text{pure}_B}{(x'_{A2} + x^\text{pure}_B)^2} V - c_A = 0.$$  

However, this further implies that

$$\frac{\partial \pi_A}{\partial x'_{A1}}|_{x'_{A1}=x'_{A2}=x^\text{pure}_2} = \frac{1}{x^\text{pure}_B} \left( \frac{x'_{A2}}{x'_{A2} + x^\text{pure}_B} W + V \right) - c_A$$

which contradicts the fact that $x'_{A1} = 0$ is optimal. Therefore, it remains to consider the case $x'_{A j} > 0$ for $j = 1, 2$, in which the two first-order conditions (18) and (19) must both be satisfied for $(x'_{A1}, x'_{A2}) = (x'_{A1}, x'_{A2})$. From these two conditions, it is clear that we only need to consider the following two cases: (i) $x'_{A1} \geq x^\text{pure}_{A1}$ and $x'_{A2} \geq x^\text{pure}_{A2}$, and (ii) $0 < x'_{A1} < x^\text{pure}_{A1}$ and $0 < x'_{A2} < x^\text{pure}_{A2}$.

In case (i), we claim that $(x^\text{pure}_A, x^\text{pure}_A)$ is the unique local maximum. To prove this, it is sufficient to show that $\pi_A$ is concave. This clearly holds given that the second-order conditions are satisfied, i.e.,

$$\frac{\partial^2 \pi_A}{\partial x'^2_{A1}}|_{x'_{A1}=x'_{A2}=x^\text{pure}_A} < 0,$$

$$\frac{\partial^2 \pi_A}{\partial x'^2_{A1}} \frac{\partial^2 \pi_A}{\partial x'_{A1} \partial x'_{A2}}|_{x'_{A1}=x'_{A2}=x^\text{pure}_A} < 0,$$

and

$$\frac{\partial^2 \pi_A}{\partial x'^2_{A2}}|_{x'_{A1}=x'_{A2}=x^\text{pure}_A} < 0.$$  

Let

$$\frac{(x^\text{pure}_B)^2}{(x'_{A1} + x^\text{pure}_B)^4(x'_{A2} + x^\text{pure}_B)^4} \left[ 4(x'_{A1} W + (x'_{A1} + x^\text{pure}_B) V)(x'_{A2} W + (x'_{A2} + x^\text{pure}_B) V) - (x^\text{pure}_B)^2 W^2 \right]$$

$$\geq \frac{(x^\text{pure}_B)^2}{(x'_{A1} + x^\text{pure}_B)^4(x'_{A2} + x^\text{pure}_B)^4} \left[ 2(W + V)x^\text{pure}_A + (W + 2V)x^\text{pure}_B \right] \left[ 2(W + V)x^\text{pure}_A + (2V - W)x^\text{pure}_B \right]$$

$$> 0,$$

where the second-to-last inequality follows from the fact that $x'_{A j} \geq x^\text{pure}_A$ for $j = 1, 2$. 

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In case (ii), from the two first-order conditions (18) and (19), which hold for \((x_{A1}, x_{A2}) = (x'_{A1}, x'_{A2})\), it is easy to derive that:

\[
\frac{x'_{A2}}{x'_{A2} + x_B^{\text{pure}}} W + V = \frac{(x'_{A1} + x_B^{\text{pure}})^2}{(x'_{A2} + x_B^{\text{pure}})^2},
\]

which holds if and only if \(x'_{A1} = x'_{A2} > 0\). Let \(x'_{Aj} = x_A'\) for \(j = 1, 2\), and \(k' = \frac{x_B^{\text{pure}}}{x_A}\), where \(k' > k^{\text{pure}}\) and \(k'\) solves the following condition

\[
\frac{(k')^2}{(1 + k')^2 x_B^{\text{pure}}} \left( \frac{1}{1 + k'} W + V \right) - c_A = 0.
\]

Then player A obtains the following profit when he chooses \((x'_A, x'_A)\):

\[
\pi_A(x'_A, x'_A) = \frac{1}{(1 + k')^2} \left( \frac{1 - k'}{1 + k'} W + 2V \right).
\]

It is clear that \(\pi_A^{\text{pure}} > \pi_A(x'_A, x'_A)\), where \(\pi_A^{\text{pure}}\) is player A’s equilibrium payoff given by (16), which follows from the fact that \(k' > k^{\text{pure}}\).

Thus, it is optimal for player A to choose \((x_A^{\text{pure}}, x_A^{\text{pure}})\) given player B’s equilibrium efforts \((x_B^{\text{pure}}, x_B^{\text{pure}})\).

Step 2: We show that it is optimal for player B to choose \((x_B^{\text{pure}}, x_B^{\text{pure}})\) given player A’s equilibrium efforts \((x_A^{\text{pure}}, x_A^{\text{pure}})\).

Provided that player A has chosen \((x_A^{\text{pure}}, x_A^{\text{pure}})\), player B’s payoff is

\[
\pi_B(x_{B1}, x_{B2}) = \frac{x_{B1}}{x_{B1} + x_A^{\text{pure}}} \left( \frac{x_{B2}}{x_{B2} + x_A^{\text{pure}}} \right) W + \frac{x_{B1}}{x_{B1} + x_A^{\text{pure}}} V + \frac{x_{B2}}{x_{B2} + x_A^{\text{pure}}} V - c_B x_{B1} - c_B x_{B2}.
\]

To show that \((x_B^{\text{pure}}, x_B^{\text{pure}})\) is a local maximum, it is clear that the following first-order conditions are satisfied when \((x_{B1}, x_{B2}) = (x_B^{\text{pure}}, x_B^{\text{pure}})\):

\[
\frac{x_A^{\text{pure}}}{(x_{B1} + x_A^{\text{pure}})^2} \left( \frac{x_{B2}}{x_{B2} + x_A^{\text{pure}}} W + V \right) - c_B = 0,
\]

\[
\frac{x_A^{\text{pure}}}{(x_{B2} + x_A^{\text{pure}})^2} \left( \frac{x_{B1}}{x_{B1} + x_A^{\text{pure}}} W + V \right) - c_B = 0.
\]
The second-order conditions are also satisfied, given that

\[
\frac{\partial^2 \pi_B}{\partial x_{B1}^2} \bigg|_{x_{B1} = x_{B2} = x_B^{pure}} = -\frac{2x_A^{pure}}{(x_B^{pure} + x_A^{pure})^3} \left( \frac{x_B^{pure}}{x_B^{pure} + x_A^{pure}} W + V \right) < 0,
\]

where the inequality follows from the fact that \(2(W + V)k^{pure} + (2V - W) > 0\). Note that this is clearly true when \(W \leq 2V\); when \(W > 2V\), this also holds given that \(k^{pure} \geq \frac{W - 2V}{W + 2V}\). Thus, \((x_B^{pure}, x_B^{pure})\) is indeed a local maximum.

We further need to show that there does not exist another local maximum, denoted by \((x'_B, x'^B)\), that yields a strictly higher payoff for player \(B\) than \(\pi_B^{pure}\). Following a similar logic as we have done for player \(A\), we just need to consider the following two cases: (i) \(x'_{B1} \geq x_B^{pure} \) and \(x'_{B2} \geq x_B^{pure}\), and (ii) \(0 < x'_{B1} < x_B^{pure}\) and \(0 < x'_{B2} < x_B^{pure}\).

In case (i), it can be shown that \((x_B^{pure}, x_B^{pure})\) is the unique local maximum. To prove this, it is sufficient to show that \(\pi_B\) is concave, which is true given that

\[
\frac{\partial^2 \pi_B}{\partial x_{B1}^2} \bigg|_{x_{B1} = x'_{B1} \geq x_B^{pure}, x_{B2} = x'_{B2} \geq x_B^{pure}} < 0,
\]

where the second-to-last inequality follows from the fact that \(x'_{Bj} \geq x_B^{pure}\) for \(j = 1, 2\).
In case (ii), following a similar logic as we have done for player $A$, we can show that if $(x'_{B1}, x'_{B2})$ is another local maximum, it must be the case that $x'_{B1} = x'_{B2} > 0$. Let $x'_B = x'_B$ for $j = 1, 2$, and $k'' = \frac{x'_B}{x''_A}$, where $0 < k'' < k''_{\text{pure}}$, and $k''$ solves

\[ \frac{1}{(1 + k'')^2 x''_A} \left( \frac{k''}{1 + k''} W + V \right) - c_B = 0. \]

Player $B$ obtains the following payoff when he chooses $(x'_B, x'_B)$:

\[ \pi_B(x'_B, x'_B) = \frac{(k'')^2}{(1 + k'')^2} \left( \frac{k''}{1 + k''} W + 2V \right), \]

It is clear that $\pi''_B > \pi_B(x'_B, x'_B)$, where $\pi''_B$ is player $B$’s equilibrium payoff given by (17), which follows from the fact that $k'' < k''_{\text{pure}}$.

Thus, it is optimal for player $B$ to choose $(x''_B, x''_B)$ given player $A$’s equilibrium efforts $(x''_A, x''_A)$.

In sum, it is the unique pure-strategy equilibrium in which player $A$ chooses $(x''_A, x''_A)$ and player $B$ chooses $(x''_B, x''_B)$.

### 5.3 Proof of Proposition 3

Consider a semi-pure equilibrium in which player $A$ chooses $(x_A, x_A)$, and player $B$ chooses $(x_B, x_B)$ with probability $p$ and remains inactive with probability $1 - p$, where $0 < p < 1$. It is clear that $x_{ij} > 0$ for any $i \in \{A, B\}$ and $j \in \{1, 2\}$, which implies that the following first-order conditions must hold:

\[ \frac{x_B}{(x_A + x_B)^2} \left( \frac{x_A}{x_A + x_B} pW + pV \right) - c_A = 0, \]

\[ \frac{x_B}{(x_A + x_B)^2} \left( \frac{x_A}{x_A + x_B} pW + pV \right) - c_A = 0, \]

\[ \frac{x_A}{(x_A + x_B)^2} \left( \frac{x_B}{x_A + x_B} W + V \right) - c_B = 0, \]

\[ \frac{x_A}{(x_A + x_B)^2} \left( \frac{x_B}{x_A + x_B} W + V \right) - c_B = 0. \]
Define $k_1 = \frac{x_{A1}}{x_{A1}} > 0$ and $k_2 = \frac{x_{A2}}{x_{A2}} > 0$. Then the first-order conditions can be written as

\[
\begin{align*}
k_1 &= \frac{\theta}{p} \left( \frac{W k_2 + V k_2 + V}{V k_2 + W + V} \right), \\
k_2 &= \frac{\theta}{p} \left( \frac{W k_1 + V k_1 + V}{V k_1 + W + V} \right), \\
x_{A1} &= \frac{k_1}{(1 + k_1)^2 c_A} \left( \frac{1}{1 + k_2} pW + pV \right), \\
x_{A2} &= \frac{k_2}{(1 + k_2)^2 c_A} \left( \frac{1}{1 + k_1} pW + pV \right).
\end{align*}
\]

Moreover, player $B$’s payoff being zero in equilibrium implies that

\[
\frac{x_{B1}}{x_{B1} + x_{A1}} \left( \frac{x_{B2}}{x_{B2} + x_{A2}} \right) W + \frac{x_{B1}}{x_{B1} + x_{A1}} V + \frac{x_{B2}}{x_{B2} + x_{A2}} V - c_B x_{B1} - c_B x_{B2} = 0,
\]

which can also be written as

\[
\frac{k_1 k_2 (k_1 k_2 - 1)}{(1 + k_1)^2 (1 + k_2)^2} W + \frac{(k_1)^2}{(1 + k_1)^2} V + \frac{(k_2)^2}{(1 + k_2)^2} V = 0.
\]

Therefore, it can be derived that there exists a unique solution to the set of first-order conditions and the condition that player $B$’s payoff is zero, such that $p = p^{semi}$, $k_1 = k_2 = k^{semi}$, $x_{A1} = x_{A2} = x^{semi}_A$ and $x_{B1} = x_{B2} = x^{semi}_B$, where $p^{semi}$, $k^{semi}$, $x^{semi}_A$ and $x^{semi}_B$ are given by Proposition 3. Note that when $\frac{W}{V} > \left( \sqrt{\frac{2 - \theta}{1 - \theta}} - 1 \right)$, we can derive that $0 < p^{semi} < 1$ and $k^{semi} < 1$. To show that the above strategy profile indeed constitutes an equilibrium, we also consider the following two steps.

Step 1: We show that it is optimal for player $A$ to choose $(x^{semi}_A, x^{semi}_A)$ given that player $B$ chooses $(x^{semi}_B, x^{semi}_B)$ with probability $p^{semi}$ and remains inactive with probability $1 - p^{semi}$.

Given player $B$’s strategy above, player $A$’s payoff is

\[
\pi_A(x_{A1}, x_{A2}) = \frac{x_{A1}}{x_{A1} + x^{semi}_B} \left( \frac{x^{semi}_A}{x_{A2} + x^{semi}_B} \right) p^{semi} W + \frac{x_{A1}}{x_{A1} + x^{semi}_B} p^{semi} V + \frac{x_{A2}}{x_{A2} + x^{semi}_B} p^{semi} V
\]

\[
+(1 - p^{semi})(W + 2V) - c_A x_{A1} - c_A x_{A2},
\]

for $x_{A1} > 0$ and $x_{A2} > 0$. Note that it is easy to rule out the case in which $x_{Aj} = 0$ for some $j$, where $j = 1, 2$. Following a similar logic as in the proof of Proposition 2, we can show $(x^{semi}_A, x^{semi}_A)$ maximizes $\pi_A$.

Step 2: We show it is optimal for player $B$ to choose $(x^{semi}_B, x^{semi}_B)$ with probability $p^{semi}$ and to stay inactive with probability $1 - p^{semi}$, given player $A$’s efforts $(x^{semi}_A, x^{semi}_A)$.
Provided that player A has chosen \((x^\text{semi}_A, x^\text{semi}_A)\), player B’s payoff is
\[
\pi_B(x_{B1}, x_{B2}) = \frac{x_{B1}}{x_{B1} + x^\text{semi}_A} \left( \frac{x_{B2}}{x_{B2} + x^\text{semi}_A} \right) W + \frac{x_{B1}}{x_{B1} + x^\text{semi}_A} \left( -c_B x_{B1} - c_B x_{B2} \right).
\]
Following a similar logic as in the proof of Proposition 2, we can show that \((x^\text{semi}_B, x^\text{semi}_B)\) is a global maximum for \(\pi_B\). Moreover, \((x_{B1}, x_{B2}) = (0, 0)\) is also a global maximum for \(\pi_B\), given that \(\pi_B(x^\text{semi}_B, x^\text{semi}_B) = \pi_B(0, 0)\). As a result, it is optimal for player B to randomize between \((x^\text{semi}_B, x^\text{semi}_B)\) and \((0, 0)\).

In sum, the semi-pure equilibrium characterized above indeed exists.

### 5.4 Proof of Lemma 2

By Corollary 1, we derive that when \(t \leq \bar{t}(\theta)\), a pure-strategy equilibrium exists, under which
\[
TE^\text{pure} = \frac{2\theta}{c_A(1 - \theta)^2 t} \left[ \sqrt{(1 - \theta)^2(t + 1)^2 + 4\theta} - (1 + \theta) \right] V,
\]
\[
TPE^\text{pure} = \frac{(1 - \theta)^2 t(t + 3) + (1 + \theta)^2 - (1 + \theta)\sqrt{(1 - \theta)^2(t + 1)^2 + 4\theta}}{(1 - \theta)^2 t} V.
\]
Given expected budget \(TPE^\text{pure} = \Gamma\), define \(TE^P = TE^\text{pure}\). Using the above two equations, we derive
\[
TE^P = \left( \frac{2\theta}{c_A} \right) \frac{\sqrt{(1 - \theta)^2(t + 1)^2 + 4\theta} - (1 + \theta)}{(1 - \theta)^2 t(t + 3) + (1 + \theta)^2 - (1 + \theta)\sqrt{(1 - \theta)^2(t + 1)^2 + 4\theta}} \Gamma. \quad (20)
\]

By Corollary 2, we derive that when \(t > \bar{t}(\theta)\), a semi-pure equilibrium exists, under which
\[
TE^\text{semi} = \frac{\theta[(1 + \theta)t + 4](t + 2)^2}{4c_A t(t + 4)} V,
\]
\[
TPE^\text{semi} = \frac{(t + 2)[2(t + 4) - \theta(t + 2)]}{2(t + 4)} V.
\]
Given expected budget \(TPE^\text{semi} = \Gamma\), define \(TE^S = TE^\text{semi}\). Using the above two equations, we derive
\[
TE^S = \left( \frac{\theta}{2c_A} \right) \frac{[(1 + \theta)t + 4](t + 2)}{t[2(t + 4) - \theta(t + 2)]} \Gamma. \quad (21)
\]

Thus, total effort \(TE\) can be expressed as \(\Gamma \times F(t)\), where \(F(t)\) is a function of \(t\) such that
\[
F(t) = \begin{cases} 
\frac{TE^P}{\Gamma}, & t \leq \bar{t}(\theta), \\
\frac{TE^S}{\Gamma}, & t > \bar{t}(\theta), 
\end{cases} \quad (22)
\]

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where $T E^p$ and $T E^s$ are given by (20) and (21), respectively. Given that both $T E^p$ and $T E^s$ are continuous in $t$ when $t \leq \bar{t}(\theta)$ and $t > \bar{t}(\theta)$, respectively, and $T E^p = T E^s$ when $t = \bar{t}(\theta)$, $F(t)$ is also continuous in $t$.

5.5 Proof of Proposition 4

To work out the optimal prize structure that maximizes $T E$ given any expected budget $\Gamma > 0$, we just need to derive the optimal $t$ that maximizes $F(t)$.

First consider $t < \bar{t}(\theta)$. We can derive that

$$F'(t) = \frac{2\theta(1-\theta)^2}{c A AB^2} \left[ (1+\theta)(2t+3)A - (1-\theta)^2t^3 - 3(1-\theta)^2t^2 - (5\theta^2 - 2\theta + 5)t - 3(1+\theta)^2 \right],$$

where

$$A = \sqrt{(1-\theta)^2(t+1)^2 + 4\theta},$$

$$B = (1-\theta)^2t(t+3) + (1+\theta)^2 - (1+\theta)A.$$

Then we can further derive that $F'(t) > 0$ if and only if $G(t) < 0$, where

$$G(t) = (1-\theta)^2t^4 + 6(1-\theta)^2t^3 + 15(1-\theta)^2t^2 + 8(1-2\theta)(2-\theta)t + 6(\theta^2 - 4\theta + 1).$$

Note that when $\theta^2 - 4\theta + 1 \geq 0$, i.e., $\theta \leq \overline{\theta}$, where $\overline{\theta} = 2 - \sqrt{3} \approx 0.2679$, we always have $G(t) > 0$ for any $t > 0$, which implies that $F'(t) < 0$ for any $0 < t < \bar{t}(\theta)$. Next consider the case in which $\theta > \overline{\theta}$. Given the above expression of $G(t)$, we derive

$$G'(t) = 4(1-\theta)^2t^3 + 18(1-\theta)^2t^2 + 30(1-\theta)^2t + 8(1-2\theta)(2-\theta).$$

Then when $\overline{\theta} < \theta \leq 1/2$, $G'(t) > 0$ for any $t > 0$; when $1/2 < \theta < 1$, we know there exists a unique threshold of $t$, denoted by $t_0 > 0$, such that $G'(t) < 0$ if $t < t_0$, and $G'(t) > 0$ if $t > t_0$. Therefore, given that $G(0) < 0$ and $G(t) > 0$ when $t$ is sufficiently large, it is clear that there always exists a unique $t > 0$, denoted by $\bar{t}(\theta)$, such that $G(t) < 0$ for $t < \bar{t}(\theta)$ and $G(t) > 0$ for $t > \bar{t}(\theta)$, where $\bar{t}(\theta)$ solves $G(\bar{t}(\theta)) = 0$. It can be shown that $\bar{t}(\theta)$ is strictly increasing in $\theta$. To show this, we rewrite $G(t)$ as $\Psi(t, \theta)$. Then by the definition of $\bar{t}(\theta)$, we get that $\Psi(\bar{t}(\theta), \theta) = 0$. Total differentiation of $\Psi$ yields

$$\frac{\partial \Psi}{\partial \theta} + \frac{\partial \Psi}{\partial \bar{t}} \frac{d \bar{t}}{d \theta} = 0,$$
which implies that \( \frac{d\tilde{t}}{dt} = -\frac{\partial \Psi}{\partial \theta} / \frac{\partial \Psi}{\partial t} \). First, \( \frac{\partial \Psi}{\partial \theta} < 0 \) for any \( t > 0 \). Next, we want to show \( \frac{\partial \Psi}{\partial t} > 0 \) for any \( t = \tilde{t}(\theta) \). Clearly, this is true for any \( \theta \leq 1/2 \). For \( \theta > 1/2 \), we get that

\[
\frac{\partial \Psi}{\partial t} \bigg|_{t=\tilde{t}(\theta)} = (1 - \theta)^2(4t^3 + 18t^2 + 30t) + 8(1 - 2\theta)(2 - \theta) > \frac{(1 - \theta)^2}{t}(t^4 + 6t^3 + 15t^2) + 8(1 - 2\theta)(2 - \theta) = -\frac{6(\theta^2 - 4\theta + 1)}{t} > 0,
\]

where the second equality follows from the definition of \( \tilde{t}(\theta) \) and the second inequality follows from \( \theta > 1/2 \). Thus, we have shown that \( \frac{d\tilde{t}}{dt} > 0 \), so \( \tilde{t}(\theta) \) is strictly increasing in \( \theta \). We can further derive the inverse function of \( \tilde{t}(\theta) \), denoted by \( \tilde{t}^{-1}(t) \), where

\[
\tilde{t}^{-1}(t) = 1 + \frac{6 + 4t - 2(t^2 + 3t + 3)\sqrt{3 + 2t}}{t^4 + 6t^3 + 15t^2 + 16t + 6}.
\]

(23)

Given that \( \tilde{t}(\theta) \) is also increasing in \( \theta \), we can also derive the inverse function of \( \tilde{t}(\theta) \), denoted by \( \tilde{t}^{-1}(t) \), where \( \tilde{t}^{-1}(t) = \frac{(t - 2)(t + 4)}{(2t + 3)(t^2 + 2t)} \). Then it is clear that \( \tilde{t}(\theta) > \tilde{t}(\theta) \) if and only if \( \tilde{t}^{-1}(t) < \tilde{t}^{-1}(t) \). Note that

\[
\tilde{t}^{-1}(t) - \tilde{t}^{-1}(t) = \frac{8t^4 + 52t^3 + 134t^2 + 140t + 48 - 2(t^2 + 2t)(t^2 + 2t)(t^2 + 3t + 3)\sqrt{3 + 2t}}{(t^4 + 6t^3 + 15t^2 + 16t + 6)(t^4 + 6t^3 + 15t^2 + 16t + 6)}
\]

which is positive if and only if the numerator is positive, i.e.,

\[
H(t) = (8t^4 + 52t^3 + 134t^2 + 140t + 48)^2 - [2(t^2 + 2t)(t^2 + 3t + 3)\sqrt{3 + 2t}]^2 > 0.
\]

It can be derived that

\[
H(t) = 4(1 + t)^2(12 + 8t - t^2)(6 + 4t + t^2)(8 + 20t + 11t^2 + 2t^3),
\]

which is positive if and only if \( 12 + 8t - t^2 > 0 \), i.e., \( t < t_1 \), where \( t_1 = 4 + 2\sqrt{7} \). Then we know that \( \tilde{t}(\theta) > \tilde{t}(\theta) \) if and only if \( t < t_1 \), or \( \theta < \theta_1 = \tilde{t}^{-1}(t_1) \approx 0.9237 \).

We summarize the monotonicity of \( F(t) \) for \( t < \tilde{t}(\theta) \): When \( \theta \leq \theta_1 \), where \( \theta_1 \approx 0.2679 \), \( F'(t) < 0 \) for any \( t < \tilde{t}(\theta) \); when \( \theta < \theta < \theta_1 \), where \( \theta_1 \approx 0.9237 \), \( F'(t) > 0 \) for \( t < \tilde{t}(\theta) \) and \( F'(t) < 0 \) when \( \tilde{t}(\theta) < t < \tilde{t}(\theta) \); when \( \theta \geq \theta_1 \), \( F'(t) > 0 \) for any \( t < \tilde{t}(\theta) \).

Next consider the case with \( t > \tilde{t}(\theta) \).\(^{11}\) We can derive that

\[
F'(t) = \left( \frac{\theta}{2c_A} \right) \frac{4[(2\theta - 1)t^2 - 4(2 - \theta)t - 4(4 - \theta)]}{[(2 - \theta)t^2 + 2(4 - \theta)t]^2}.
\]

\(^{11}\)It can be checked that \( F'(t) \) is not continuous in \( t \) when \( t = \tilde{t}(\theta) \), which implies that \( F(t) \) is not differentiable in \( t \) when \( t = \tilde{t}(\theta) \).
Clearly, when \( \theta \leq 1/2 \), \( F'(t) < 0 \) for any \( t \); when \( \theta > 1/2 \), we have \( F'(t) < 0 \) if and only if \( t < \hat{t}(\theta) \). Note that \( \hat{t}(\theta) \) is strictly decreasing in \( \theta \) when \( \theta > 1/2 \). Given that \( \tilde{t}(\theta) \) is increasing in \( \theta \), we know that there is a unique \( \theta \), denoted by \( \theta_2 \), that solves \( \tilde{t}(\theta) = \hat{t}(\theta) \). It can be derived that \( \theta_2 = 5 - 2^{\frac{3}{2}} - 2^{\frac{5}{2}} \approx 0.8928 \). Moreover, we have \( \tilde{t}(\theta) > \hat{t}(\theta) \) if and only if \( \theta > \theta_2 \).

We also summarize the monotonicity of \( F(t) \) when \( t > \tilde{t}(\theta) \): When \( \theta \leq 1/2 \), \( F'(t) < 0 \) for any \( t > \tilde{t}(\theta) \); when \( 1/2 < \theta < \theta_2 \), \( F'(t) < 0 \) for \( \tilde{t}(\theta) < t < \hat{t}(\theta) \), and \( F''(t) > 0 \) for \( t > \hat{\theta} \); when \( \theta \geq \theta_2 \), \( F'(t) > 0 \) for \( t > \tilde{t}(\theta) \).

Combining the above two cases, the following general results on the relationship between \( F(t) \) and \( t \in [0, +\infty) \) can be obtained: When \( \theta \leq 2 - \sqrt{3} \), \( F(t) \) is always decreasing in \( t \); when \( 2 - \sqrt{3} < \theta \leq 1/2 \), \( F(t) \) is increasing in \( t \) for \( t < \tilde{t}(\theta) \), and decreasing in \( t \) afterward; when \( 1/2 < \theta < \theta_1 \), \( F(t) \) is increasing in \( t \) for \( t < \tilde{t}(\theta) \), then decreasing in \( t \), and finally increasing in \( t \);\(^{12}\) when \( \theta \geq \theta_1 \), \( F(t) \) is always increasing in \( t \).

Given the monotonicity of \( F(t) \), we can show the optimal prize structure: When \( \theta \leq \theta_1 \), \( F(t) \) is maximized at \( t = 0 \) and it is optimal to set \( W = 0 \) and \( V > 0 \); when \( \theta_1 < \theta < 1/2 \), \( F(t) \) is maximized at \( t = \tilde{t}(\theta) > 0 \) and it is optimal to set \( W > 0 \) and \( V > 0 \); when \( 1/2 < \theta < \theta_1 \), \( F(t) \) is maximized at \( t = \tilde{t}(\theta) \) or \( t = +\infty \),\(^{13}\) and it is optimal to set \( W > 0 \) and \( V \geq 0 \); when \( \theta \geq \theta_1 \), \( F(t) \) is maximized when \( t = +\infty \), and it is optimal to set \( W > 0 \) and \( V = 0 \).

The optimal prize structure has been identified for any \( \theta \) except when \( 1/2 < \theta < \theta_1 \), in which case we need to compare two local maxima \( t = \tilde{t}(\theta) \) and \( t = +\infty \). Using \( \lim_{t \to +\infty} F(t) = \frac{\theta(1+\theta)}{2(2-\theta)} \left( \frac{1}{t_4} \right) \) and (20), we derive that \( \lim_{t \to +\infty} F(t) < F(\tilde{t}(\theta)) \) if and only if

\[
\frac{\sqrt{(1-\theta)^2(t+1)^2 + 4\theta} - (1+\theta)}{(1-\theta)^2(t+3) + (1+\theta)^2 - (1+\theta)\sqrt{(1-\theta)^2(t+1)^2 + 4\theta}} > \frac{(1+\theta)}{4(2-\theta)}, \tag{24}
\]

where \( \theta = \tilde{t}^{-1}(t) \), which is given by (23). Substituting (23) into (24), we can show that (24) holds if and only if

\[
\frac{9 + 24a + 15a^2 + 3a^4 + 8a^5 - 3a^6}{(a^4 + 12a + 3)(a^4 + 6a^2 - 3)} > 0, \tag{25}
\]

where \( a = \sqrt{3 + 2\tilde{t}} \) and \( a \geq \sqrt{3} \) given that \( t \geq 0 \). It is clear that for any \( a \geq \sqrt{3} \), the denominator of (25) is always positive, which implies that (25) is positive if and only if the numerator is positive. Define \( h(a) = 9 + 24a + 15a^2 + 3a^4 + 8a^5 - 3a^6 \). First, it is clear that

\(^{12}\)When \( \frac{1}{2} < \theta < \theta_2 \), \( F(t) \) is increasing in \( t \) for \( t < \tilde{t}(\theta) \), decreasing in \( t \) for \( \tilde{t}(\theta) < t < t' \), where \( t' > \tilde{t}(\theta) \), and increasing in \( t \) afterward; when \( \theta_2 \leq \theta < \theta_1 \), \( F(t) \) is increasing in \( t \) for \( t < \tilde{t}(\theta) \), decreasing in \( t \) for \( \tilde{t}(\theta) < t < \hat{t}(\theta) \), and increasing in \( t \) afterward.

\(^{13}\)Note that \( t = +\infty \) corresponds to the case with \( W > 0 \) and \( V = 0 \).
$h(a) > 0$ when $\sqrt{3} \leq a \leq 3$, since

$$h(a) \geq 9 + 24a + 15a^2 + 3a^4 - a^5 \geq 9 + 24a + 15a^2 > 0.$$ 

Note that $h(a)$ is strictly decreasing in $a$ when $a > 3$ due to the fact that

$$h'(a) = 24 + 30a + 12a^3 + 40a^4 - 18a^5 < 24 + 30a + 12a^3 - 14a^4$$

$$< 24 + 30a - 30a^3 < 24 - 240a < 0.$$ 

Given that $h(a) < 0$ when $a$ is sufficiently large, it is clear that there exists a unique $a$, denoted by $\tilde{a}$, such that $h(a) > 0$ if and only if $a < \tilde{a}$, where $\tilde{a} \approx 3.2126$. Using $a = \sqrt{3} + 2t$, we obtain that (24) holds when $t < \tilde{t}$, where $\tilde{t} \approx 3.6604$. Using (23), we can further derive that (24) holds when $\theta < \tilde{\theta}$, where $\tilde{\theta} \approx 0.7900$. Thus, when $1/2 < \theta < \tilde{\theta}$, it is optimal to set $W > 0$ and $V > 0$; when $\tilde{\theta} \leq \theta < \theta_1$, it is optimal to set $W > 0$ and $V = 0$. 

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References


