Optimal Orchestration of Rewards and Punishments in Rank-Order Contests

Bin Liu† Jingfeng Lu‡

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Abstract

We allow negative prizes and investigate effort-maximizing prize design in rank-order contests. Endogenous participation arises due to less-efficient types’ incentive to avoid punishments. If prize sequences are independent of the number of entrants, the optimum features winner-take-all for the best performer coupled with a potential punishment for the worst performer only when all contestants participate. If prize sequences can be contingent on the number of entrants, the optimum integrates winner-take-all and egalitarian rule: When all contestans participate, all non-top performers are penalized by a budget-augmenting negative prize and the top performer collects the augmented prize budget; otherwise, an egalitarian rule applies.

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1 Introduction

The practice of utilizing “sticks” jointly with “carrots” in incentivizing agents has long and widely been observed. In a contest environment, a positive prize can be viewed as a carrot and a negative prize can be viewed as a stick. Prize allocation has long been recognized as a main instrument to incentivize contestants to exert productive effort. However, despite the fact that negative prizes

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†Bin Liu: School of Management and Economics and Shenzhen Finance Institute, The Chinese University of Hong Kong, Shenzhen (CUHK-Shenzhen), China 518172. Email: binliu@cuhk.edu.cn.
‡Jingfeng Lu: Department of Economics, National University of Singapore, Singapore 117570. Email: ecsljf@nus.edu.sg.
(sticks) are widely adopted in practice and they are a powerful tool in eliciting effort supply,\(^1\) a vast majority of the prize design literature has been focusing on analyzing positive prizes (carrots), with only a few exceptions. A celebrated result is established by Lazear and Rosen [8], who show that rank-order contests with negative prizes can achieve the first best in an environment of complete information. In environments with incomplete information, however, introducing negative prizes into the analysis inevitably entails the issue of endogenous entry of contestants,\(^2\) which would often make the analysis cumbersome and lack of tractability and elegance.

A complete characterization of the optimal rank-based prize design while allowing negative prizes in all-pay contests with incomplete information has remained an open question since Moldovanu and Sela’s [12] seminal work, which largely focuses on positive prizes. Nevertheless, several studies have made important progress by considering special prize structures. Moldovanu, Sela, and Shi [15] study both cases with exogenous and endogenous entry. For the case of endogenous entry, they assume a single positive prize for the top performer and a fixed number of uniform negative prizes for the bottom entrants.\(^3\) Thomas and Wang [20] and Kamijo [7] focus on a single positive prize for the top performer and a single negative prize for the worst performer among all entrants in their analysis. Hammond et al. [6] study prize-augmenting entry fees, which can be viewed as uniform negative prizes. The goal of this paper is to provide a complete and elegant answer to this important question by coming up with a tractable procedure and fully characterizing the optimal design of positive prizes and negative prizes in rank-order contests. In our study, we focus on an environment of incomplete information all-pay auction with linear bidding cost, which is a well adopted workhorse model in the contest literature.

Our baseline analysis essentially allows prizes to be negative in the incomplete information all-pay auction model of Moldovanu and Sela [12]. Each contestant is endowed with his private type, which is his marginal cost of exerting effort. Negative prizes in general lead to endogenous entry of contestants because of the participation constraint, so the actual number of entrants can be endogenous. The organizer, who has a fixed budget, designs a prize structure \(v = (v_1, \ldots, v_N)\) with \(v_1 \geq v_2 \geq \cdots \geq v_N\) to maximize the total effort from \(N\) contestants, where \(v_n\), which can be nonnegative or negative, is the prize for the participant with the \(n\)th highest effort. A nonparticipant receives no prize. The organizer must observe ex post budget constraints for all possible numbers of entrants, and she is allowed to value any leftover budget.\(^4\)

\(^1\)Negative prizes (punishments) in contests are prevalent in practice. As noted by Liu et al. [11] and Hammond et al. [6], the widely used entry fees in contests are negative prizes—for example, a contestant, who pays 1 dollar entry fee and wins nothing from the contest, effectively gets \(-1\) dollar from the contest ex post. Examples of punishments in contests also include, for instance, firings or demotion in internal labor markets, an F grade for students, relegation in sports competitions, among others.


\(^3\)They assume that if there is a single entrant, then he gets both the positive prize and a negative prize.

\(^4\)In our baseline analysis with non-contingent prizes, ex post budget constraints mean that negative prizes collected cannot be used to augment the positive prizes.
The equilibrium endogenous participation induced by a prize structure is a threshold-entry, in which only contestants with types higher than a certain threshold enter the contest. We first identify the equilibrium bidding function and entry threshold for any given prize structure. Based on these, we find that the organizer’s problem, for any fixed entry threshold, is essentially a linear programming, in the sense that all of the functions in the problem—the objective function, the budget constraint, and the participation constraint—are linear in the \( N \) prizes. Nevertheless, coefficients in this linear programming problem vary in a highly intractable and nonlinear way as the induced entry threshold changes, which makes solving the problem challenging.

A typical procedure is to first identify the constrained optimum among all feasible prize structures that induce the same entry threshold, and then vary across all possible entry thresholds in the type space to pin down the universal optimum. However, this approach is rather infeasible because of the aforementioned arbitrary and intractable behavior of the coefficients in the linear programming problem, which generates complicacy in characterizing the optimum for a fixed entry threshold. We illustrate that the number of positive prizes and the number of negative prizes at the optimum for a fixed entry threshold can vary with the threshold. In particular, in general, multiple positive prizes and multiple negative prizes can arise as the optimum for fixed thresholds. Such arbitrariness in the number of prizes constitutes the main challenge of pinning down the optimal design.

We manage to tackle this difficulty by developing an innovative procedure. A first key step relies on discovering useful relations among positive coefficients of prizes in the total effort function. This is made possible by observing a hazard rate dominance result of relevant order statistics. Specifically, these positive coefficients must be associated with higher prizes, and the ratios of these coefficients to their counterparts in the equilibrium entry condition are decreasing in ranks. Moreover, more coefficients become positive as the entry threshold gets higher. A second key step shows that when an entry threshold is sufficiently high such that all coefficients become positive, then the corresponding optimal prize structure can have only one negative prize. A third key step shows that for any fixed prize structure, we can construct an alternative prize structure by varying only the last prize, which can induce the minimal entry threshold \( t^* \) that makes all coefficients in the total effort function positive. Therefore, the entry threshold \( t^* \) must be optimal and there can be only one negative prize at the optimum. A final step establishes that a single positive prize equal to the whole initial budget is universally optimal, relying on the key relations among ratios of coefficients in the objective function and the equilibrium entry condition, which is discovered in step one.

The optimal prize structure thus takes an elegant form of \( \mathbf{v}^* = (v_1^*, 0, \ldots, 0, v_N^*) \): A single positive prize \( v_1^* > 0 \) equal to the budget and a single negative prize \( v_N^* \leq 0 \) that supports the optimal entry threshold \( t^* \). In other words, the best performer always receives all the money and
the worst performer is punished only when all \( N \) contestants enter the contest. We find that the opposite of the coefficient of \( v_N \) in the objective function can be interpreted as the marginal revenue gained by imposing an extra unit of punishment through the last negative prize. Therefore, the optimum is achieved when this marginal revenue is precisely zero. Based on this observation, we further provide a necessary and sufficient condition for the optimality of pure winner-take-all (i.e., no negative prize).

Since the number of entrants is endogenous, a further question naturally raises: What would the optimal design be when the prize allocation rule can be contingent on the number of entrants? We also fully pin down the optimum for this case. A key feature is that the optimal prize allocation rule must induce endogenous entry by excluding less efficient types. The prizes also vary dramatically upon the number of entrants. When everyone participates, all non-top performers are penalized by a finite uniform budget-augmenting negative prize and the top performer collects all available prize budget; whenever the entry is partial, an egalitarian rule of sharing the initial budget applies to entrants. The distribution of more efficient types depends on the number of entrants. The optimal prize design provides an enhanced incentive for more efficient types, when the entry is full and thus it is more likely that there are more efficient contestants participating. When the number of entrants is low, the egalitarian rule ensures that entrants enjoy more surplus, which functions to maintain their participation incentive. The optimal prize design can be implemented by a winner-take-all contest with an appropriately set entry fee. With full entry, the best performer is rewarded the initial budget and all entry fees collected; with partial entry, the contest is cancelled and all entrants share the available budget equally regardless of their effort.

Our baseline analysis of the non-contingent prize structure addresses the open question of optimal prize design when negative prizes are introduced in Moldovanu and Sela’s [12] setting. In the literature of rank-order contests with incomplete information, to the best of our knowledge, only Moldovanu, Sela, and Shi [15], Thomas and Wang [20], Kamijo [7], and Hammond et al. [6] study this question. However, as mentioned above, all these papers focus on specific prize allocation rules and thus have not yet fully addressed the optimal design. Our optimal non-contingent design reveals that there is no loss of generality by assuming a single positive prize that equals the initial budget, which has been adopted by Moldovanu, Sela, and Shi [15], Thomas and Wang [20], and Kamijo [7]. However, our findings reveal that the restrictions they impose on negative prizes are indeed restrictive. Differing from these studies, our paper allows full flexibility on the specification of prizes in both non-contingent and contingent prize structures. Moreover, in our analysis of the contingent prize design, a negative prize can be used to augment the prize budget, as in Fullerton and McAfee [4], Liu et al. [11], and Hammond et al. [6].

In a closely related paper, Liu et al. [11] adopt a mechanism design approach to investigate the same question as our paper. They find that imposing an arbitrarily high entry fee and a minimum
bid can extract nearly all surplus from contestants and achieve a level of total effort that is inducible in an environment in which all contestants are of the most efficient type with certainty. In their optimal contest, all collected entry fees, together with the initial budget, are awarded to the highest bidder if the bid is above the minimum bid. If everyone bids zero, the initial budget and collected entry fees are randomly allocated to each contestant to maintain the entry incentive of all types. Therefore, their optimal design crucially relies on the information on the level of bids, which is exactly the key difference between their model and ours—they allow information on the level of contestants’ performance to be used in the mechanism, which is not feasible in rank-order contests. As such, all types of contestants enter the contest at the optimum, so their analysis does not involve the issue of endogenous entry.

Moldovanu and Sela [12] conjecture (in their Section IV) that the effort level of Myerson’s optimal seller revenue should be implementable by charging an appropriate entry fee while maintaining a single prize for the highest bidder among entrants, which equals the organizer’s initial prize budget. Since their prize design with entry fees is a non-contingent prize structure, our analysis of the non-contingent prize structure reveals that their prize design with entry fees is generally suboptimal, unless the optimal entry fee is zero. Moreover, our investigation of the optimal contingent prize structure indicates that the entry fees in Moldovanu and Sela [12] can be better employed to top up the initial budget to reward top performers, as in Hammond et al. [6]. However, the rule of Hammond et al. [6] can be further improved. In particular, when there is no full entry, a more lenient prize allocation rule should be adopted: All entrants should be treated equally.

Our paper contributes to the literature on optimal prize allocation in all-pay auctions with incomplete information. In addition to the pioneering work of Moldovanu and Sela [12] and the papers mentioned above, other important contributions include the investigation of a two-stage all-pay auction framework (Moldovanu and Sela [13]), the environment in which contestants care about their relative status (Moldovanu, Sela, and Shi [14]), endogenous contest success functions (Polishchuk and Tonis [18]), the analysis of optimal crowdsourcing contests (Chawla, Hartline, and Sivan [1]), innovation contests (Erkal and Xiao [2]), contests with entry costs (Liu and Lu [10]), and large contests (Olszewski and Siegel [17]). All of these papers assume positive prizes.

Our paper is also related to the literature on prize design in all-pay contests with complete information. Recently, Ghosh and Hummel [5] introduce cardinal information on contestants’ performance to rank-order contests; Xiao [21] studies ability grouping and characterizes the optimal prize structure; Fang, Noe, and Strack [3] find that more unequal prize structures lead to lower effort provision, when contestants are symmetric and the effort cost is convex; Letina, Liu, and Netzer [9] provide a general approach to study the contest design when the organizer can choose both the prize profile and the contest success function.

The rest of the paper is organized as follows. In Section 2, we set up the model. Section 3
presents the equilibrium analysis. We provide the analysis of non-contingent optimal design with negative prizes in Section 4. Section 5 further investigates the contingent prize structures, and Section 6 concludes. Technical proofs are relegated to the appendix.

2 The Model

A risk-neutral contest organizer has a fixed budget $V(>0)$ to elicit effort from $N (\geq 2)$ risk-neutral potential contestants. For contestant $i$, his cost of exerting effort $e_i(\geq 0)$ is $e_i/t_i$, where $t_i$ is his private information. We assume that $t_i$’s are independently and identically distributed with cumulative distribution function $F(\cdot)$ on the support $[a,b]$ with $a > 0$. The corresponding probability density function $f(\cdot)$ is strictly positive on $[a,b]$.

A prize structure is an $N$-vector $v = (v_1, v_2, \ldots, v_N)$ with $v_1 \geq v_2 \geq \cdots \geq v_N$, where $v_n$ is the prize for the contestant with the $n$th highest effort. A key feature is that, similar to Liu et al. [11], prizes can be negative. The possibility of ending up with negative prizes would, in general, result in endogenous entry of contestants. Throughout the paper, we use scenario $n$ to refer to the situation that $n$ entrants participate in the competition. The prize allocation rule goes as follows: When there are $n$ entrants, they win the respective first $n$ prizes $v_1, v_2, \ldots, v_n$ from $v$ according to their performance—that is, the highest entrant wins $v_1$, the second highest entrant wins $v_2$, ... , and the last entrant wins $v_n$; ties are broken randomly and fairly. The budget constraint requires that $\sum_{i=1}^{n} v_i \leq V$ for any $n = 1, 2, \ldots, N$.

A participant’s payoff is equal to the prize he receives minus the cost of exerting effort; if a contestant does not enter the contest, he receives his outside option, which is normalized as 0. The contest organizer’s goal is to design a prize structure to maximize the expected total effort using her budget; at the same time, if there is money left in the budget, she values that money as well. Assume that there is a linear relationship between effort and money: 1 dollar is worth $t_0 (\geq 0)$ units of effort for the organizer. Note that the cost of 1 unit of effort for the maximum ability (b) type is $1/b$, which needs to be less than $1/t_0$; otherwise, it is optimal for the organizer not to spend any of the prize budget. Therefore, we assume that $t_0 < b$.

We call the organizer’s benefit from the leftover budget effort-equivalent. Thus, her goal is to maximize the sum of the expected total effort induced and the effort-equivalent of the leftover budget. For simplicity and clarity, we will just say that the organizer maximizes the expected overall effort, which covers these two components.

$F(\cdot)$, $t_0$, and $N$ are public information. The timing of the game is as follows.

Time 0: Each contestant privately learns his type.

Time 1: The organizer chooses $v$ and commits to it. The prize structure $v$ is announced.
Time 2: All potential contestants decide whether to participate in the contest. If a contestant decides to enter, he observes the number of rival(s) and then exerts effort.\(^5\)

Time 3: The contest rule is implemented according to the one announced at time 1.

As a special case of our model setup, the prize structure \(v = (V - E, -E, \ldots, -E)\) corresponds to the discussion of using an entry fee \(E \geq 0\) to screen contestants in Section IV of Moldovanu and Sela [12]: That is, the organizer charges an entry fee \(E\) from every entrant and all prize budget \(V\) is awarded to the highest bidder of all entrants. Our formulation of \(t_0\) captures the organizer’s payoff from entry fees collected in this case. In fact, suppose that there are \(n\) entrants, then the leftover budget is \(V - (V - nE) = nE\), which is precisely the amount of entry fees collected and generates \(t_0nE\) units of effort for the organizer.

Define \(J(t) = t - \frac{1 - F(t)}{f(t)}\). We make the following standard assumption, as in the mechanism design literature.

**Assumption 1** (Regularity). \(J(t)\) is strictly increasing in \(t\).

Since entry is generally endogenous, the analysis of the optimal design can be complicated. We tackle the problem in the following way: We first characterize the symmetric monotone bidding equilibrium for any given prize structure. We then characterize several important features of the problem for any fixed entry threshold. Finally, we pin down the optimal entry threshold and thus the optimal prize structure.

### Equilibrium Analysis

As a first step, we characterize the entry and bidding equilibrium for any given prize structure \(v\), which forms the base for the search of the optimal prize structure. We focus on symmetric monotone bidding equilibria in our analysis.

#### 3.1 Equilibrium Entry

A given prize structure \(v\) induces an entry threshold \(t^e\) such that types higher than \(t^e\) enter with probability one, while types lower than it enter with probability zero. It is clear that the threshold type must bid zero at equilibrium and enjoys a zero expected payoff.\(^6\) Note that type \(t^e\) obtains the lowest prize \(v_n\) for sure in scenario \(n\) for any \(n\) in any monotone bidding equilibrium. Therefore, \(t^e\)

\(^5\)It does not matter whether the entrant knows the number of rival(s) when exerting effort, because both the organizer and contestants are risk neutral. See Remark 1.

\(^6\)The threshold type may enjoy a positive payoff when the prize structure induces full entry. However, this is clearly not optimal. See the discussion after Lemma 1.
is characterized by

$$
\sum_{n=1}^{N} p_n(t^c)v_n = 0, \quad (1)
$$

where $p_n(t^c) = \binom{N-1}{n-1}(1 - F(t^c))^{n-1}F^{N-n}(t^c)$ is the probability that there are $n - 1$ rival(s) from an entrant’s perspective.

There are two special cases of equilibrium entry, which are full entry ($t^c = a$) and no entry ($t^c = b$). If the prize structure induces full entry, it must be the case that $v_N \geq 0$, so $v_1 \geq \ldots \geq v_N \geq 0$—i.e., all prizes are nonnegative. This full-entry (with nonnegative prizes) case is studied in Moldovanu and Sela [12]. If the prize structure induces no entry, then this means that the organizer cancels the contest and enjoys a $t_0V$ overall effort. In general, when the entry is stochastic—i.e., the prize structure $v$ induces a threshold $t^c \in (a,b)$—the induced entry threshold must be unique. The following result provides details about these. (All proofs are relegated to the Appendix.)

**Lemma 1.** For any prize structure $v$, it induces a unique entry threshold, which can be full entry, no entry, or a unique interior threshold $t^c \in (a,b)$; more specifically, it induces: (i) full entry if and only if $v_N \geq 0$; (ii) no entry if and only if $v_1 \leq 0$; (iii) a unique interior threshold if and only if $v_1 > 0$ and $v_N < 0$.

Notice that when the prize structure $v$ induces an interior entry threshold, (1) must hold, as the threshold type must enjoy a zero expected payoff. However, when $v$ induces full entry, (1) may not hold because the threshold type, type $a$, may enjoy a strictly positive payoff—i.e., $\sum_{n=1}^{N} p_n(a)v_n > 0$. However, this means that $v_N > 0$, which is obviously suboptimal, as the organizer can induce a strictly higher expected overall effort by reducing $v_N$ by $\varepsilon = v_N$ and increasing $v_1$ by $\varepsilon$, which still induces full entry.\(^7\) Therefore, whenever the prize structure induces entry (full entry or an interior entry threshold), there is no loss of generality to assume that (1) holds.

### 3.2 Scenario Bidding Functions and The Overall Effort

For any given prize structure $v$, suppose that it induces an entry threshold $t^c \in [a,b]$ determined by (1). An entrant knows that his rivals’ types are independently drawn from the truncated cumulative distribution function $G(t, t^c) = \frac{F(t) - F(t^c)}{1 - F(t^c)}$, with density function $g(t, t^c) = \frac{f(t)}{1 - F(t^c)}$, $t \in [t^c, b]$.

Suppose that there are $n$ entrants. Lemma 1 in Liu and Lu [10] characterizes the entrant’s bidding function, given prizes $(v_1, v_2, \ldots, v_n)$ in scenario $n$. We restate it below. Note that the difference is that in this paper, the organizer values the leftover budget, which does not affect the

\(^7\)This argument of the full-entry case follows from the analysis in Moldovanu and Sela [12], because all $N$ prizes must be nonnegative.
equilibrium analysis for given prizes. However, the expected overall effort takes a different form, since we introduce $t_0$ in the analysis.

**Proposition 1.** Suppose that the prize structure $v$ induces the (unique) threshold $t^c$. In scenario $n \geq 1$ with prizes $(v_1, v_2, \ldots, v_n)$,

(i) The unique symmetric monotone bidding function $e^{(n)}(t, v, t^c)$ for type $t \in [t^c, b]$ is

$$e^{(n)}(t, v, t^c) = tV^{(n)}(t) - \int_{t^c}^{t} V^{(n)}(s)ds - t^c v_n,$$

where

$$V^{(n)}(t) = \sum_{j=1}^{n} v_{n+1-j} \left( \frac{n - 1}{j - 1} \right) G^{j-1}(t, t^c)(1 - G(t, t^c))^{n-j}$$

is the expected prize an entrant with type $t$ obtains.

(ii) The corresponding scenario-$n$ expected overall effort is

$$TE^{(n)}(v, t^c) = n \int_{t^c}^{b} J(t)V^{(n)}(t)g(t, t^c)dt - nt^c v_n + t_0(V - \sum_{j=1}^{n} v_j).$$

In scenario-0, no one enters, so the organizer’s payoff is simply the benefit from her budget. Thus, the scenario-0 expected overall effort is $TE^{(0)}(v, t^c) = t_0 V$.

The expected overall effort is simply the weighted average of scenario-$n$ expected overall effort across all scenarios, which is summarized below.

**Lemma 2.** Suppose that $v = (v_1, \ldots, v_N)$ with the induced entry threshold $t^c \in [a, b)$, then the expected overall effort is

$$TE(v, t^c) = \sum_{n=0}^{N} \binom{N}{n} (1 - F(t^c))^n F^{N-n}(t^c)TE^{(n)}(v, t^c),$$

where $TE^{(n)}(v, t^c)$ is given in Proposition 1 when $n \geq 1$, and $TE^{(0)}(v, t^c) = t_0 V$.

**Remark 1.** If the entrant bids without knowing the number of rival(s), one can easily verify that since the entrant is risk neutral, the bidding function is exactly the weighted average of the scenario bidding functions. More precisely, type $t \geq t^c$ bids $\sum_{n=1}^{N} p_n(t^c)e^{(n)}(t, v, t^c)$. This further implies that the expected overall effort is the same as that in Lemma 2.
4 Analysis of the Optimal Design

4.1 The Organizer’s Problem for a Fixed Threshold

Equipped with Lemma 2, we are ready to investigate the problem for any given entry threshold $t_c$. For any given $t_c$, the set of prize structures that induce it can be characterized by equation (1). Applying Lemma 2, the organizer’s problem for a given $t_c \in [a, b]$ can be expressed as

$$\max_{v} TE(v, t_c) = \sum_{n=0}^{N} \binom{N}{n} (1 - F(t_c))^n F^{N-n}(t_c) TE^{(n)}(v, t_c)$$

subject to

$$v_1 \geq v_2 \geq \ldots \geq v_N,$$  

$$\sum_{j=1}^{n} v_j \leq V, \forall n,$$  

$$\sum_{n=1}^{N} p_n(t_c)v_n = 0,$$

where (4) are the budget constraints for all scenarios, and (5) is the binding participation constraint for the threshold type.

4.2 Rewriting the Problem Using Order Statistics

We use $G_{(i,n)}(t, t_c)$ to denote the CDF of the $i$th order statistics of $n$ independent random variables, with each following CDF $G(t, t_c)$. It is well known that the CDF of the $i$th order statistics is $G_{(i,n)}(t, t_c) = \sum_{j=i}^{n} \binom{n}{j} G_j(t, t_c)(1-G(t, t_c))^{n-j}$, with density function $g_{(i,n)}(t, t_c) = n(n-1)\ldots(i-1)G^{i-1}(t, t_c)(1-G(t, t_c))^{n-i}g(t, t_c)$. Notice that when $t_c = a$, $G$ reduces to $F$, so we use $F_{(i,n)}(t)$ and $f_{(i,n)}(t)$ to represent $G_{(i,n)}(t, a)$ and $g_{(i,n)}(t, a)$, respectively; that is,

$$F_{(i,n)}(t) = \sum_{j=i}^{n} \binom{n}{j} F_j(t)(1-F(t))^{n-j}, \text{ and } f_{(i,n)}(t) = n(n-1)\ldots(i-1)F^{i-1}(t)(1-F(t))^{n-i}f(t).$$

Obviously, $F_{(i,n)}(\cdot)$ is the CDF of $X_{(i,n)}$, where $X_{(i,n)}$ denotes the random variable corresponding to the $i$th order statistics of $n$ independent random variables, with each following CDF $F(t)$.

The following result shows that the objective function can be expressed as a linear function in all $N$ prizes, with coefficients related to all these order statistics across different scenarios and different orders.
Lemma 3. The objective function (2) can be rewritten as

$$TE(v, t^c) = N(1 - F(t^c)) \sum_{n=1}^{N} \frac{p_n(t^c)}{n} \sum_{j=1}^{n} v_{n+1-j} \left( \int_{t^c}^{b} J(t)g(j,n)(t, t^c)dt - t_0 \right) + t_0 V.$$ 

Therefore, the objective function is linear in $v_1, v_2, \ldots, v_N$; denote the coefficient associated with $v_k$ as $\beta_k(t^c)$. That is,

$$\beta_k(t^c) = N(1 - F(t^c)) \sum_{n=k}^{N} \frac{p_n(t^c)}{n} \left( \int_{t^c}^{b} J(t)g(n+1-k,n)(t, t^c)dt - t_0 \right), \ k = 1, 2, \ldots, N.$$ 

These coefficients are quite complicated, with binomials and weighted average of various order statistics. However, it turns out that these coefficients can be rewritten in a remarkably simple way, as shown in the next lemma.

Lemma 4. The coefficients can be rewritten as

$$\beta_k(t^c) = \int_{t^c}^{b} [J(t) - t_0] f_{(N-k+1,N)}(t)dt, \ k = 1, 2, \ldots, N,$$

where $f_{(N-k+1,N)}(t)$ is the density function of the $(N-k+1)$th order statistics of $N$ random draws that follow CDF $F(\cdot)$, as defined in (6).

For any random variable $Z$ and an event $A$, we use $[Z|A]$ to denote any random variable that has its distribution the conditional distribution of $Z$ given $A$. The above Lemma says that $\beta_k(t^c)$—the coefficient associated with the $k$th prize when the induced threshold is $t^c$—is precisely the product of the mean of the random variable $[J(X_{(N-k+1,N)}) - t_0|X_{(N-k+1,N)} \geq t^c]$ (conditional on $X_{(N-k+1,N)} \geq t^c$) and the probability that $X_{(N-k+1,N)} \geq t^c$; that is:

$$\beta_k(t^c) = \left[ 1 - F_{(N-k+1,N)}(t^c) \right] \cdot \mathbb{E} [J(X_{(N-k+1,N)}) - t_0|X_{(N-k+1,N)} \geq t^c].$$  \hspace{1cm} (7)$$

Observing the structure of $\beta_k(t^c)$ as relating to the random variable $X_{(N-k+1,N)}$ revealed in (7) is crucial, as the stochastic orders among $X_{(N-k+1,N)}$ across $k$ (hazard rate dominance) will greatly facilitate the analysis of the comparison among $\beta_k(t^c)$’s. This will be clearer in Lemma 5 below.

Now the organizer’s problem for a fixed entry threshold $t^c$ can be conveniently restated as

$$\max_v TE(v, t^c) = \sum_{n=1}^{N} \beta_n(t^c)v_n + t_0 V$$ \hspace{1cm} (8)
subject to
\begin{align*}
  v_1 & \geq v_2 \geq \ldots \geq v_N, \\
  \sum_{j=1}^{n} v_j & \leq V, \forall n, \\
  \sum_{n=1}^{N} p_n(t^c)v_n &= 0.
\end{align*}

4.3 Features of Coefficients

As a linear programming problem, it is critical to investigate the relation among coefficients \( \beta_k(t^c) \) and \( \beta_k(t^c)/p_k(t^c) \). The following definition would be helpful for the presentation.

**Definition 1.** \( n(t^c) = \max\{n \in \{1, \ldots, N\} : \beta_n(t^c) \geq 0\} \); that is, \( n(t^c) \) is the largest integer in \( \{1, \ldots, N\} \) such that \( \beta_n(t^c) \geq 0 \). If \( \beta_n(t^c) < 0 \) for all integers \( n \in \{1, \ldots, N\} \), define \( n(t^c) = 0 \).

We have the following important observation regarding the comparison among these coefficients.

**Lemma 5.** (i) For any \( t^c \in [a, b] \) with \( n(t^c) \geq 1 \),
\[ \beta_1(t^c) > \beta_2(t^c) > \cdots > \beta_{n(t^c)}(t^c) \geq 0. \]

(ii) For any \( t^c \in (a, b) \) with \( n(t^c) \geq 1 \),
\[ \frac{\beta_1(t^c)}{p_1(t^c)} > \frac{\beta_2(t^c)}{p_2(t^c)} > \cdots > \frac{\beta_{n(t^c)}(t^c)}{p_{n(t^c)}(t^c)} \geq 0. \]

(iii) **Single crossing**: On the interval \([a, b]\), \( \beta_n(t^c) \) crosses 0 at most once; and if it does, it crosses 0 from below. Moreover, \( n(t^c) \) is weakly increasing in \( t^c \in [a, b] \) and \( n(t^c) = N \) when \( t^c < b \) is large enough—i.e., all \( \beta \) coefficients are positive when \( t^c \) is large enough.

By Theorem 1.B.26 (page 31) in Shaked and Shanthikumar [19], \( X_{(N-n+1,N)} \leq_{hr} X_{(N-n+2,N)} \), which further implies that \( [X_{(N-n+1,N)}]X_{(N-n+1,N)} \geq t^c \] \( \leq_{st} [X_{(N-n+2,N)}]X_{(N-n+2,N)} \geq t^c \) for any \( t^c \) by (1.B.7) on page 17 of Shaked and Shanthikumar [19]. Here, \( \leq_{hr} \) and \( \leq_{st} \) refer to hazard rate dominance and first-order stochastic dominance, respectively.\(^8\) With an additional observation that \( Np_n(t^c) = f_{(N-n+1,N)}(t^c) \), Part (i) and (ii) then readily follow from these observations. The details can be found in the proof of Lemma 5 in the Appendix.

\(^8\)For any two nonnegative random variables \( X \) and \( Y \) with absolutely continuous distribution functions, \( X \leq_{hr} Y \) if \( \frac{f_X(t)}{1-F_X(t)} \geq \frac{f_Y(t)}{1-F_Y(t)} \), for any \( t \in \mathbb{R} \), where \( F_X(.) \) and \( F_Y(.) \) are the CDF of \( X \) any \( Y \), respectively, with \( f_X(.) \) and \( f_Y(.) \) the corresponding density function. On the other hand, \( X \leq_{st} Y \) if \( F_X(t) \geq F_Y(t) \) for any \( t \in \mathbb{R} \).
From the objective function (8), $\beta_k(t^c)$ can be interpreted as the marginal revenue in terms of units of effort gained by increasing the $k$th prize by 1 dollar, taking into account the cost of losing 1 dollar (as the organizer values it as $t_0$ units of effort); from (11), $p_k(t^c)$ can be regarded as the shadow price of the $k$th prize. Lemma 5 reveals that the coefficients $\beta_k(t^c)$ and $\frac{\beta_k(t^c)}{p_k(t^c)}$ are decreasing in $k$ whenever these coefficients are nonnegative—i.e., $\beta_k(t^c) < \beta_{k-1}(t^c)$ and $\frac{\beta_k(t^c)}{p_k(t^c)} < \frac{\beta_{k-1}(t^c)}{p_{k-1}(t^c)}$ if $\beta_k(t^c) \geq 0$—however, it is silent on the case when the coefficients are negative. In fact, when $\beta_k(t^c) < 0$ for some integer $k \leq N - 1$, all possibilities can arise: The signs of $\beta_k(t^c) - \beta_{k+1}(t^c)$ and $\frac{\beta_k(t^c)}{p_k(t^c)} - \frac{\beta_{k+1}(t^c)}{p_{k+1}(t^c)}$ can be arbitrary. The following example illustrates this.

**Example 1.** Suppose that $N = 3$, $F(\cdot)$ is the uniform distribution on $[1, 2]$, and $t_0 = 1.5$. Then

<table>
<thead>
<tr>
<th>$t^c$</th>
<th>$\beta_2(t^c)$</th>
<th>$\beta_3(t^c)$</th>
<th>$\beta_2(t^c) - \beta_3(t^c)$</th>
<th>$\frac{\beta_2(t^c)}{p_2(t^c)} - \frac{\beta_3(t^c)}{p_3(t^c)}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.1</td>
<td>-0.46</td>
<td>-0.62</td>
<td>0.16</td>
<td>-1.8</td>
</tr>
<tr>
<td>1.2</td>
<td>-0.37</td>
<td>-0.36</td>
<td>-0.01</td>
<td>-0.6</td>
</tr>
<tr>
<td>1.55</td>
<td>-0.029</td>
<td>-0.016</td>
<td>-0.013</td>
<td>0.02</td>
</tr>
<tr>
<td>1.6</td>
<td>-0.003</td>
<td>-0.006</td>
<td>0.003</td>
<td>0.03</td>
</tr>
</tbody>
</table>

As can be seen from the last two columns of the table, when $\beta_2(t^c) < 0$, all four possibilities regarding the signs of $\beta_2(t^c) - \beta_3(t^c)$ and $\frac{\beta_2(t^c)}{p_2(t^c)} - \frac{\beta_3(t^c)}{p_3(t^c)}$ can arise. Moreover, one can easily verify numerically that when $t^c = 1.1$ or $1.2$—i.e., when $\frac{\beta_2(t^c)}{p_2(t^c)} < \frac{\beta_3(t^c)}{p_3(t^c)}$—awarding a single negative prize cannot be optimal. In fact, this is not a coincidence, as shown in Lemma 6 below.

The subtlety of relations among coefficients when they are negative can lead to the optimality of both multiple positive prizes and multiple negative prizes for certain entry thresholds. The following result provides a set of sufficient conditions under which multiple positive prizes and/or multiple negative prizes must arise at the optimum for certain thresholds.

**Lemma 6.** Fix $t^c \in (a, b)$. Suppose that $\hat{v} = (\hat{v}_1, \ldots, \hat{v}_N)$ is the optimum among all prize structures that induce $t^c$.

(i) If $\frac{\beta_{N-1}(t^c)}{p_{N-1}(t^c)} < \frac{\beta_N(t^c)}{p_N(t^c)}$, then multiple negative prizes must arise at the optimum—i.e., $\hat{v}_{N-1}, \hat{v}_N < 0$.

(ii) If $\beta_1(t^c) - \beta_2(t^c) + \frac{(p_2(t^c) - p_1(t^c))\beta_N(t^c)}{p_N(t^c)} < 0$ and $p_2(t^c) > p_1(t^c)$, then multiple positive prizes must arise at the optimum—i.e., $\hat{v}_1, \hat{v}_2 > 0$.

(iii) If both assumptions in (i) and (ii) are satisfied and $N \geq 4$, then multiple positive prizes and multiple negative prizes arise at the optimum.

Therefore, Example 1 is an example where the condition in part (i) of Lemma 6 is satisfied. Here, we provide another example where all conditions in Lemma 6 are satisfied.
**Example 2.** Suppose that $N = 4$, $F(\cdot)$ is the uniform distribution on $[1, 2]$, and $t_0 = 1.85$. Then when $t^c = 1.65$,

<table>
<thead>
<tr>
<th>$\beta_1(t^c)$</th>
<th>$\beta_2(t^c)$</th>
<th>$\beta_3(t^c)$</th>
<th>$\beta_4(t^c)$</th>
<th>$\frac{\beta_3(t^c)}{p_2(t^c)} - \frac{\beta_4(t^c)}{p_3(t^c)}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>-0.105</td>
<td>-0.12</td>
<td>-0.046</td>
<td>-0.006</td>
<td>-0.048</td>
</tr>
</tbody>
</table>

Note that in this example, $\beta_2(t^c) < \beta_1(t^c) < \beta_3(t^c) < \beta_4(t^c)$ and $\frac{\beta_1(t^c)}{p_1(t^c)} < \frac{\beta_2(t^c)}{p_2(t^c)} < \frac{\beta_3(t^c)}{p_3(t^c)} < \frac{\beta_4(t^c)}{p_4(t^c)}$, which further suggests that the behavior of $\beta$ coefficients can be arbitrary when they are negative. More importantly, this example satisfies all conditions in Lemma 6. Therefore, at the optimum, there must be two positive prizes and two negative prizes for $t^c = 1.65$.

**4.4 The Optimal Contest Structure**

Lemma 5 implies that for any given $t^c \in [a, b)$, $\beta_k(t^c) > \beta_{k+1}(t^c)$ whenever $\beta_{k+1}(t^c) \geq 0$ and $\beta_k(t^c)$ must be positive when $t^c$ is large enough. A particular threshold is $t^*$, defined as the smallest threshold such that all $\beta$ coefficients are nonnegative—i.e.,

$$t^* = \begin{cases} 
\text{the unique threshold such that } \beta_N(t^*) = 0, & \text{if } \beta_N(a) \leq 0 \\
\text{a, otherwise}
\end{cases}$$

Alternatively, by Definition 1, $t^*$ is the smallest threshold $t^c$ such that $n(t^c) = N$.

When $t^c > t^*—i.e., n(t^c) = N—we have the following observation that there is only one negative prize at the optimum.

**Proposition 2.** Fix any $t^c \in (t^*, b)$. Suppose that $\hat{v} = (\hat{v}_1, \ldots, \hat{v}_N)$ is the optimum among all prize structures that induce $t^c$. Then $\hat{v}_{N-1} \geq 0$, so that at the optimum, there can be only one negative prize.

Nevertheless, Examples 1 and 2 illustrate the complicacy in characterizing the optimum for an arbitrary entry threshold, as they show that the optimality in general demands multiple positive and negative prizes for entry thresholds below $t^*$. It is thus technically challenging to fully pin down the optimum for every $t^c$. If follows that a procedure of first identifying the optimum for each fixed $t^c$ and then pinning down the optimal entry threshold by comparing across all thresholds would be quite cumbersome if not infeasible.

We overcome the above-mentioned difficulties by first pinning down the optimal entry threshold
without first fully characterizing the optimal prizes for every entry threshold. This is made feasible
by an innovative procedure detailed as follows. Consider an arbitrary entry threshold \( t^*_0 \in [a,b) \) and
a (non-zero) prize structure \( v \) which induces it. We construct a particular class of prize structures
by fixing the first \( N - 1 \) prizes of \( v \) and varying only the last prize. Such class of prize structures
can induce any entry threshold in the interval \([t^*_0, b)\) when \( t^*_0 > t^* \); and it can induce any entry
threshold in the interval \([t^*_0, b)\) when \( t^*_0 \leq t^* \). Moreover, for each threshold in these two intervals,
the corresponding prize structure satisfies the required monotonicity constraint.

In the following Proposition, we establish that the optimal prize design for a given entry thresh-
old \( t^*_0 \in [a,b) \) is strictly dominated by a particular prize structure in the class mentioned above,
which induces entry threshold \( t^* \). Note that by Proposition 2, for \( t^*_0 > t^* \), we must have that only
the last prize is negative at the optimum.

**Proposition 3.** Let \( v = (v_1, \ldots, v_N) \neq 0 \) be an arbitrary non-zero prize structure with \( t^*_0 \in [a,b) \)
being its corresponding entry threshold. Fixing the first \( N - 1 \) prizes, for any \( t^c \in [a,b) \), construct
the vector

\[
v(t^c) = (v_1, \ldots, v_{N-1}, -\frac{\sum_{j=1}^{N-1} p_j(t^c)v_j}{p_N(t^c)}).
\]

Then:

(i) If \( t^*_0 \leq t^* \), then \( v(t^c) \) satisfies constraints (9)-(11) when \( t^c \in [t^*_0, b) \), and the expected overall
effort \( TE(v(t^c)) \) induced by \( v(t^c) \) achieves its unique optimum over \([t^*_0, b)\) at \( t^c = t^* \);

(ii) If \( t^*_0 > t^* \) and \( v_{N-1} \geq 0 \), then \( v(t^c) \) satisfies constraints (9)-(11) when \( t^c \in [a,b) \), and the
expected overall effort \( TE(v(t^c)) \) induced by \( v(t^c) \) achieves its unique optimum over \([a,b)\) at \( t^c = t^* \).

Propositions 2 and 3 mean that the optimal prize design for any entry threshold \( t^*_0 \in [a,b) \)
must be strictly dominated by that for the entry threshold \( t^* \). In the following Theorem, we fully
establish the optimality of \( t^* \) by further allowing \( t^*_0 = b \).

**Theorem 1.** The unique optimal entry threshold is \( t^* \).

Provided that \( t^* \) is the unique optimal threshold, we only need to identify the optimal prize
design for \( t^* \) to characterize the global optimum. At the threshold \( t^* \), by definition, all coefficients
\( \beta_n(t^*) \) are nonnegative, which, by Lemma 5, further entails useful monotonicity properties. We are
then ready to characterize the optimal prize design as in the following Theorem.

**Theorem 2.** The unique optimal prize structure is

\[
v^* = (V, \underbrace{0, \ldots, 0}_{N-2 \text{ times}}, -\frac{p_1(t^*)V}{p_N(t^*)}).
\]
The optimal design features (i) winner-take-all for the best performer among all entrants and (ii) a potential punishment for the worst performer that realizes if and only if all contestants enter the competition. The intuitions are revealed as below.

Theorem 1 reveals that the optimal entry threshold must be \( t^* \). In particular, Proposition 3 shows that starting from an optimal prize structure that induces an entry threshold \( t_0^* \in [a, b] \), one can always adjust only the punishment \( v_N \) \( (c < 0) \) to induce entry threshold \( t^* \) and at the same time generates a higher payoff for the organizer. In fact, when \( t_0^* < t^* \), a harsher punishment \( v_N \) that induces \( t^* \) reduces entry but increases the effort supply of an entrant; when \( t_0^* > t^* \), a more lenient punishment \( v_N \) that induces \( t^* \) increases entry but reduces the effort supply of an entrant. The optimality of \( t^* \) means that the positive effects of the above adjustment in punishment \( v_N \) always dominate its negative effects.

To further understand why \( t^* \) is the optimal entry threshold, recall that \( -\beta_N(t^c) \) can be interpreted as the marginal revenue gained by using an additional unit of punishment \( v_N \) when \( t^c \) is fixed, and that the organizer’s payoff can be expressed as \( \sum_{n=1}^{N} \beta_n(t^c)v_n + t_0V \) by (8). We will explain how this marginal revenue links to optimality of \( t^* \). In fact, when fixing the first \( N-1 \) prizes and varying the threshold \( t^c \), the participation constraint implies that \( v_N(t^c) = -\sum_{j=1}^{N-1} p_j(t^c)v_j/p_N(t^c) \) as in Proposition 3. When increasing \( t^c \) by \( \varepsilon \), such change leads to changes in all \( \beta_n(t^c), n \in \{1, \ldots, N\} \), and also in the last prize \( v_N(t^c) \). As such, the marginal change in the organizer’s payoff is 
\[
\varepsilon \left[ \sum_{n=1}^{N-1} \beta'_n(t^c)v_n + \beta'_N(t^c)v_N(t^c) + \beta_N(t^c)v'_N(t^c) \right].
\]

However, \( \sum_{n=1}^{N-1} \beta'_n(t^c)v_n + \beta'_N(t^c)v_N(t^c) = 0 \) (see the proof of Proposition 3), which implies that the marginal change is \( \varepsilon \beta_N(t^c)v'_N(t^c) \). Since \( v'_N(t^c) < 0 \), the marginal change has the same sign as \( -\beta_N(t^c) \). Therefore, an \( \varepsilon \) increase in \( t^c \) leads to a marginal change in the organizer’s payoff, which has the same sign as \( -\beta_N(t^c) \)—the marginal revenue gained by using an additional unit of punishment \( v_N \). Since the marginal revenue is positive when \( t^c < t^* \) and is negative when \( t^c > t^* \), the optimum must stop exactly at the point when the marginal revenue is 0, which is \( t^* \).

Given the optimality of the entry threshold \( t^* \), the organizer’s payoff can be expressed as \( \sum_{n=1}^{N} \beta_n(t^*)v_n + t_0V \) by (8), in which \( \mathbf{v} = (v_n) \) is an arbitrary prize structure that induces entry \( t^* \). Without loss of generality, we assume \( t^* \in (a, b) \). Thus, we have \( \beta_1(t^*) > \beta_2(t^*) > \cdots > \beta_{N-1}(t^*) \geq \beta_N(t^* \) = 0 by Lemma 5. Constructing \( \tilde{\mathbf{v}} = (\tilde{v}_n) \) by setting \( \tilde{v}_1 = V, \tilde{v}_n = 0 \) for \( n \in \{2, \ldots, N-1\} \) and \( \tilde{v}_N = -p_1(t^*)V/p_N(t^*) \) \( (c < 0) \). Note that \( \tilde{\mathbf{v}} = (\tilde{v}_n) \) still induces entry \( t^* \). Clearly, we have \( \sum_{n=1}^{N} \beta_n(t^*)\tilde{v}_n + t_0V = \sum_{n=1}^{N} \beta_n(t^*)v_n + t_0V \), since the sum of all positive prizes in \( \mathbf{v} \) is no greater than \( V \) and \( \beta_1(t^*) > \beta_2(t^*) > \cdots > \beta_{N-1}(t^*) > \beta_N(t^* \) = 0. We thus have that \( \tilde{\mathbf{v}} \) is the optimum.

Note that the punishment prevails if and only if \( p_1(t^*) > 0 \), i.e., \( t^* > a \). When \( t^* = a \), we go back to the optimal design of Moldovanu and Sela [12] who do not allow punishments. However, our result shows that in this case, there is no loss of generality in their analysis.
By the definition of \( t^* \) and Lemma 5, it is clear that \( t^* = a \) if and only if \( \beta_N(a) \geq 0 \), which is the case when, for example, \( J(a) \geq t_0 \). In this case, the optimum induces full entry and is winner-take-all \( v = (V, 0, \ldots, 0) \). Thus, we have the following immediate observation.

**Corollary 1.** Winner-take-all is optimal if and only if \( \beta_N(a) = \int_0^a [J(t) - t_0] f_{(1, N)}(t) dt \geq 0 \).

When \( \beta_N(a) < 0 \), we have \( t^* \in (a, b) \), which corresponds to stochastic entry. Note that this can be the case even when \( t_0 = 0 \); in this case, the organizer still imposes a negative prize, although she does not value it at all (as she does not value any leftover budget).

According to well-received insights from the auction design literature, when the virtual values of less efficient types contestants are negative, excluding them would enhance the seller’s revenue. Therefore, in their Section IV, Moldovanu and Sela [12] further investigate using entry fees to exclude less efficient types of bidders to boost contest performance while maintaining the winner-take-all prize allocation rule—i.e., awarding all prize budget \( V \) to the highest bidder of all entrants.\(^{10}\) Moldovanu and Sela [12] conjecture that such winner-take-all rule with appropriate entry fees is the optimal prize structure. As we mentioned in the model setup, such contest rule corresponds to the prize structure \( v = (V - E, -E, \ldots, -E) \) when then entry fee is \( E \geq 0 \). Theorem 2 and Corollary 1 imply that the Moldovanu and Sela type of entry fees are suboptimal, unless the optimal entry fee is \( 0 \)—i.e., when \( \beta_N(a) \geq 0 \).

### 5 Analysis of the Optimal Contingent Prize Structure

So far, our analysis focuses on prize structures that are independent of the number of entrants. The general prize structure allows such possibility. This section devotes to the characterization of the optimal contingent prize structure.

A contingent prize structure is a set \( W \) of scenario prize vectors. Specifically, \( W = \{W_1, W_2, \ldots, W_N\} \) is the set of scenario prize vectors with \( W_n \) being the prize allocation rule in scenario \( n \) (i.e., when there are \( n \) entrants). In \( W_n = (w_{n,1}, w_{n,2}, \ldots, w_{n,n}) \in \mathbb{R}^n \), we have \( w_{n,1} \geq w_{n,2} \geq \ldots \geq w_{n,n} \) and \( \sum_{j=1}^n w_{n,j} \leq V \). Here, \( w_{n,j} \), which is allowed to be negative, is the \( j \)th prize in scenario \( n \), which is for the \( j \)th highest effort among the \( n \) entrants. Ties are broken randomly and fairly. Let \( V_n = \sum_{j=1}^n w_{n,j} \), which is the sum of prizes in scenario \( n \). We use \( V = (V_1, V_2, \ldots, V_N) \in \mathbb{R}^N \) to denote the scenario budget vector.\(^{11}\) Our formulation of the contingent prize structures covers the

\(^{9}\)We illustrate that with \( t_0 = 0 \), both \( \beta_N(a) < 0 \) and \( \beta_N(a) > 0 \) are possible. It is clear that if \( J(a) \geq 0 \), \( \beta_N(a) > 0 \). On the other hand, for example, if \( F(\cdot) \) is the uniform distribution on \([0.25, 1.25]\) and \( N = 3 \), then \( \beta_N(a) = \beta_3(0.25) = -0.25 < 0 \).

\(^{10}\)Note that in rank-order contests, a minimum bid is not an eligible design instrument since its implementation requires information on the level of bids.

\(^{11}\)Note that once the prize structure \( W \) is given, \( V \) is pinned down accordingly. Introducing \( V \) is thus purely for notational convenience.
modelling of negative prizes in Moldovanu, Sela, and Shi [15], Thomas and Wang [20], Kamijo [7], and Hammond et al. [6] as special cases.

The analysis of the contingent case proceeds as follows. We first characterize the symmetric monotone bidding equilibrium for any given prize allocation rule. Then, we identify the optimal allocation rule for any fixed entry threshold. Finally, we vary across all possible entry thresholds to pin down the optimum.

5.1 Entry Threshold

A given prize structure \( W \) induces an entry threshold \( t^c \in [a, b] \) such that types higher than \( t^c \) enter with probability one, while types lower than it enter with probability zero. At equilibrium, type \( t^c \) bids zero and enjoys a zero expected payoff.\(^{12}\) Since type \( t^c \) obtains the lowest prize \( w_{n,n} \) for sure in every scenario in any monotone bidding equilibrium, it satisfies

\[
\sum_{n=1}^{N} p_n(t^c)w_{n,n} = 0. \tag{12}
\]

Note that given \( W \), the solution to (12) may not be unique. However, this does not affect our analysis; see footnote 14.

5.2 Scenario Bidding Functions

For any given prize structure \( W \), suppose that it induces an entry threshold \( t^c \in [a, b] \) determined by (12). Similar to the analysis of the non-contingent case, we have the following results.

**Lemma 7.** Suppose that the induced entry threshold is \( t^c \). In scenario \( n \geq 1 \) with prizes \( W_n = (w_{n,1}, w_{n,2}, \ldots, w_{n,n}) \),

(i) The unique symmetric monotone bidding function \( e^{(n)}(t, W_n, t^c) \) for type \( t \in [t^c, b] \) is

\[
e^{(n)}(t, W_n, t^c) = tV^{(n)}(t) - \int_{t^c}^{t} V^{(n)}(s)ds - t^c w_{n,n},
\]

where

\[
V^{(n)}(t) = \sum_{j=1}^{n} w_{n,n+1-j} \left( \begin{array}{c} n-1 \\ j-1 \end{array} \right) G^{j-1}(t, t^c)(1 - G(t, t^c))^{n-j}
\]

is the expected prize an entrant with type \( t \) obtains.

---

\(^{12}\)Similar to the non-contingent case, the threshold type may enjoy a positive payoff when the prize structure induces full entry; however, this is clearly not optimal.
(ii) The corresponding scenario-0 expected overall effort is

$$TE^{(n)}(W_n, V_n, t^c) = n \int_{t^c}^{b} J(t)V^{(n)}(t)g(t, t^c)dt - nt^c_w_n + t_0(V - V_n).$$

(iii) The scenario-0 expected overall effort is \(TE^{(0)}(W_0, V_0, t^c) = t_0V\).

**Lemma 8.** Suppose that \(W = \{W_1, W_2, \ldots, W_N\}\) where \(W_n = (w_{n,1}, w_{n,2}, \ldots, w_{n,n})\), and the entry threshold is \(t^c\), then the expected overall effort is

$$TE(W, V, t^c) = \sum_{n=0}^{N} \left( \binom{N}{n} (1 - F(t^c))^n F^{N-n}(t^c)TE^{(n)}(W_n, V_n, t^c) \right),$$

where \(TE^{(n)}(W_n, V_n, t^c)\) is given in Lemma 7 when \(n \geq 1\), and \(TE^{(0)}(W_0, V_0, t^c) = t_0V\).

We search for the optimum in two steps. In the first step, we fix the entry threshold \(t^c\) and find the optimum in the set of prize structures \(W\) that induce \(t^c\) through (12). In the second step, we vary across all entry thresholds in \([a, b]\) to pin down the optimum. Note that for any given prize structure \(W\), it must induce an entry threshold determined by (12). Thus, this two-step procedure indeed characterizes the optimum. The next subsection presents the first-step analysis.

### 5.3 The Optimum for a Fixed Entry Threshold

Equipped with Lemma 8, we are ready to derive the optimum for any given entry threshold \(t^c \in [a, b]\). The organizer’s problem for a given \(t^c\) can be written as

$$\max_{W,V} TE(W, V, t^c) = \sum_{n=0}^{N} \left( \binom{N}{n} (1 - F(t^c))^n F^{N-n}(t^c)TE^{(n)}(W_n, V_n, t^c) \right).$$

---

13 There is no loss of generality to assume that (12) always has a solution. This is because if not, then \(\sum_{n=1}^{N} p_n(t^c)w_{n,n} > 0\) for all \(t^c \in [a, b]\), or \(\sum_{n=1}^{N} p_n(t^c)w_{n,n} < 0\) for all \(t^c \in [a, b]\). The former case means full entry, which is obviously suboptimal given that type a obtains a strictly positive payoff; the latter case just means that there is no entry, which is a trivial case—the expected overall effort is simply \(t_0V\), which is not optimal according to Lemma 9.

14 A given prize structure \(W\) induces a unique bidding equilibrium with the corresponding entry threshold \(t^c\) that satisfies (12) and the bidding functions given by Lemma 7. However, conversely, fixing \(W\), the solution to (12) may not be unique—i.e., there can be more than one entry threshold that satisfies (12). For example, consider a prize structure with \(w_{n,1} = 0\) for all \(n\). Then, all \(t^c \in [a, b]\) satisfy (12). It is obvious that in this case the prize structure induces full entry so the only induced entry threshold is \(t^c = a\). However, this is not an issue in the above-mentioned two-step procedure. This is because in the first step, we start with a fixed entry threshold \(\tilde{t}^c\) and find the optimum in the set \(S\) of prize structures \(W\) that satisfies (12) with \(t^c = \tilde{t}^c\). The set \(S'\) of prize structures that induce equilibria with entry threshold \(\tilde{t}^c\) is clearly a subset of \(S\), as the multi-solution issue we just mentioned. Thus, in the first step, we search for the optimum in \(S\), so the optimum identified is at least as good as that in \(S'\). Nevertheless, the optimum in \(S\), as in Proposition 4, is indeed an element in \(S'\).
subject to
\[\sum_{j=1}^{n} w_{n,j} = V_n \leq V, \forall n \geq 1, \tag{13}\]
\[\sum_{n=1}^{N} p_n(t^c)w_{n,n} = 0, \tag{14}\]

where (13) is the budget constraint, and (14) is the binding participation constraint for the threshold type.

For each positive integer \(n\), denote
\[\delta_n(t^c) = n \int_{t^c}^{b} J(t)G_n^{-1}(t, t^c)g(t, t^c)dt, \quad t^c \in [a, b).\]

After presenting Proposition 4, we will show that \(\delta_n(t^c)\) is the marginal benefit of increasing the first prize by 1 dollar in scenario \(n\). Lemma 13 in the appendix shows the following observation.

**Observation.** (i) For all \(k \geq 1\), \(\delta_k(t^c) < \delta_{k+1}(t^c)\); (ii) For all \(k \geq 1\), \(\delta_k(t^c)\) is strictly increasing in \(t^c \in [a, b)\) and \(\lim_{t^c \to a^+} \delta_k(t^c) = b\); and (iii) \(\delta_1(t^c) = t^c\).

Thus, there exists a unique threshold \(\hat{t} \in [a, b)\) such that \(\delta_N(\hat{t}) = t_0\) (when \(\delta_N(a) > t_0, \) define \(\hat{t} = 0\)). Then \(\delta_N(t^c) \geq t_0\) if and only if \(t^c \geq \hat{t}\), with equality only when \(t^c = \hat{t}\). Note that since \(\delta_N(t^c) > t^c\) for all \(t^c\), we have \(\hat{t} < t_0\).

The following proposition presents the solution to the organizer’s problem for a given entry threshold \(t^c\).

**Proposition 4.** Fix any entry threshold \(t^c \in [a, b)\), the optimal budget vector \(V^*(t^c) = (V_1^*(t^c), \ldots, V_N^*(t^c))\), and prize vectors \(W^*_m(t^c) = (w^*_{m,1}(t^c), w^*_{m,2}(t^c), \ldots, w^*_{m,m}(t^c))\), \(m \geq 1\), are given by:15

(i) If \(t^c \leq \hat{t}\), cancelling the contest is optimal, which can be achieved by, for example, setting \(V^*_m(t^c) = 0\) and \(W^*_m(t^c) = 0\) for all \(m \geq 1\).

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15The optimum \(W^*(t^c) = \{W^*_1(t^c), \ldots, W^*_N(t^c)\}\) in part (ii) of Proposition 4 belongs to \(S'\), as mentioned in footnote 14. In fact, given \(w^*_{n,n}(t^c)\), the solution to the equation \(\sum_{n=1}^{N} p_n(t)w^*_{n,n}(t^c) = 0\), with the unknown variable \(t \in [a, b]\), is unique; the unique solution must be \(t = t^c\). Therefore, \(t^c\) is indeed the equilibrium threshold induced by \(W^*(t^c)\). To see the uniqueness, notice that when \(t \in (a, b)\),
\[\frac{d}{dt} \sum_{j=1}^{N-1} p_n(t)w^*_{n,n}(t^c) = V\frac{d}{dt} \sum_{n=1}^{N-1} \frac{p_n(t)}{n} = \frac{Vf(t)}{1-F(t)}(p_N(t) + \sum_{j=2}^{N-1} \frac{p_j(t)}{j}) > 0,\]
and \(\frac{d}{dt}[p_N(t)w^*_{N,N}(t^c)] = -w^*_{N,N}(t^c)(N-1)(1-F(t))^{N-2}f(t) \geq 0\). Thus, \(\sum_{n=1}^{N} p_n(t)w^*_{n,n}(t^c)\) is strictly increasing in \(t \in [a, b]\).
(ii) If $t^c > \hat{t}$, $V^*(t^c) = (V, V, \ldots, V)$ and

$$W^*_m(t^c) = \begin{cases} \frac{(V/m, V/m, \ldots, V/m)}{m} \text{ if } m < N, \\ (V - (N - 1)K^*_N(t^c), K^*_N(t^c), \ldots, K^*_N(t^c)) \text{ if } m = N, \end{cases}$$

where $K^*_N(t^c) = \frac{V}{N} \left( 1 - \frac{1 - F^N(t^c)}{(1 - F(t^c))^N} \right)$, which is negative.

Thus, it is never optimal to induce an endogenous entry with entry threshold $t^c \leq \hat{t}$; the organizer should cancel the contest in this case, which yields $t_0 V$ overall effort. However, when the organizer wishes to induce an endogenous entry with $t^c > \hat{t}$, cancelling the contest can never be optimal; moreover, the organizer uses all of her budget regardless of the number of entrants. In the next subsection, we will show that it is never optimal to cancel the contest—i.e., the optimum must involve endogenous entry and the optimal entry threshold is strictly higher than $\hat{t}$. Note that when $t^c \neq \hat{t}$, the optimum is unique; when $t^c = \hat{t}$, it is easy to verify that part (i) and part (ii) give the same expected overall effort. In this case, the optimum is not unique.

### 5.4 Proof Strategy

We briefly explain the roadmap to prove the above proposition and the intuitions behind the results. Call $w_{n,n}$ the minimum prize in scenario $n$ and denote $W_{\text{min}} = (w_{1,1}, w_{2,2}, \ldots, w_{N,N})$ as the minimum prize vector. To characterize the optimum for a fixed $t^c$, one needs to search all possible prize allocation rules that implement the entry threshold $t^c$. However, notice that only the minimum prize vector $W_{\text{min}}$ matters for implementing $t^c$, as seen from (14). This leads to the following two-step procedure to prove Proposition 4, which relies on the principle of cross-rank transfer and the principle of cross-scenario transfer, which are first established in Liu and Lu [10]. However, note that in Liu and Lu [10], $t_0$ is not involved. Therefore, the tradeoff in our paper is further complicated.

In the first step, we characterize the optimum assuming that the budget vector $V$ and the minimum prize vector $W_{\text{min}} = (w_{1,1}, w_{2,2}, \ldots, w_{N,N})$ are fixed (and the entry threshold induced is $t^c$). It turns out that the optimum is winner-take-all with a minimum-prize requirement—in scenario $m$, all prizes except the first prize are set as the minimum prize $w_{m,m}$, while the first prize is $V_m - (m - 1)w_{m,m}$. This is the principle of cross-rank transfer: Transferring prizes from low ranks to enlarge the prize to be awarded to the highest rank induces more effort. Thus, the organizer should make the prize spread (or equivalently, the first prize) as large as possible in all scenarios.
Mathematically, this is because the scenario-$m$ expected overall effort is a linear function in all $m$ prizes and the budget left over, in which the coefficient associated with the first prize (the highest of all $m$ prizes) is precisely $\delta_m(t^c)$, and the coefficient associated with the leftover budget is simply $t_0$. Thus, $\delta_m(t^c)$ is the marginal benefit of increasing the first prize by 1 dollar in scenario $m$, while $t_0$ can be interpreted as the marginal benefit of increasing the leftover budget by 1 dollar. Thus, the principle of cross-rank transfer follows, which gives the optimum as in the previous paragraph, given $V_m$ and $w_{m,m}$ fixed.

In the second step, given that all other prizes are set optimally as in the first step, we then find the optimal budget vector $V$ and the minimum prize vector $W_{\text{min}}$, subject to the entry threshold implemented by $W_{\text{min}}$ being $t^c$. Ignoring the participation constraint, given a fixed $V$, to induce more effort the organizer should set the minimum prizes as low as possible regardless of the number of entrants. However, type $t^c$ must win minimum prizes in all scenarios. Therefore, the participation constraint of type $t^c$ implies that if type $t^c$ obtains a negative prize in some scenario, he must be compensated by a positive prize in another scenario. The remaining question is: In which scenario should the organizer sacrifice effort induction for the purpose of providing the needed entry subsidy that facilitates a more negative minimum prize in another scenario?

The principle of cross-scenario transfer answers this question. It means that the larger the number of entrants, the more effective it is to induce efforts using a certain prize spread. More precisely, a drop in $w_{n,n}$ by $\varepsilon/p_n(t^c)$ must be compensated for by an increase in $w_{m,m}$ by $\varepsilon/p_m(t^c)$ due to the binding participation constraint, $m \neq n$. The cross-scenario transfer implies that whenever $n > m$, such a transfer of minimum prizes between the two scenarios concerned must enhance the expected overall effort. Thus, in the extreme, the organizer should set $w_{N,N}$ as low as possible, which is accomplished by setting minimum prizes in the rest of the scenarios as high as possible—obviously, the extreme is $V_k/k$ when the number of entrants is $k$, for all $k < N$. This means that at the optimum, it must be the case that whenever the entry is partial, the organizer’s payoff solely comes from the leftover budget.

However, notice that the above argument assumes that the budget vector $V$ is fixed. What is left is to find the optimal $V$. In fact, in the first case, when $t^c \leq \hat{t}$, the marginal benefit of the first prize in scenario $n$, $\delta_n(t^c)$, is (weakly) lower than the marginal benefit of the leftover budget, $t_0$, for all $n \geq 1$. Thus, it is optimal to make the leftover budget as large as possible. In other words, cancelling the contest is optimal. When $t^c > \hat{t}$, since $\delta_N(t^c) > t_0$, the organizer should use up her budget in scenario $N$, i.e., $V_N = V$. Furthermore, a similar principle of cross-scenario transfer applies here. A drop in $w_{N,N}$ must be compensated for by an increase in $w_{m,m}$ for some $m \neq N$, which means the budget used in scenario $m$ must increase, given the argument in the previous

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16 Note that the sum of all prizes in any scenario-$n$ need not equal the original budget $V$; the organizer can keep a positive amount.
paragraph that $V_m = mw_{m,m}$ at the optimum. Thus, a drop in $w_{N,N}$ must be compensated by a decrease in the scenario-$m$ leftover budget $V - V_m$. The marginal cost of an increase in $mw_{m,m}$, which is equivalent to the marginal cost of a decrease in the leftover budget $V - V_m$, is simply $t_0$ (because the organizer’s payoff in scenario $m$ comes solely from the leftover budget). But this marginal cost is strictly lower than the marginal benefit of a drop in $w_{N,N}$, which is $\delta_N(t^e)$. Thus, the organizer should “transfer” any unused budget $V - V_n$ in scenarios $n \neq N$ to decrease $w_{N,N}$. The optimum is obviously to use up her budget whenever the entry is partial. This leads to part (ii) in Proposition 4.

5.5 The Optimal Prize Design

Having characterized the optimal prize design for any fixed entry threshold, we proceed to identify the optimal entry threshold, and hence the globally optimal prize design. In fact, as shown in (32), when $t^e > \hat{t}$, the highest expected overall effort for fixed $t^e$ is

$$TE^*(t^e) = V(1 - F^N(t^e))\delta_N(t^e) + F^N(t^e)t_0V.$$ (15)

Recall that $\delta_N(t^e) > t_0$ when $t^e > \hat{t}$. Thus,

$$TE^*(t^e) = V(1 - F^N(t^e))\delta_N(t^e) + F^N(t^e)t_0V$$

$$> V(1 - F^N(t^e))t_0 + F^N(t^e)t_0V = t_0V,$$

which implies that it is never optimal to cancel the contest and the organizer uses all her budget regardless of the number of entrants. Furthermore, when $t^e \to b^-$, $TE^*(t^e) \to t_0V$. This implies that the optimal threshold $\hat{t}^*$ must lie strictly in between $\hat{t}$ and $b$.

The above argument implies that when $\hat{t} \geq a$, we have $\hat{t}^* \in (\hat{t}, b)$, which further implies that endogenous entry arises at the optimum. One still needs to be careful for the case of $\hat{t} < a$—i.e., when $t_0$ is small enough (e.g., $t_0 = 0$). In this case, does endogenous entry necessarily arise at the optimum? The following lemma encapsulates the results and answers this question.

**Lemma 9.** The optimal prize design must involve endogenous entry. That is, $\max\{a, \hat{t}\} < \hat{t}^* < b$, where $\hat{t}^*$ is the optimal entry threshold.

Since $\hat{t}^* > \max\{a, \hat{t}\}$, the optimal design must entail endogenous entry and the adoption of negative prizes. Since Lemma 9 holds for any $t_0 < b$, it immediately implies that the organizer uses all of her budget regardless of the number of entrants, no matter how the organizer values the leftover budget.
Let the corresponding minimum prize in scenario $N$ for $\hat{t}^*$ be $K_N^\ast(\hat{t}^*)$, which is negative. Proposition 4 immediately implies the following result.

**Theorem 3.** (i) The optimal design is $W^\ast(\hat{t}^*) = (W_1^\ast(\hat{t}^*), W_2^\ast(\hat{t}^*), \ldots, W_N^\ast(\hat{t}^*))$, where $W_N^\ast(\hat{t}^*) = (V - (N - 1)K_N^\ast(\hat{t}^*), K_N^\ast(\hat{t}^*), \ldots, K_N^\ast(\hat{t}^*))$ and $W_n^\ast(\hat{t}^*) = (\frac{V}{n}, \frac{V}{n}, \ldots, \frac{V}{n})$ for all $n \leq N - 1$. (ii) The optimal design can be implemented by entry fees. An entry fee $|K_N^\ast(\hat{t}^*)|$ is required for participation. When there is full entry, a winner-take-all contest with a grand prize of $V + N|K_N^\ast(\hat{t}^*)|$ prevails; when there are less than $N$ entrants, an egalitarian rule applies: All entrants equally share the grand prize $V + n|K_N^\ast(\hat{t}^*)|$, where $n(<N)$ is the number of entrants.

Moldovanu and Sela [12] show that winner-take-all is optimal when the effort cost function is linear or concave, while multiple prizes can be optimal when the cost function is convex. The extreme case of multiple prizes is the egalitarian rule. Therefore, the two extremes of prize allocation are winner-take-all and egality. By allowing negative prizes, we illustrate that with linear effort cost, the optimal prize structure is an interesting combination of these two extremes. Moreover, the adoption of negative prizes is essential, and endogenous entry must arise at the optimum.

### 5.6 Discussion

Moldovanu and Sela [12] conjecture that a winner-take-all rule coupled with appropriate entry fees is the optimal prize structure. As we mentioned at the end of Section 4, our Theorem 2 reveals that their prize design with entry fees is however generally suboptimal even within the non-contingent prize structures. Allowing the prize allocation to be contingent on the number of entrants, our Theorem 3 further discloses the optimal way to use entry fees: The entry fees collected should be used to reward the top performer, as in Hammond et al. [6]. Nevertheless, our finding also implies that the contest rule of winner-take-all with budget-augmenting entry fees of Hammond et al. [6] can also be further improved by making the design contingent on the number of entrants, which harmonically integrates two seemingly conflicting features of both winner-take-all and egality rule.

Liu et al. [11] also introduce budget-augmenting negative prizes in contest design. They show that full surplus extraction can be achieved in the limit so that the highest total effort can be arbitrarily close to an utmost level, when the bound on negative prizes approaches infinity. In their paper, optimal negative prizes do not exist (or are infinite). However, in our paper, the optimal negative prize is finite. The crucial difference between our paper and theirs is that an (endogenous) minimum effort requirement is imposed in their paper. Thus, in their paper the prize allocation rule depends not only on the rank information of efforts but also on the information on the level of effort. In our paper, the prize allocation rule only relies on the rank information of efforts (and the number of entrants). This is why we do not have full surplus extraction in our setting, and

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17 Full entry is optimal in Liu et al. [11], so endogenous entry is not an issue there.
the optimal negative prizes must be finite.

6 Concluding Remarks

Understanding how to optimally orchestrate carrots and sticks in contests is important in both theory and practice. The optimal use of rewards and punishments in rank-order contests with incomplete information has remained an open question. In this paper, we provide a complete and elegant characterization using a workhorse model of all-pay auction. We find that the non-contingent optimal design must feature a single positive prize equal to the budget, and potentially a single negative prize that determines the optimal endogenous entry threshold. The optimal negative prize is set at a level such that the marginal revenue of imposing an extra unit of punishment through the last prize vanishes. We find that this marginal revenue is precisely captured by the coefficient of the last prize in the total effort function. As a direct implication, a necessary and sufficient condition for the optimality of pure winner-take-all is established.

We further characterize the optimal design when prize structures can be contingent on the number of entrants. We find that the optimal design must entail endogenous entry and must be contingent on the number of entrants in an interesting but rather unexpected manner. When all contestants participate, all non-top performers are penalized by a finite uniform budget-augmenting negative prize; the top performer collects all available prize budget, including that contributed by the imposed negative prizes. With full participation, more efficient types are more likely to exist. The enhanced prize for the top performer in this scenario incentivizes the more efficient types to work harder. Whenever the entry is partial, an egalitarian rule applies to entrants. The egalitarian rule functions to maintain the participation incentive of more efficient types. Therefore, the optimal contingent design harmonically integrates two seemingly conflicting prize allocation rules: winner-take-all and egalitarian rule.

Our analysis on contingent prizes discloses the optimal way of using entry fees in rank-order contests, which shows that the rules investigated in the literature, such as in Moldovanu and Sela [12] and Hammond et al. [6], can be further improved. Unlike Liu et al. [11] who must utilize the level information of effort, when only rank-order information is used, the optimal level of negative prizes must be finite, and their full surplus extraction result—which crucially relies on the feasibility of minimum bids—does not extend to our environment.

Our study assumes that the contestants are risk neutral and their bidding cost function is linear. Further investigating the impacts of risk aversion and/or nonlinearity of bidding cost function on the optimal designs is a highly meaningful but nontrivial task. We leave these studies to future work.
7 Appendix

Proof of Lemma 1: We first show that \(\sum_{n=1}^{N} p_n(t^c)v_n\) is strictly increasing in \(t^c \in [a, b]\) for any \(v = (v_1, \ldots, v_N) \neq ve\) with \(v_1 \geq \cdots \geq v_N\), where \(v \in \mathbb{R}\) and \(e = (1, \ldots, 1)\) is the \(N\)-vector with all its elements being 1. To see this, notice first that for any integer \(n \in \{1, \ldots, N-1\}\) and any \(t^c \in (a, b)\),

\[
\frac{d}{dt^c} \sum_{k=1}^{n} p_k(t^c) = \frac{d}{dt^c} \sum_{k=1}^{N} \binom{N-1}{k-1} (1 - F(t^c))^{k-1} F^{N-k}(t^c)
\]

\[
= f(t^c) \sum_{k=1}^{n} \left[ (N-k)\binom{N-1}{k-1}(1 - F(t^c))^{k-1} F^{N-k-1}(t^c) \right]
\]

\[
= (N-1)f(t^c) \left[ \sum_{k=1}^{n} \binom{N-2}{k-1}(1 - F(t^c))^{k-1} F^{N-k-1}(t^c) \right]
\]

\[
= (N-1)f(t^c) \left( \frac{N-2}{n-1} \right) (1 - F(t^c))^{n-1} F^{N-1}(t^c) > 0. \quad (16)
\]

Therefore, \(\sum_{k=1}^{n} p_k(t^c)\) is strictly increasing in \(t^c \in [a, b]\), for any integer \(n \in \{1, \ldots, N-1\}\). Define \(v_{N+1} = 0\) and notice that

\[
\sum_{n=1}^{N} p_n(t^c)v_n = \sum_{n=1}^{N} p_n(t^c) \left[ \sum_{k=n}^{N} (v_k - v_{k+1}) \right] = \sum_{k=1}^{N} (v_k - v_{k+1}) \left[ \sum_{n=1}^{k} p_n(t^c) \right]
\]

\[
= \sum_{k=1}^{N-1} (v_k - v_{k+1}) \left[ \sum_{n=1}^{k} p_n(t^c) \right] + v_N, \quad (17)
\]

where the last equality uses \(\sum_{n=1}^{N} p_n(t^c) = 1\). Note that \(v_k - v_{k+1} \geq 0\) for any integer \(k \in \{1, \ldots, N-1\}\), with strict inequality for some \(k' \in \{1, \ldots, N-1\}\) (because \(v \neq ve\)). Since \(\sum_{n=1}^{k} p_n(t^c)\) is strictly increasing in \(t^c \in [a, b]\), it follows that \(\sum_{n=1}^{N} p_n(t^c)v_n\) is strictly increasing in \(t^c \in [a, b]\) for any \(v \neq ve\).

Now, for any prize structure \(v \neq ve\), (i) if \(\sum_{n=1}^{N} p_n(a)v_n < 0\) (i.e., \(V_N < 0\)) and \(\sum_{n=1}^{N} p_n(b)v_n > 0\) (i.e., \(v_1 > 0\)), then the monotonicity established above and the Intermediate Value Theorem imply that there exists a unique \(t^c_0 \in (a, b)\) such that \(\sum_{n=1}^{N} p_n(t^c_0)v_n = 0\), which means that such prize structure induces a unique interior entry threshold; (ii) if \(\sum_{n=1}^{N} p_n(a)v_n \geq 0\) (i.e., \(v_N \geq 0\)), then such prize structure induces full entry; (iii) if \(\sum_{n=1}^{N} p_n(b)v_n \leq 0\) (i.e., \(v_1 \leq 0\)), then such prize structure induces no entry. If the prize structure \(v = ve\), then \(\sum_{n=1}^{N} p_n(t^c)v_n = v\), so such prize structure induces no entry. Definition \(\sum_{j=m}^{n} H_j(\cdot) = 0\) when \(m > n\) for any function \(H_j(\cdot)\).
structure induces either full entry (when $v \geq 0$) or no entry (when $v < 0$). Note that by definition, a prize structure satisfies $v_1 \geq v_N$, so these cases exhaust all possibilities, which further implies that the reverse is also true—the characterization is indeed a necessary and sufficient one. This completes the proof. $\square$

**Proof of Lemma 3:** By Proposition 1,

$$\begin{align*}
TE^{(n)}(v, t^c) &= n \int_{t^c}^{b} J(t) V^{(n)}(t) g(t, t^c) dt - nt^c v_n + t_0 \left( V - \sum_{j=1}^{n} v_j \right) \\
&= n \int_{t^c}^{b} J(t) \left( \sum_{j=1}^{n} v_{n+1-j} \binom{n-1}{j-1} G^j(t, t^c)(1 - G(t, t^c))^{n-j} \right) g(t, t^c) dt \\
&\quad - nt^c v_n + t_0 \left( V - \sum_{j=1}^{n} v_j \right) \\
&= \sum_{j=1}^{n} v_{n+1-j} \left( \int_{t^c}^{b} J(t) g(j, n)(t, t^c) dt - t_0 \right) - nt^c v_n + t_0 V. \quad (18)
\end{align*}$$

Therefore,

$$\begin{align*}
TE(v, t^c) &= \sum_{n=0}^{N} \binom{N}{n} (1 - F(t^c))^n F^{N-n}(t^c) TE^{(n)}(v, t^c) \\
&= \sum_{n=1}^{N} \binom{N}{n} (1 - F(t^c))^n F^{N-n}(t^c) \left[ \sum_{j=1}^{n} v_{n+1-j} \left( \int_{t^c}^{b} J(t) g(j, n)(t, t^c) dt - t_0 \right) - nt^c v_n \right] + F^N(t^c) t_0 V \\
&= (1 - F(t^c)) \sum_{n=1}^{N} \frac{N p_n(t^c)}{n} \left[ \sum_{j=1}^{n} v_{n+1-j} \left( \int_{t^c}^{b} J(t) g(j, n)(t, t^c) dt - t_0 \right) - t^c v_n \right] + t_0 V \\
&= N(1 - F(t^c)) \sum_{n=1}^{N} \frac{p_n(t^c)}{n} \left[ \sum_{j=1}^{n} \frac{v_{n+1-j}}{n} \left( \int_{t^c}^{b} J(t) g(j, n)(t, t^c) dt - t_0 \right) - t^c v_n \right] + t_0 V \\
&= N(1 - F(t^c)) \sum_{n=1}^{N} \frac{p_n(t^c)}{n} \sum_{j=1}^{n} v_{n+1-j} \left( \int_{t^c}^{b} J(t) g(j, n)(t, t^c) dt - t_0 \right) + t_0 V,
\end{align*}$$

where the last equality uses (5). $\square$
Proof of Lemma 4: Recall that

\[
\beta_k(t^c) = N(1 - F(t^c)) \sum_{n=k}^{N} \frac{p_n(t^c)}{n} \left( \int_{t^c}^{b} J(t)g_{(n+1-k,n)}(t, t^c)dt - t_0 \right)
\]

\[
= N(1 - F(t^c)) \sum_{n=k}^{N} \binom{N-1}{n-1} (1 - F(t^c))^{n-1} F^{N-n}(t^c)
\]

\[
\cdot \left( \int_{t^c}^{b} J(t) \binom{n-1}{n-k} G^{n-k}(t, t^c)(1 - G(t, t^c))^{k-1} g(t, t^c)dt - \frac{t_0}{n} \right).
\]

For convenience, denote \( \beta_k(t^c) = A - B \), where

\[
A = N(1 - F(t^c)) \sum_{n=k}^{N} \binom{N-1}{n-1} (1 - F(t^c))^{n-1} F^{N-n}(t^c) \int_{t^c}^{b} J(t) \binom{n-1}{n-k} G^{n-k}(t, t^c)(1 - G(t, t^c))^{k-1} g(t, t^c)dt,
\]

and

\[
B = N(1 - F(t^c)) \sum_{n=k}^{N} \frac{(N-1)!}{(n-1)! (n-k)! (k-1)!} \binom{N-1}{n-1} (1 - F(t^c))^{n-1} F^{N-n}(t^c) t_0.
\]

Recall that \( G(t, t^c) = \frac{F(t) - F(t^c)}{1 - F(t^c)} \) and \( g(t, t^c) = \frac{f(t)}{1 - F(t^c)} \). Then,

\[
A = N \sum_{n=k}^{N} \binom{N-1}{n-1} F^{N-n}(t^c) \int_{t^c}^{b} J(t) \binom{n-1}{n-k} (F(t) - F(t^c))^{n-k} (1 - F(t))^{k-1} f(t)dt.
\]

Notice further that for any integer \( n \in \{k, \ldots, N\} \),

\[
\binom{N-1}{n-1} \binom{n-1}{n-k} = \frac{(N-1)!}{(n-1)! (n-k)!} \cdot \frac{(N-1)!}{(k-1)!} = \binom{N-1}{N-k} \binom{N-k}{n-k}.
\]

Thus,

\[
A = N \binom{N-1}{N-k} \sum_{n=k}^{N} \frac{N-k}{n-k} F^{N-n}(t^c) \int_{t^c}^{b} J(t) (F(t) - F(t^c))^{n-k} (1 - F(t))^{k-1} f(t)dt
\]

\[
= N \binom{N-1}{N-k} \int_{t^c}^{b} J(t) F^{N-k}(t) (1 - F(t))^{k-1} f(t)dt = \int_{t^c}^{b} J(t) f_{(N-k+1,N)}(t)dt.
\]
On the other hand, since $F_{N-k+1,N}(t) = \sum_{j=N-k+1}^{N} \binom{N}{j} F^j(t)(1 - F(t))^{N-j}$, we have

\[
B = N(1 - F(t^c)) \sum_{n=k}^{N} \frac{(N-1)_n}{(n-1)} F^n(t^c) = t_0 \sum_{n=k}^{N} \binom{N}{n} (1 - F(t^c))^n F^{N-n}(t^c)
\]

which further implies that

\[
0 \leq E \left[ J(X_{N-n+1,N}) - t_0 | X_{N-n+1,N} \geq t^c \right] < E \left[ J(X_{N-n+2,N}) - t_0 | X_{N-n+2,N} \geq t^c \right],
\]

Therefore,

\[
\beta_k(t^c) = A - B = \int_{t^c}^{b} J(t) f_{N-k+1,N}(t) dt - t_0 \int_{t^c}^{b} f_{N-k+1,N}(t) dt = \int_{t^c}^{b} [J(t) - t_0] f_{N-k+1,N}(t) dt.
\]

**Proof of Lemma 5**: (i) It suffices to show that for if $\beta_n(t^c) \geq 0$ for some integer $n \in \{2, \ldots, N\}$, then $\beta_{n-1}(t^c) > \beta_n(t^c)$. To this end, notice that from Lemma 4 and (7)

\[
\beta_n(t^c) = \left[ 1 - F_{N-n+1,N}(t^c) \right] \cdot E \left[ J(X_{N-n+1,N}) - t_0 | X_{N-n+1,N} \geq t^c \right].
\]

By Theorem 1.B.26 (page 31) in Shaked and Shanthikumar [19], $X_{N-n+1,N} \leq_{st} X_{N-n+2,N}$, which further implies that $[X_{N-n+1,N} | X_{N-n+1,N} \geq t^c] \leq_{st} [X_{N-n+2,N} | X_{N-n+2,N} \geq t^c]$ for any $t^c$ by (1.B.7) on page 17 of Shaked and Shanthikumar [19]. Since $J(\cdot)$ is strictly increasing, the definition of first-order stochastic dominance further implies that

\[
E \left[ J(X_{N-n+1,N}) - t_0 | X_{N-n+1,N} \geq t^c \right] < E \left[ J(X_{N-n+2,N}) - t_0 | X_{N-n+2,N} \geq t^c \right].
\]

Since $\beta_n(t^c) \geq 0$, we have

\[
0 \leq E \left[ J(X_{N-n+1,N}) - t_0 | X_{N-n+1,N} \geq t^c \right] < E \left[ J(X_{N-n+2,N}) - t_0 | X_{N-n+2,N} \geq t^c \right],
\]

Here it is a strictly inequality, because the CDFs of $X_{N-n+1,N}$ and $X_{N-n+2,N}$ obviously differ at least one point in $(t^c, b)$ (so they differ over a set with strictly positive measure, because of continuity).

29
so

\[
0 \leq \left[ 1 - F_{N-n+1,N}(t^c) \right] \cdot E \left[ J(X_{N-n+1,N}) - t_0 \mid X_{N-n+1,N} \geq t^c \right]_{\beta_n(t^c)} < \left[ 1 - F_{N-n+2,N}(t^c) \right] \cdot E \left[ J(X_{N-n+2,N}) - t_0 \mid X_{N-n+2,N} \geq t^c \right]_{\beta_{n-1}(t^c)}
\]

where we use the fact that \( F_{N-n+1,N}(t^c) \geq F_{N-n+2,N}(t^c) \), which is due to \( X_{N-n+1,N} \leq_{st} X_{N-n+2,N} \).\(^{20}\)

(ii) It suffices to show that if \( \beta_n(t^c) \geq 0 \) for some integer \( n \in \{2, \ldots, N\} \), then \( \frac{\beta_n(t^c)}{p_n(t^c)} > \frac{\beta_{n-1}(t^c)}{p_{n-1}(t^c)} \). Notice that \( Np_k(t^c) = f_{N-k+1,N}(t^c) \) for any \( k \). Since \( X_{N-n+1,N} \leq_{hr} X_{N-n+2,N} \), we have, by definition, \( f_{N-n+1,N}(t^c) = \frac{1}{1-F_{N-n+1}(t^c)} \) for any \( t^c \in (a, b) \), which further implies that

\[
\frac{\beta_n(t^c)}{p_n(t^c)} = \frac{1-F_{N-n+1,N}(t^c)}{p_n(t^c)} \cdot E \left[ J(X_{N-n+1,N}) - t_0 \mid X_{N-n+1,N} \geq t^c \right] < \frac{1-F_{N-n+2,N}(t^c)}{p_{n-1}(t^c)} \cdot E \left[ J(X_{N-n+2,N}) - t_0 \mid X_{N-n+2,N} \geq t^c \right] \leq \frac{1-F_{N-n+2,N}(t^c)}{p_{n-1}(t^c)} \cdot E \left[ J(X_{N-n+2,N}) - t_0 \mid X_{N-n+2,N} \geq t^c \right] = \frac{\beta_{n-1}(t^c)}{p_{n-1}(t^c)},
\]

where the first inequality is from part (i).

(iii) Since \( \beta_n(t^c) = \int_a^b [J(t) - t_0] f_{N-n+1,N}(t)dt \), \( \beta_n(t^c) = -[J(t^c) - t_0] f_{N-n+1,N}(t^c) \). Together with the fact that \( J(\cdot) \) is strictly increasing and that \( \int_a^b [J(t) - t_0] f_{N-n+1,N}(t)dt > 0 \) for any \( t^c \) with \( J(t^c) \geq 0 \) (note that \( J(b) = b > t_0 \)), \( \beta_n(t^c) \) crosses zero at most once and \( \beta_n(t^c) > 0 \) when \( t^c \) satisfies \( J(t^c) > t_0 \). Moreover, if \( \beta_n(t^c) \) crosses zero, it must cross from below. \( n(t^c) \) being increasing in \( t^c \) is obvious from part (i). \( \square \)

**Proof of Lemma 6:** Notice first that Lemma 1 implies that \( \hat{v}_1 > 0 \) and \( \hat{v}_N < 0 \).

(i) Suppose, to the contrary, that \( \hat{v}_{N-1} < 0 \). Then the prize structure

\[
\hat{v} = (\hat{v}_1, \hat{v}_2, \ldots, \hat{v}_{N-2}, \hat{v}_{N-1} - \frac{\varepsilon}{p_{N-1}(t^c)}, \hat{v}_N + \frac{\varepsilon}{p_N(t^c)}),
\]

which differs from \( \bar{v} \) only at the last two prizes, still satisfies constraints (9)–(11), when \( \varepsilon > 0 \) is

\(^{20}\)As mentioned above, \( [X_{N-n+1,N}] X_{N-n+1,N} \geq t^c \) \( \leq_{st} [X_{N-n+2,N}] X_{N-n+2,N} \geq t^c \) for any \( t^c \); in particular, this holds when \( t^c = a \).
For any non-zero prize structure $v$, Lemma 10.

Proof of Proposition 3:

(iii) This part is obvious from part (i) and part (ii).

Proof of Proposition 2: Notice first that Lemma 1 implies that $\hat{v}_1 > 0$. Suppose, to the contrary, that $\hat{v}_{N-1} < 0$. Due to monotonicity (9), there exists some integer $j \in \{2, \ldots, N-1\}$ such that $\hat{v}_{j-1} \geq 0$ and $\hat{v}_j < 0$—that is, $j$ is the smallest integer such that $\hat{v}_j < 0$. Consider the prize structure

$$\tilde{v} = (\hat{v}_1 - \varepsilon, \hat{v}_2 + \varepsilon, \hat{v}_3, \ldots, \hat{v}_{N-1}, \hat{v}_N - \frac{(p_2(t^c) - p_1(t^c))}{p_N(t^c)} \varepsilon),$$

which differs from $\hat{v}$ only at the $j$th and $N$th prizes. Obviously, since $\hat{v}_j < 0$, $\tilde{v}$ satisfies constraints (9)–(11) for threshold $t^c$. However, since $t^c > t^*$, by Lemma 5 and the definition of $t^*$, the expected overall effort induced by $\tilde{v}$ is strictly higher than that by $\hat{v}$:

$$TE(\tilde{v}) - TE(\hat{v}) = -\varepsilon \left[ \beta_1(t^c) - \beta_2(t^c) + \frac{(p_2(t^c) - p_1(t^c))}{p_N(t^c)} \beta_N(t^c) \right] > 0,$$

contradicting the optimality of $\tilde{v}$. Thus, $\hat{v}_{N-1} < 0$.

(ii) Suppose, to the contrary, that $\hat{v}_2 \leq 0$. Then, when $\varepsilon > 0$ is small enough, $\hat{v}_1 - \varepsilon > 0$, $\hat{v}_2 + \varepsilon < \hat{v}_1 - \varepsilon$ and $\hat{v}_N - \frac{(p_2(t^c) - p_1(t^c))}{p_N(t^c)} \varepsilon < \hat{v}_N$ (because $p_2(t^c) > p_1(t^c)$). As such, when $\varepsilon > 0$ is small enough, the prize structure

$$\tilde{v} = (\hat{v}_1 - \varepsilon, \hat{v}_2 + \varepsilon, \hat{v}_3, \ldots, \hat{v}_{N-1}, \hat{v}_N - \frac{(p_2(t^c) - p_1(t^c))}{p_N(t^c)} \varepsilon),$$

which differs from $\hat{v}$ only at the 1st, 2nd, and the last prizes, still satisfies constraints (9)–(11). However, the expected overall effort induced by $\tilde{v}$ is strictly higher than that by $\hat{v}$:

$$TE(\tilde{v}) - TE(\hat{v}) = -\varepsilon \left[ \beta_1(t^c) - \beta_2(t^c) + \frac{(p_2(t^c) - p_1(t^c))}{p_N(t^c)} \beta_N(t^c) \right] > 0,$$

contradicting the optimality of $\tilde{v}$. Thus, $\hat{v}_2 > 0$.

(iii) This part is obvious from part (i) and part (ii). □

Proof of Proposition 3: The following observation is useful, the proof of which is relegated to the end of the appendix.

Lemma 10. (i) For any non-zero prize structure $v = (v_1, \ldots, v_N)$ with induced entry threshold.
t_0^c < b \text{ (i.e., } \sum_{n=1}^{N} p_n(t_0^c)v_n = 0 \text{) and } \sum_{n=1}^{N-1} p_n(t_0^c)v_n \geq 0, \]
\[
d\over dt \left( \frac{\sum_{n=1}^{N-1} p_n(t^c)v_n}{p_N(t^c)} \right) > 0 \text{ for any } t^c \in (t_0^c, b).
\]

(ii) If the prize structure \( \nu = (v_1, \ldots, v_N) \) satisfies \( v_1 > 0 \) and \( v_{N-1} \geq 0 \), then
\[
d\over dt \left( \frac{\sum_{n=1}^{N-1} p_n(t^c)v_n}{p_N(t^c)} \right) > 0 \text{ for any } t^c \in [a, b].
\]

Now we are ready to prove the proposition. Notice that by construction, \( \nu(t^c) \) satisfies the participation constraint (11) and
\[
TE(\nu(t^c)) = \sum_{j=1}^{N-1} \beta_j(t^c)v_j - \frac{\beta_N(t^c) \sum_{j=1}^{N-1} p_j(t^c)v_j}{p_N(t^c)} + t_0V.
\]

Therefore,
\[
TE'(\nu(t^c)) = \sum_{j=1}^{N-1} \beta'_j(t^c)v_j - \beta_N(t^c) \sum_{j=1}^{N-1} \frac{p_j(t^c)v_j}{p_N(t^c)} - \beta_N(t^c) \frac{d}{dt} \left( \frac{\sum_{j=1}^{N-1} p_j(t^c)v_j}{p_N(t^c)} \right).
\]

From Lemma 4,
\[
\beta'_j(t^c) = -[J(t^c) - t_0] f_{(N-j+1, N)}(t^c) = -[J(t^c) - t_0] N \binom{N-1}{N-j} F^{N-j}(t^c)(1 - F(t^c))^{j-1}
\]
\[
= -N [J(t^c) - t_0] p_j(t^c).
\]

Thus, when \( t^c < b \),
\[
TE'(\nu(t^c)) = -N [J(t^c) - t_0] \left( \sum_{j=1}^{N-1} p_j(t^c)v_j - p_N(t^c) \right) \sum_{j=1}^{N-1} \frac{p_j(t^c)v_j}{p_N(t^c)} - \beta_N(t^c) \frac{d}{dt} \left( \frac{\sum_{j=1}^{N-1} p_j(t^c)v_j}{p_N(t^c)} \right)
\]
\[
= -\beta_N(t^c) \frac{d}{dt} \left( \frac{\sum_{j=1}^{N-1} p_j(t^c)v_j}{p_N(t^c)} \right).
\]
For part (i), if \( \sum_{j=1}^{N-1} p_j(t_0^c)v_j \geq 0 \), then Lemma 10 implies that

\[
\frac{d}{dt^c} \left( \frac{\sum_{j=1}^{N-1} p_j(t^c)v_j}{p_N(t^c)} \right) > 0 \text{ for any } t^c \in (t_0^c, b),
\]

which, combined with (19), further implies that \( TE'(v(t^c)) \) has the opposite sign to \( \beta_N(t^c) \) in \((t_0^c, b)\). Thus, if \( \sum_{j=1}^{N-1} p_j(t_0^c)v_j \geq 0 \), then \( TE(v(t^c)) \) achieves its unique optimum over \([t_0^c, b]\) at \( t^c = t^* \); this is because \( TE(v(t^c)) \) is differentiable in \([t_0^c, b]\) and \( TE'(v(t^c)) \) has the opposite sign to \( \beta_N(t^c) \) in \((t_0^c, b)\), which, by definition, satisfies\(^{21}\)

\[
\beta_N(t^c) = \begin{cases} 
< 0, & \text{if } t^c < t^* \\
0, & \text{if } t^c = t^* \\
> 0, & \text{if } t^c > t^*
\end{cases}
\]

Thus, it remains to show that \( \sum_{j=1}^{N-1} p_j(t_0^c)v_j \geq 0 \). If the induced entry threshold \( t_0^c = a \), \( \sum_{j=1}^{N-1} p_j(t_0^c)v_j = \sum_{j=1}^{N-1} p_j(a)v_j = 0 \geq 0 \). If \( t_0^c > a \), then, by definition,

\[
v_N = -\frac{\sum_{j=1}^{N-1} p_j(t_0^c)v_j}{p_N(t_0^c)}.
\]

Lemma 1 implies that \( v_N < 0 \), which further implies that \( \sum_{j=1}^{N-1} p_j(t_0^c)v_j > 0 \).

Finally, what is left is to show that \( v(t^c) \) satisfies constraints (9)—(11) when \( t^c \in [t_0^c, b) \). To see this, note that \( v(t^c) \) satisfies (11) by construction; for the rest two constraints, it suffices to verify that \( -\sum_{j=1}^{N-1} p_j(t^c)v_j/p_N(t^c) \leq v_N \) when \( t^c \in [t_0^c, b) \), because \( v \) is a prize structure (so \( v_1 \geq \cdots \geq v_N \) and \( \sum_{j=1}^{N-1} v_j \leq V \) for all \( n \)). Note that by definition, \( -\sum_{j=1}^{N-1} p_j(t_0^c)v_j/p_N(t_0^c) = v_N \), so it remains to verify that \( \sum_{j=1}^{N-1} p_j(t^c)v_j/p_N(t^c) \geq \sum_{j=1}^{N-1} p_j(t_0^c)v_j/p_N(t_0^c) \). To see why this is true, notice that \( \sum_{j=1}^{N-1} p_j(t_0^c)v_j/p_N(t_0^c) = -v_N \geq 0 \) by Lemma 1. Then, Lemma 10 implies that \( \sum_{j=1}^{N-1} p_j(t)v_j/p_N(t) \) is increasing in \( t \in [t_0^c, b) \). \( \sum_{j=1}^{N-1} p_j(t^c)v_j/p_N(t^c) \geq \sum_{j=1}^{N-1} p_j(t_0^c)v_j/p_N(t_0^c) \) when \( t^c \in [t_0^c, b) \) then follows from \( t^c \geq t_0^c \).

For part (ii), if

\[
\frac{d}{dt^c} \left( \frac{\sum_{j=1}^{N-1} p_j(t^c)v_j}{p_N(t^c)} \right) > 0 \text{ for any } t^c \in [a, b),
\]

then, similarly, (19) directly implies the claim of optimality in this proposition. However, notice that it is assumed that \( v_{N-1} \geq 0 \) and \( t_0^c > t^* \), which, by Lemma 1, implies that \( v_1 > 0 \). (20) then follows directly from part (ii) of Lemma 10. Finally, what is left is to show that \( v(t^c) \) satisfies constraints (9)—(11) when \( t^c \in [a, b) \). To see this, note that \( v(t^c) \) satisfies (11) by construction; for

\(^{21}\)If \( t^* = a \), then \( \beta_N(t^c) \geq 0 \) for all \( t^c \in [a, b) \) with equality only possibly when \( t^c = a \).
the rest two constraints, it suffices to verify that \( -\sum_{j=1}^{N-1} p_j(t^*) v_j/p_N(t^*) \leq 0 \), because \( v_{N-1} \geq 0 \). This is obvious, because \( v_1 \geq \cdots \geq v_{N-1} \geq 0 \) immediately implies \( -\sum_{j=1}^{N-1} p_j(t^*) v_j/p_N(t^*) \leq 0 \).

This completes the proof of the proposition. \( \Box \)

**Proof of Theorem 1:** We show that \( t^* \) is the unique optimal entry threshold. Suppose, to the contrary, that \( t^{**} \) is an optimal entry threshold with \( t^{**} \neq t^* \). Let \( v = (v_1, \ldots, v_N) \) be the corresponding optimal prize structure to threshold \( t^{**} \). There are three cases.

Case 1: \( t^{**} < t^* \). In this case, part (i) of Proposition 3 implies that the expected overall effort induced by \( v \) is strictly less than that by

\[
v(t^*) = (v_1, \ldots, v_{N-1}, -\frac{\sum_{j=1}^{N-1} p_j(t^*) v_j}{p_N(t^*)}),
\]

which induces threshold \( t^* \). The contradiction will arise if we can show that \( v(t^*) \) satisfies constraints (9)–(11), which is true by Proposition 3.

Case 2: \( t^{**} \in (t^*,b) \). In this case, Proposition 2 implies that \( v_{N-1} \geq 0 \). Since \( v_{N-1} \geq 0 \), part (ii) of Proposition 3 implies that the expected overall effort induced by \( v \) is strictly less than that by

\[
v(t^*) = (v_1, \ldots, v_{N-1}, -\frac{\sum_{j=1}^{N-1} p_j(t^*) v_j}{p_N(t^*)}),
\]

which induces threshold \( t^* \). The contradiction will arise if we can show that \( v(t^*) \) satisfies constraints (9)–(11), which is true by Proposition 3.

Case 3: \( t^{**} = b \). In this case, the mechanism induces no entry, so that the expected overall effort is \( t_0 V \). However, for the prize structure

\[
v^* = (V, 0, \ldots, 0, -\frac{p_1(t^*) V}{p_N(t^*)}),
\]

which induces the entry threshold \( t^* \) and satisfies constraints (9)–(11), it yields an expected overall effort of

\[
TE(v^*) = \beta_1(t^*) V - \beta_N(t^*) \frac{p_1(t^*) V}{p_N(t^*)} + t_0 V > t_0 V.
\]

The inequality follows from Lemma 5 and the fact that if \( t^* > a \), then \( \beta_N(t^*) = 0 \) and \( \beta_1(t^*) > 0 \); if \( t^* = a \), then by the definition of \( t^* \), \( \beta_N(a) \geq 0 \), so \( \beta_1(a) > 0 \) (note \( p_1(a) = 0 \) and \( p_N(a) = 1 \)). Thus, the no-entry case is strictly dominated by a prize structure inducing \( t^* \). \( \Box \)

**Proof of Theorem 2:** If \( t^* = a \)—i.e., \( \beta_N(a) \geq 0 \)—then \( v^* = (V, 0, \cdots, 0) \) is optimal. To see this, note that in this case \( \beta_1(a) > \cdots > \beta_N(a) \geq 0 \) by Lemma 5. Lemma 1 implies that for
any prize structure \( v = (v_1, \ldots, v_N) \) inducing threshold \( a \), we have \( v_N \geq 0 \). Now, suppose, to the contrary, that \( \tilde{v} = (\hat{v}_1, \ldots, \hat{v}_N) \neq v^* \) is optimal. Since \( \hat{v}_1 \geq \ldots \geq \hat{v}_N \geq 0 \), there exists some integer \( k \in \{2, \ldots, N\} \) such that \( \hat{v}_k > 0 \). Let \( j \in \{k, \ldots, N\} \) be the largest integer such that \( \hat{v}_j > 0 \). Consider the prize structure

\[
\tilde{v} = (\hat{v}_1 + \hat{v}_j, \hat{v}_2, \ldots, \hat{v}_{j-1}, 0, \hat{v}_{j+1}, \ldots, \hat{v}_N),
\]

which differs from \( \tilde{v} \) only at the 1st and \( j \)th prizes. Obviously, \( \tilde{v} \) satisfies constraints (9)–(11) for threshold \( t^* = a \). However,

\[
TE(\tilde{v}) - TE(\tilde{v}) = (\beta_1(a) - \beta_j(a))v_j > 0,
\]

which contradicts the optimality of \( \tilde{v} \). Therefore, when \( t^* = a \), \( v^* = (V, 0, \ldots, 0) \) is optimal.

We proceed to the case when \( t^* > a \). Suppose, to the contrary, that \( \tilde{v} = (\hat{v}_1, \ldots, \hat{v}_N) \neq v^* \) is optimal. Then there must exist some integer \( k \in \{2, \ldots, N-1\} \) such that \( \hat{v}_k \neq 0 \). Note that \( \hat{v}_N < 0 \) and \( \hat{v}_1 > 0 \) by Lemma 1. There are two cases to consider.

Case 1: \( \hat{v}_k < 0 \). Due to monotonicity (9), there exists some integer \( j \in \{2, \ldots, k\} \) such that \( \hat{v}_{j-1} \geq 0 \) and \( \hat{v}_j < 0 \)—that is, \( j \) is the smallest integer such that \( \hat{v}_j < 0 \). Consider the prize structure

\[
\tilde{v} = (\hat{v}_1, \ldots, \hat{v}_{j-1}, 0, \hat{v}_{j+1}, \ldots, \hat{v}_{N-1}, \hat{v}_N + \frac{p_j(t^*)}{p_N(t^*)} \hat{v}_j),
\]

which differs from \( \tilde{v} \) only at the \( j \)th and \( N \)th prizes. Obviously, since \( \hat{v}_j < 0 \), \( \tilde{v} \) satisfies constraints (9)–(11) for threshold \( t^* \). However,

\[
TE(\tilde{v}) - TE(\tilde{v}) = \hat{v}_j \left( \beta_N(t^*) \frac{p_j(t^*)}{p_N(t^*)} - \beta_j(t^*) \right) = \hat{v}_j p_j(t^*) \left( \frac{\beta_N(t^*)}{p_N(t^*)} - \frac{\beta_j(t^*)}{p_j(t^*)} \right) > 0,
\]

which contradicts the optimality of \( \tilde{v} \).

Case 2: \( \hat{v}_k > 0 \). Due to monotonicity (9), there exists some integer \( j \in \{k, \ldots, N-1\} \) such that \( \hat{v}_j > 0 \) and \( \hat{v}_{j+1} \leq 0 \)—that is, \( j \) is the largest integer such that \( \hat{v}_j > 0 \). Note that, if \( j \leq N - 2 \), then the argument in Case 1 implies that that \( \hat{v}_i = 0 \) for all \( i \in \{j + 1, \ldots, N - 1\} \).

Consider the prize structure

\[
\tilde{v} = (\hat{v}_1 + \hat{v}_j, \hat{v}_2, \ldots, \hat{v}_{j-1}, 0, \hat{v}_{j+1}, \ldots, \hat{v}_{N-1}, \hat{v}_N + \frac{p_j(t^*)}{p_N(t^*)} \hat{v}_j - \frac{p_1(t^*)}{p_N(t^*)} \hat{v}_j),
\]

which differs from \( \tilde{v} \) only at the 1st, \( j \)th and \( N \)th prizes. Note that since \( \hat{v}_i = 0 \) for all \( i \in \{j + 1, \ldots, N - 1\} \),
\{j + 1, \ldots, N - 1\} (if \( j \leq N - 2 \)), to verify that \( \tilde{\nu} \) satisfies constraints (9)–(11) for threshold \( t^* \), one only needs to show
\[
\hat{v}_N + \frac{p_j(t^*)}{p_N(t^*)} \hat{v}_j - \frac{p_1(t^*)}{p_N(t^*)} \hat{v}_j < 0.
\]
This is true, because with \( \hat{v}_i = 0 \) for all \( i \in \{ j + 1, \ldots, N - 1 \} \) (if \( j \leq N - 2 \)), we have
\[
\hat{v}_N + \frac{p_j(t^*)}{p_N(t^*)} \hat{v}_j - \frac{p_1(t^*)}{p_N(t^*)} \hat{v}_j = \frac{1}{p_N(t^*)} \left( - \sum_{i=1}^{N-1} p_i(t^*) \hat{v}_i + p_j(t^*) \hat{v}_j - p_1(t^*) \hat{v}_j \right)
\]
\[
= \frac{1}{p_N(t^*)} \left( - \sum_{i=1}^{j-1} p_i(t^*) \hat{v}_i - p_1(t^*) \hat{v}_j \right) < 0,
\]
(Obviously, this argument also applies to the case when \( j = N - 1 \)). Thus, \( \tilde{\nu} \) satisfies constraints (9)–(11) for threshold \( t^* \). However,
\[
TE(\tilde{\nu}) - TE(\tilde{\nu}) = \hat{v}_j (\beta_1(t^*) - \beta_j(t^*)) + \beta_N(t^*) \left( \frac{p_j(t^*)}{p_N(t^*)} \hat{v}_j - \frac{p_1(t^*)}{p_N(t^*)} \hat{v}_j \right) > 0,
\]
which contradicts the optimality of \( \tilde{\nu} \).

The proof completes. \( \square \)

**Proof of Proposition 4:** To solve for the optimal \( W \), step 1: We first fix the scenario budget vector \( V \) and the minimum prize vector \( W_{\text{min}} \) that satisfies (14) to solve for the optimal prize vectors \( \{ W_n \} \); step 2: We then search across all possible \( V \) and \( W_{\text{min}} \) to obtain the optimum for the fixed \( t^c \). For notational clarity, let \( K_n = w_{n,n} \), and thus \( K = (K_1, K_2, \ldots, K_N) \) is the minimum prize vector \( W_{\text{min}} \).

**Step 1:** Notice that the expected overall effort \( TE(W, V, t^c) \) is a weighted sum of \( TE^{(n)}(W_n, V_n, t^c) \), \( n = 0, 1, 2, \ldots, N \). When both \( w_{n,n} \) and the budget \( V_n \) for scenario \( n \) are fixed for all \( n \), maximizing \( TE(W, V, t^c) \) is equivalent to maximizing each \( TE^{(n)}(W_n, V_n, t^c) \) separately subject to (13).

Denote the density of the \( i \)th order statistics of the \( n \) random draws from a distribution with cumulative distribution function \( G(t, t^c) \) by \( g_{(i,n)}(t, t^c) \). As is well known, \( g_{(i,n)}(t, t^c) = n(n-1)G^{i-1}(t, t^c)(1 - G(t, t^c))^{n-i}g(t, t^c) \). By Lemma 7, when \( n \geq 1 \), we have
\[
TE^{(n)}(W_n, V_n, t^c) = \sum_{j=1}^{n} w_{n,n+1-j} \int_{t^c}^{b} J(t) g_{(j,n)}(t, t^c) dt - nt^c w_{n,n} + t_0(V - V_n).
\]

(21)
Lemma 4 in Liu and Lu [10] shows that the coefficient associated with the \( n \)th prize is decreasing in \( n \), which is restated below.

**Lemma 11** (Liu and Lu [10]). \( \forall t^c < b, \int_{t^c}^{b} J(t)g(n,n)(t, t^c)dt > \int_{t^c}^{b} J(t)g(i,n)(t, t^c)dt \) for any \( i \) with \( 1 \leq i \leq n - 1 \). Moreover, \( \int_{t^c}^{b} J(t)g(n,n)(t, t^c)dt > 0 \).

Thus, when the minimum prize \( K_n \) and the sum of prizes \( V_n \) are given, the following result follows directly (it is similar to Lemma 3 in Liu and Lu [10] and is called the principle of cross-rank transfer).

**Lemma 12** (Liu and Lu [10]). When \( V_n \) and \( K_n \) are given \( (n \geq 1) \), the optimal solution to

\[
\max_{W_n} TE^{(n)}(W_n, V_n, t^c)
\]

s.t. \( \sum_{j=1}^{n} w_{n,j} = V_n \), with \( w_{n,n} = K_n \leq \frac{V_n}{n} \),

is given by \( w_{n,1} = V_n - (n - 1)K_n \) and \( w_{n,j} = K_n \) for all \( j \geq 2 \).

To see the above lemma, notice that \( TE^{(n)}(W_n, V_n, t^c) \) is linear in the \( n \) prizes \( w_{n,1}, \ldots, w_{n,n} \). The principle of cross-rank transfer results from the fact that the coefficient associated with the first prize \( w_{n,1} \) is positive and the highest. Therefore, transferring prizes from low ranks of effort to enlarge the prize to be awarded to the highest rank of effort induces more effort. Constrained by (22), the lowest prize an entrant can have is \( K_n \). Therefore, the prizes specified in lemma 12 must be optimal.

Equipped with Lemma 12, with \( W_n^* = (V_n - (n - 1)K_n, K_n, \ldots, K_n) \) and \( V_n \), the scenario-\( n \) expected overall effort is

\[
TE^{(n)}(W_n^*, V_n, t^c) = \sum_{j=1}^{n} w_{n,n+1-j} \int_{t^c}^{b} J(t)g(j,n)(t, t^c)dt - nt^cK_n + t_0(V - V_n)
\]

\[
= n \int_{t^c}^{b} J(t) \left\{ \frac{[V_n - (n - 1)K_n]G^{n-1}(t, t^c)}{G_n[1 - G^{n-1}(t, t^c)]} \right\} g(t, t^c)dt - nt^cK_n + t_0(V - V_n)
\]

\[
= n \left\{ \int_{t^c}^{b} J(t)(V_n - nK_n)G^{n-1}(t, t^c)g(t, t^c)dt + K_n \int_{t^c}^{b} J(t)g(t, t^c)dt \right\} - nt^cK_n + t_0(V - V_n)
\]

\[
= n \int_{t^c}^{b} J(t)(V_n - nK_n)G^{n-1}(t, t^c)g(t, t^c)dt + t_0(V - V_n)
\]

where the last equality follows from \( \int_{t^c}^{b} J(t)g(t, t^c)dt = t^c \) (which is \( \delta_1(t^c) \), as will be shown in
Lemma 13, and we have $\delta_1(t^c) = t^c$ there). For convenience, for each positive integer $k$, define

$$\delta_k(t^c) = k \int_{t^c}^{b} J(t) G^{k-1}(t, t^c) g(t, t^c) dt.$$  

The following lemma states the properties of $\delta_k(t^c)$, the proof of which is relegated to the end of the appendix.

**Lemma 13.** (i) For all $k \geq 1$, $\delta_k(t^c) < \delta_{k+1}(t^c)$.

(ii) For all $k \geq 1$, $\delta_k(t^c)$ is strictly increasing in $t^c \in [a, b)$ and $\lim_{t^c \to b^-} \delta_k(t^c) = b$.

(iii) $\delta_1(t^c) = t^c$.

Thus, $TE^{(n)}(W_n^*, V_n, t^c)$ of (23) can be expressed as

$$TE^{(n)}(W_n^*, V_n, t^c) = (V_n - nK_n)\delta_n(t^c) + t_0(V - V_n).$$

Therefore, by Lemma 12, the highest expected overall effort inducible under scenario budgets $V = (V_1, V_2, \ldots, V_N)$ and scenario minimum prizes $K = (K_1, K_2, \ldots, K_N)$ can be expressed as follows:

$$TE(V, K, t^c) = \sum_{n=0}^{N} \binom{N}{n} (1 - F(t^c))^n F^{N-n}(t^c) TE^{(n)}(W_n^*, V_n, t^c)
\begin{align*}
&= \sum_{n=1}^{N} \binom{N}{n} (1 - F(t^c))^n F^{N-n}(t^c) \left[ (V_n - nK_n)\delta_n(t^c) + t_0(V - V_n) \right] + F^N(t^c)t_0V \\
&= F^N(t^c)t_0V + \sum_{n=1}^{N} \binom{N}{n} (1 - F(t^c))^n F^{N-n}(t^c) n \left[ \frac{(V_n - nK_n)\delta_n(t^c)}{t_0(V - V_n)} \right] \\
&= F^N(t^c)t_0V + N(1 - F(t^c)) \sum_{n=1}^{N} \frac{p_n(t^c)}{n} [(V_n - nK_n)\delta_n(t^c) + t_0(V - V_n)].
\end{align*}$$

**Step 2:** The final step is to find the optimal budget vector $V$ and minimum prize vector $K$. Formally, the organizer’s problem now reduces to

$$\max_{V,K} TE(V, K, t^c) = N(1 - F(t^c)) \sum_{n=1}^{N} \frac{p_n(t^c)}{n} [(V_n - nK_n)\delta_n(t^c) + t_0(V - V_n)] + F^N(t^c)t_0V \quad (24)$$

subject to

$$V_n \leq V, \forall n; \quad (25)$$

38
\[ K_n \leq \frac{V_n}{n}, \forall n, \text{ with equality when } n = 1; \]
\[ \sum_{n=1}^{N} p_n(t^c) K_n = 0. \]

For convenience, call this Problem (O). There are two cases to consider.

**Case 1**: \( \delta_N(t^c) \leq t_0 \), i.e., \( t^c \leq \hat{t} \). Recall that \( \delta_N(t^c) > \delta_n(t^c) \), for all \( n \) with \( 1 \leq n < N \), from Lemma 13. Thus, in this case, \( \delta_n(t^c) \leq t_0 \) for all \( n \geq 1 \) (with strict inequality when \( n < N \)). Imagine the following two-step maximization. In the first step, solve the optimal \( V \), assuming that \( K \) is fixed; in the second step, find the optimal \( K \).

In the first step, when \( K \) is fixed, the term in the square bracket of \( TE(V, K, t^c) - (V_n - nK_n)\delta_n(t^c) + t_0(V - V_n) \) is maximized by setting \( V_n = nK_n \), \( \forall n \geq 1 \). The reason is that the organizer should make the coefficient associated with \( t_0 \), which is \( V - V_n \), as large as possible, because \( \delta_n(t^c) \leq t_0 \); constrained by (26), with \( K_n \) fixed, the best she can do is to set \( V_n = nK_n \). Substituting \( V_n = nK_n \) into the objective function of Problem (O), we have

\[
TE(K, t^c) = N(1 - F(t^c)) \sum_{n=1}^{N} p_n(t^c) \frac{t_0}{n} V_n - nK_n + N(t^c) t_0 V
\]

\[
= N(1 - F(t^c)) \sum_{n=1}^{N} p_n(t^c) \left( \frac{t_0 V_n}{n} - K_n \right) + N(t^c) t_0 V
\]

\[
= t_0 V - N(1 - F(t^c)) \sum_{n=1}^{N} p_n(t^c) K_n = t_0 V,
\]

where the third equality uses (27).

Thus, in the second step when searching for the optimal \( K \), the highest possible total effort must be \( t_0 V \). This can be achieved by cancelling the contest. Thus, cancelling the contest is optimal, which can be done by, for example, setting \( V_n(t^c) = 0 \) and \( W_n(t^c) = 0 \) for all \( n \geq 1 \).

**Case 2**: \( \delta_N(t^c) > t_0 \), i.e., \( t^c > \hat{t} \). For notational simplicity, let \( \tilde{V} = (\tilde{V}_1, \ldots, \tilde{V}_N) \) and \( \tilde{K} = (\tilde{K}_1, \tilde{K}_2, \ldots, \tilde{K}_N) \) be the optimal budget vector and the minimum prize vector, respectively. By Lemma 12, to prove Proposition 4, it suffices to show that the budget vector \( \tilde{V} = V^* = (V, \ldots, V) \) and that

\[
\tilde{K} = K^*(t^c) = (V, \frac{V}{2}, \ldots, \frac{V}{N - 1}, 0).
\]

(28)

It is obvious that \( \tilde{V}_1 = \tilde{K}_1 \). Since \( \tilde{K} \) satisfies (27), we have \( \sum_{j=1}^{N} p_j(t^c) \tilde{K}_j = 0 \).

If \( t^c = a \), then only scenario-N matters, and so \( K_N \) must be zero in Problem (O). Therefore,
Problem (O) becomes
\[
\max_{V,K} TE(V,K,t^c) = V_N \delta_N(t^c) + t_0(V - V_N), \text{ s.t. } 0 = NK_N \leq V_N \leq V.
\]
Since \( \delta_N(t^c) > t_0 \), \( V_N = V \) is optimal. Thus, in this case, the optimum must be \( K_N = 0 \) and \( V_N = V \). Notice that \( V^* = (V, \ldots, V) \) and \( K = K^*(a) \) in Proposition 4 clearly induce \( t^c = a \); moreover, \( K^*_N(a) = 0 \), which is consistent with the fact that \( K_N \) must be 0 when \( t^c = a \). Finally, with \( K = K^*(a) \), \( t^c = a \) is the only solution to (27) (see footnote 15 for relevant discussion). Thus, if \( t^c = a \), the solution of \( V^* = (V, \ldots, V) \) and \( K = K^*(a) \) in Proposition 4 is optimal.

Now assume \( t^c > a \). The proof consists of three steps.

Step 2.1: We first show that \( \bar{V}_N = V \). Notice that in Problem (O), fixing the minimum prize vector \( K \) and the budgets \( V_1, \ldots, V_{N-1} \), the only relevant part for optimization in the objective function (24) of Problem (O) is
\[
N(1 - F(t^c)) \frac{p_N(t^c)}{N} [V_N - NK_N] \delta_N(t^c) + t_0(V - V_N)].
\]
Since \( \delta_N(t^c) > t_0 \), the organizer should make \( V_N = NK_N \) as large as possible, which is achieved by setting \( V_N = V \), given that \( K_N \) is fixed. Therefore, in Problem (O), fixing the minimum prize vector \( K \) and the budgets \( V_1, \ldots, V_{N-1} \), it is optimal to set \( V_N = V \). It follows that this must also be true for the optimal solution to Problem (O)—that is, \( \bar{V}_N = V \).

Step 2.2: Having shown that \( \bar{V}_N = V \), we proceed to show that fixing the budgets \( V_1, \ldots, V_{N-1} \), it is optimal to set \( \bar{K}_n = \frac{V_n}{n} \) for all \( n = 1, \ldots, N - 1 \). Let \( \bar{K} = (\bar{K}_1, \bar{K}_2, \ldots, \bar{K}_N) \) be the optimal solution, when fixing \( V_1, \ldots, V_{N-1} \). Our goal is to show that \( \bar{K}_n = \frac{V_n}{n} \) for all \( n \leq N - 1 \) (and the rest \( K_N \) is determined by (27)).

To see this, suppose, to the contrary, that there is some \( n < N \) such that \( \bar{K}_n < \frac{V_n}{n} \). Note that \( n \) must be strictly larger than 1, because, by definition, \( \bar{K}_1 = V_1 \). Consider the minimum prize vector \( \bar{K}' = (\bar{K}_1, \ldots, \bar{K}_{n-1}, \bar{K}'_n, \bar{K}_{n+1}, \ldots, \bar{K}_{N-1}, \bar{K}'_N) \) constructed as follows. Choose \( \varepsilon > 0 \) such that \( \bar{K}'_n = \bar{K}_n + \varepsilon \leq \frac{V_n}{n} \). Define \( \bar{K}'_N = \bar{K}_N - \frac{p_n(t^c)}{p_N(t^c)} \varepsilon \). Obviously, \( \bar{K}' \) is also feasible in Problem (O). The difference in expected overall efforts induced by \( \bar{K}' \) and \( \bar{K} \) is \( TE(t^c; \bar{K}') - TE(t^c; \bar{K}) \). Our goal is to show that this difference is strictly positive.

To this end, notice that the difference can be expressed as (dropping the common factor \( N(1 - F(t^c)) \))
\[
p_N(t^c) \delta_N(t^c) \frac{p_n(t^c)}{p_N(t^c)} \varepsilon - p_n(t^c) \delta_n(t^c) \varepsilon = p_n(t^c) \varepsilon (\delta_N(t^c) - \delta_n(t^c)) > 0,
\]
where the inequality follows from Lemma 13. This further implies that when \( t^c > a \) and fixing the budgets \( V_1, \ldots, V_{N-1} \), at the optimum, \( \bar{K}_n = \frac{V_n}{n}, \forall n < N \). \( \bar{K}_N \) is then determined by (27) by
substituting $\tilde{K}_n = \frac{V_n}{n}$, $\forall n < N$.

Step 2.3: Given step 2.1 and step 2.2, Problem (O) is equivalent to the following Problem (O-E), which is done by substituting $V_N = V$ and $K_n = \frac{V_n}{n}$ for all $n < N$ into Problem (O).

$$\max_{V_{-N}, K_N} TE(V_{-N}, K_N, t^c) = N(1 - F(t^c)) \left[ \frac{p_N(t^c)(V - NK_N)\delta_N(t^c)}{n} + \sum_{n=1}^{N-1} p_n(t^c)t_0(V_n - V_0) \right] + F^N(t^c)t_0V \quad (29)$$

subject to

$$V_n \leq V, \, \forall n \leq N - 1,$$

$$p_N(t^c)K_N + \sum_{n=1}^{N-1} p_n(t^c)\frac{V_n}{n} = 0,$$

where $V_{-N}$ denotes the $(N-1)$-vector $(V_1, \ldots, V_{N-1})$.

The last step of proving Proposition 4 is to show that $V_n = V$ for all $n \leq N - 1$ is optimal in Problem (O-E), and the corresponding $K_N$ must be $\frac{V_N}{N}(1 - \frac{1 - F^N(t^c)}{(1 - F(t^c))^N})$ by (31). The idea is similar to step 2.2. Slightly abusing notation, we use $(\tilde{V}_{-N}, \tilde{K}_N)$ to denote the solution to Problem (O-E), where $\tilde{V}_{-N} = (\tilde{V}_1, \ldots, \tilde{V}_{N-1})$. Our goal is to show that $\tilde{V}_{-N} = (V, \ldots, V)$ (and the corresponding $\tilde{K}_N = \frac{V_N}{N}(1 - \frac{1 - F^N(t^c)}{(1 - F(t^c))^N})$ is optimal in Problem (O-E).

To see this, suppose, to the contrary, that there is some $n \leq N - 1$ such that $\tilde{V}_n < V$. Consider $\tilde{V}_N' = (\tilde{V}_1, \ldots, \tilde{V}_{n-1}, \tilde{V}_n + \varepsilon, \tilde{V}_{n+1}, \ldots, \tilde{V}_{N-1})$ and $\tilde{K}_N' = \tilde{K}_N - \frac{p_n(t^c)\varepsilon}{p_N(t^c)n}$, where $\varepsilon > 0$ is small enough such that $\tilde{V}_n + \varepsilon \leq V$. It is easy to verify that $(\tilde{V}_N', \tilde{K}_N')$ is feasible in Problem (O-E). The difference in expected overall efforts induced by $(\tilde{V}_N, \tilde{K}_N)$ and $(\tilde{V}_N', \tilde{K}_N')$ is $TE(\tilde{V}_N, \tilde{K}_N, t^c) - TE(\tilde{V}_N', \tilde{K}_N', t^c)$. Our goal is to show that this difference is strictly positive.

To this end, notice that the difference can be expressed as (dropping the common factor $N(1 - F(t^c))$)

$$p_N(t^c)\delta_N(t^c)\frac{p_n(t^c)\varepsilon}{p_N(t^c)n} - p_n(t^c)t_0\frac{\varepsilon}{n} = p_n(t^c)\frac{\varepsilon}{n} (\delta_N(t^c) - t_0) > 0,$$

where the inequality follows from Lemma 13. Thus, in Problem (O-E), $\tilde{V}_{-N} = (V, \ldots, V)$ and $\tilde{K}_N = \frac{V_N}{N}(1 - \frac{1 - F^N(t^c)}{(1 - F(t^c))^N})$. The proof completes. \(\square\)

**Proof of Lemma 9:** We only need to show the case when $\hat{t} < a$, as the other case (when $\hat{t} \geq a$)
has been shown in the main text. Notice that when $t^c > \hat{t}$, we have

$$TE^*(t^c) = N \left( 1 - F(t^c) \right) \left[ \frac{p_N(t^c)}{N^2} \left( \frac{V - NK_N}{N - 1} \delta_N(t^c) \right) + F^N(t^c) t_0 V \right]$$

$$= N \left( 1 - F(t^c) \right) \left[ \frac{p_N(t^c)}{N^2} \left( \frac{V - NK_N}{N - 1} \delta_N(t^c) \right) + F^N(t^c) t_0 V \right]$$

$$= V \left( 1 - F^N(t^c) \right) \delta_N(t^c) + F^N(t^c) t_0 V. \quad (32)$$

Thus, when $t^c > a > \hat{t}$,

$$\frac{dTE^*(t^c)}{dt^c}$$

$$= V f(t^c) \left[ -NF^{N-1}(t^c) \delta_N(t^c) + \frac{(1 - F^N(t^c)) d\delta_N(t^c)}{dt^c} + NF^{N-1}(t^c) t_0 \right]$$

$$= NV f(t^c) \left[ -NF^{N-1}(t^c) \delta_N(t^c) + \frac{(1 - F^N(t^c)) d\delta_N(t^c)}{dt^c} + NF^{N-1}(t^c) t_0 \right]$$

$$= NV f(t^c) \left[ -NF^{N-1}(t^c) \delta_N(t^c) + \frac{(1 - F^N(t^c)) d\delta_N(t^c)}{dt^c} + NF^{N-1}(t^c) t_0 \right]$$

where the second equality uses (35). Thus, given the continuous differentiability of $TE^*(t^c)$ in $[a, b)$, the right derivative of $TE^*(t^c)$ at $t^c = a$:

$$TE^a(a^+) = N(N - 1) V f(a) \int_a^b F^{N-2}(t)(1 - F(t))^2 dt > 0,$$

which implies that $t^c = a$ is not optimal. \Box

**Proof of Lemma 10:** (i) Notice that

$$\frac{d}{dt^c} \left( \sum_{n=1}^{N-1} \frac{p_n(t^c) v_n}{p_N(t^c)} \right) = \frac{d}{dt^c} \left[ \sum_{n=1}^{N-1} \left( N - 1 \right) \left( \frac{F(t^c)}{1 - F(t^c)} \right)^{N-n} v_n \right]$$

$$= \frac{d}{dt^c} \left( \frac{F(t^c)}{1 - F(t^c)} \right) \cdot \sum_{n=1}^{N-1} \left( N - n \right) \left( \frac{F(t^c)}{1 - F(t^c)} \right)^{N-n-1} v_n,$$

so it suffices to show that

$$\sum_{n=1}^{N-1} \left( N - n \right) \left( \frac{F(t^c)}{1 - F(t^c)} \right)^{N-n-1} v_n > 0$$

for any $t^c \in (t_0^b, b). \quad (34)$
Proof of Lemma 13
which is obviously true, because
To this end, note that since \( \sum_{n=1}^{N-1} p_n(t_0^c)v_n \geq 0 \) and \( v_1 \geq \cdots \geq v_{N-1} \), either \( v_1 > 0 \) and \( v_{N-1} \geq 0 \) or there exists some integer \( j \in \{2, \ldots, N-1\} \) such that \( v_k \geq 0 \) and \( v_{k'} < 0 \) for all \( k \in \{1, \ldots, j-1\} \) and all \( k' \in \{j, \ldots, N-1\} \). In the former case, (34) obviously holds. For the latter case, \( v_1 \) must be strictly positive and notice that

\[
\sum_{n=1}^{N-1} (N-n) \left( \frac{F(t^c)}{1-F(t^c)} \right)^{n-1} v_n
\]

\[
= \frac{\sum_{n=1}^{N-1} (N-n)p_n(t^c)v_n}{F(t^c)(1-F(t^c))^{N-2}} = \frac{\sum_{n=1}^{N-1} (N-n)p_n(t^c)v_n + \sum_{n=j}^{N-1} (N-n)p_n(t^c)v_n}{F(t^c)(1-F(t^c))^{N-2}}
\]

\[
> \frac{\sum_{n=1}^{N-1} (N-j)p_n(t^c)v_n + \sum_{n=j}^{N-1} (N-n)p_n(t^c)v_n}{F(t^c)(1-F(t^c))^{N-2}} = \frac{(N-j) \sum_{n=1}^{N-1} p_n(t^c)v_n}{F(t^c)(1-F(t^c))^{N-2}}.
\]

Thus, in this case, one only needs to show

\[
\sum_{n=1}^{N-1} p_n(t^c)v_n > 0 \text{ for any } t^c \in (t_0^c, b),
\]

which must be true, because in this case, \( v_N < 0 \) (because \( v_{k'} < 0 \) for all \( k' \in \{j, \ldots, N-1\} \)), which implies that if \( \sum_{n=1}^{N-1} p_n(t_1^c)v_n \leq 0 \) for some \( t_1^c \in (t_0^c, b) \), then \( \sum_{n=1}^{N-1} p_n(t_1^c)v_n < 0 \), which contradicts the fact that \( \sum_{n=1}^{N-1} p_n(t_0^c)v_n = 0 \) and \( \sum_{n=1}^{N-1} p_n(t_1^c)v_n \geq \sum_{n=1}^{N-1} p_n(t_0^c)v_n \) (because \( t_1^c > t_0^c \) and \( \sum_{n=1}^{N-1} p_n(t^c)v_n \) is increasing in \( t^c \), as established in the proof of Lemma 1).

(ii) Similar to the proof of part (i), it suffices to show that

\[
\sum_{n=1}^{N-1} (N-n) \left( \frac{F(t^c)}{1-F(t^c)} \right)^{n-1} v_n > 0 \text{ for any } t^c \in [a, b),
\]

which is obviously true, because \( v_1 > 0 \) and \( v_{N-1} \geq 0 \) (so \( v_2 \geq \cdots \geq v_{N-1} \geq 0 \)). \( \square \)

Proof of Lemma 13: Part (i) follows from equation (11) in Liu and Lu [10]. For part (ii) and
part (iii), note that
\[
\delta_k(t^c) = \int_{t^c}^b J(t)kG^{k-1}(t, t^c)g(t, t^c)dt = k \int_{t^c}^b (t - \frac{1 - G(t, t^c)}{g(t, t^c)})G^{k-1}(t, t^c)g(t, t^c)dt
\]
\[
= \int_{t^c}^b tdG^k(t, t^c) - k \int_{t^c}^b G^{k-1}(t, t^c)(1 - G(t, t^c))dt
\]
\[
= tG^k(t, t^c)|_{t=t^c} - \int_{t^c}^b G^k(t, t^c)dt - k \int_{t^c}^b G^{k-1}(t, t^c)(1 - G(t, t^c))dt
\]
\[
= b - k \int_{t^c}^b G^{k-1}(t, t^c)dt + (k - 1) \int_{t^c}^b G^k(t, t^c)dt.
\]
Thus, \(\lim_{t^c \to b^-} \delta_k(t^c) = b\) for all \(k \geq 1\). Moreover, \(\delta_1(t^c) = t^c\), which is strictly increasing in \(t^c\).

When \(k \geq 2\),
\[
\frac{d\delta_k(t^c)}{dt^c} = k(k - 1)f(t^c) \int_{t^c}^b G^{k-2}(t, t^c)(1 - G(t, t^c))^\frac{1 - F(t)}{(1 - F(t^c))^2}dt > 0. \tag{35}
\]
so \(\delta_k(t^c)\) is strictly increasing when \(k \geq 2\). \(\Box\)

References


