Optimal Prize Design in Team Contests*

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Abstract

This paper studies effort-maximizing prize designs in team contests with an arbitrary number of pairwise battles. The organizer rewards the teams contingent on battle outcomes subject to budget balance constraints. Our analysis fully accommodates heterogeneities across players and battles. We discover an innovative measure of the teams’ strengths. The optimal design is a majority-score rule with a headstart score for the weaker team: all battles are assigned team-invariant scores; the team collecting higher total scores from its winning battles wins the entire prize. The optimal design is robust to homogeneity-of-degree-zero contest technologies and contest temporal structure.

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1 Introduction

In many competitive circumstances, contenders from different teams compete in pairs on multiple fronts, and the winning team is determined by their overall performance in a series of battles. Such team contests with pairwise battles can be found in R&D races, sporting events with team titles, political campaigns, and many other environments (see, e.g., Fu, Lu, and Pan (2015), Häfner (2017)). In many of these competitions, a central question is how to appropriately design the prize allocation rule to incentivize a more productive effort supply. In this paper, we study the effort-maximizing prize design in an environment of multi-battle team contests.

In such contests, the best-of-$N$ rule (also called majority rule) is typically adopted, which treats opposed teams equally and allocates the entire prize to the team winning the majority of battles. This prize allocation rule depends neither on the teams’ identities nor on the order of wins, which can be found in sporting events with team titles and also in an election for the House of Representatives between Republicans and Democrats (see, e.g., Snyder (1989), Klumpp and Polborn (2006)).

However, in general, a prize rule can be contingent on both the teams’ identities and the full history of battle outcomes. For example, in an R&D race between a local research alliance and a foreign team, the foreign team usually has to outperform the local team by a sufficient margin to win the competition held by a local government. Moreover, the result may be determined by the composition of winning battles rather than the number of winning battles, as intuitively exemplified by the US presidential election. Interesting questions thus arise: Is there any theoretical rationale to adopt the majority rule beyond simplicity and fairness? What do the optimal prize rules look like in general? How does the optimal design react to the degree of asymmetry between teams and the heterogeneity across battles? How does the optimal rule respond to different contest environments, such as the number of battles, contest technology, and the contest’s temporal structure?

To address these questions, we study effort-maximizing prize allocation rules by
granting a contest organizer full flexibility to reward a team based on the full path of battle outcomes and its identity. We restrict our attention to the prize allocations that satisfy nonnegativity, monotonicity, and budget balance conditions, which imply that the prizes are nonnegative, additional victory is never detrimental, and the prize budget is always wholly awarded. We assume that two teams compete with each other, and each team consists of an equal (odd or even) number of players. Each player from one team is exogenously matched to his counterpart from the rival team, and the matched pairs compete head-to-head on their battlefields. A battle is modeled as a generalized Tullock contest. The team prize is a public good among its members and each player chooses his own effort to maximize his payoff. We say that a setting accommodates full-fledged heterogeneity when the contest technologies differ across battles and players are heterogeneous within and across teams in their marginal effort costs.

For ease of exposition and transparency in analysis, we begin our analysis by a baseline setting, in which the contest technology is uniform across battles and players on each team are homogeneous. Two teams can be asymmetric in terms of their players’ marginal effort costs. This partial-heterogeneity setting facilitates our understanding of how asymmetry between the two teams plays a role in the optimal prize design. There are two crucial steps in characterizing the optimal prize design. First, we show that a path-independent rule can replicate the expected total effort resulting from any path-dependent prize rule. In that sense, there is no loss of generality to restrict our search to the prize rules, which are solely contingent on the number of wins. Second, we reformulate the path-independent prize design problem to an incremental prize design problem, and the latter can be solved using the linear programming technique. We then recover the optimal prize structure in our original prize design problem.

The aforementioned procedure renders a closed-form solution with intuitive interpretations. The optimal prize design is a majority rule with a headstart, which
allocates the entire prize to the team winning a sufficient number of battles and favors the weaker team by awarding it a headstart in terms of an initial number of wins. Equivalently, the optimal rule rewards the entire prize to the stronger team when it wins at least $K_S (> N/2)$ battles; otherwise, the weaker team obtains the entire prize (Theorem 1). Our comparative statics analysis reveals that the winning threshold $K_S$ increases with the degree of asymmetry between teams and the discriminatory power associated with the Tullock contest. When two teams are sufficiently homogeneous, the optimal prize rule boils down to the conventional best-of-$N$ prize allocation rule.\footnote{When the number of battles is odd, the effort-maximizing allocation rule coincides with a traditional majority rule; when the number of battles is even, the traditional majority rule is not well-defined.} When two teams are sufficiently asymmetric, we find that it is even possible that the weaker team is more likely to win the whole competition at the optimum.

We proceed to the general analysis by accommodating \textit{full-fledged heterogeneity}, i.e., players on each team can be heterogeneous and contest technology can differ across battles. Due to the battle heterogeneity, the optimality of path-independent rules no longer holds. Moreover, it is also difficult to tell which team is stronger intuitively. Nevertheless, we discover a measurement to evaluate a team’s strength by aggregating its players’ strengths. We then adopt an iterative adjustment method to derive the optimal rule, which takes two steps. First, we show that the optimal design must be a vertex solution through the linear programming technique, which renders the structures that the optimal rules must take. The entire reward must be fully assigned to one team while the other team receives nothing. Second, we iteratively eliminate the sub-optimal prize rules, which leads to the closed-form optimal prize rule.

With \textit{full-fledged heterogeneity}, we first provide an innovative measure to determine the teams’ strengths.\footnote{Theorem 2 provides a measurement to determine a team’s strength by aggregating its players’ strengths.} While it looks like a daunting task to solve the optimal design problem, the optimal design takes a surprisingly simple and elegant form of a
majority-score rule with a headstart score to the weaker team. All battles are assigned scores, which are in general different across battles. Both teams earn the same score from winning a battle. The weaker team is given an initial score to start, and the team collecting higher total scores from its winning battles wins the entire prize (Theorem 2). The optimal prize rule works through two instruments: each battle’s score and the headstart score for the weaker team, which manage battle heterogeneity and team asymmetry, respectively. The battle score is proportional to the effectiveness in effort inducement and the unbalanceness of the battle. Assigning a higher score to a battle with higher effectiveness and higher unbalanceness better incentivizes more effective battle and counterbalances the asymmetry of players. In particular, when a battle grows more unbalanced in favor of the stronger (resp. weaker), on one hand, the optimal rule assigns a larger score to this battle so that the matched pair can be better incentivized; on the other hand, the optimal rule raises (resp. reduces) the headstart to the weaker team, in order to balance the overall contest. Our results generalize the traditional majority rule while incorporating full-fledged heterogeneity, and rationalize the prevalence of (generalized) majority rules in team contests. These findings can provide guidance on how to design incentive schemes in competitions between teams.

To check the robustness of our results, we extend our analysis in two dimensions. First, we validate our main results under contest technologies that satisfy the homogeneity-of-degree-zero property. Second, we allow for a completely flexible temporal structure of the contest: battles can be carried out (completely or partially) simultaneously. We therefore validate the generality of the central insights of our results.

Our paper primarily belongs to the literature on multi-battle contests between

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3. The effectiveness is measured by ratio of induced effort to prize spread and the unbalancedness of the battle is measured by reciprocal of product of players’ winning chances.
4. When the battle becomes more unbalanced, the team of the weaker player needs to win lower total scores to win the contest. At the same time, winning the concerned battle earns a higher score. Thus, the weaker player is better incentivized, which further disciplines the stronger player.
individuals or teams. In the setting of dynamic contests between individuals, many studies validate the existence of a strategic momentum/discouragement effect, including Harris and Vickers (1987); Ferrall and Smith (1999); Klumpp and Polborn (2006); Konrad and Kovenock (2009); Gelder (2014); and Gauriot and Page (2019), among others. Other papers focus on prize designs in dynamic contests between individual players, including Feng and Lu (2018); Jiang (2018); Sela and Tsahi (2020); and Clark and Nilssen (2020), among others. In the team contest setting, Fu, Lu, and Pan (2015) establish history independence under a majority prize allocation rule.\(^5\)

We generalize the history-independence result by allowing an arbitrary number of battles and general prize allocation rules. Hähner (2017, 2020) analyzes tug-of-war contests between two teams. Barbieri and Serena (2019) analyze the winning team’s effort and find that a simultaneous contest maximizes winners’ effort. Konishi, Pan, and Simeonov (2019) analyze equilibrium player ordering in team contests. We complement these studies by characterizing the optimal prize allocation rules in team contests.

Our paper is also close to the studies on single-battle group contests. Many of these assume that a team’s win is a public good among its team members, including Baik, Kim, and Na (2001); Barbieri, Malueg, and Topolyan (2014); Topolyan (2014), Chowdhury, Lee, and Topolyan (2016); and Eliaz and Wu (2018). In their settings, team performance is either evaluated by the minimum (maximum) performance of individual group members or measured by an additively separable (possibly non-linear) function that aggregates group members’ effort into group output. Crutzen, Flamand, and Sahuguet (2020) further study intra-team prize allocation rules regarding players’ incentives and teams’ winning chances. In their setting, teams’ outputs are determined by a constant-elasticity-of-substitution production function. While in ours, a team’s performance is instead evaluated by full path of battle outcomes, which

\(^5\)Klumpp, Konrad, and Solomon (2019) study Blotto games with sequential battles and a majoritarian objective. They find that the history-independence result extends to their setting with two individual contestants.
differs structurally from most studies in this stream of the literature. Moreover, we focus on the effort-maximizing prize design in team contests.

Our paper also speaks to the literature on biased contests and their optimal design, including Li and Yu (2012); Pastine and Pastine (2012); Franke, Kanzow, Leininger, and Schwartz (2013); Seel and Wasser (2014); Fu and Wu (2020), among others. These studies mainly concern the design of multiplicative biases and additive headstarts in Tullock contests. Our paper differs from the literature in two aspects. First, we consider a team contest setting with multiple pairwise battles. Second, we study the optimal prize allocation rule based on battle outcomes. We find that a biased prize allocation rule can be optimal when teams are sufficiently heterogeneous. Our finding thus complements the literature on biased contests.

The rest of the paper is organized as follows. In Section 2, we set up the model. We study the optimal prize design with homogeneous battles in Section 3 and further accommodate full heterogeneity in battles and players in Section 4. Section 5 discusses implications and extensions. Section 6 concludes. The appendix collects the technical proofs. Some technical details are relegated to an online appendix.

2 The Model

Two teams, indexed by $A$ and $B$, compete in a contest with $N$ pairwise battles. Each team consists of $N$ risk-neutral players, and each player on one team is matched to his opponent from the rival team. The matched players compete head-to-head on $N$ disjoint battlefields. A player on team $i \in \{A, B\}$ is indexed by $i(t)$ if he is assigned to battle $t$, where $t \in \mathcal{N}$ and $\mathcal{N} \triangleq \{1, 2, ..., N\}$ denote the set of all battles. In each component battle $t$, two matched players simultaneously exert their efforts, $x_{A(t)}$ and $x_{B(t)}$. Player $i(t)$’s effort entry $x_{i(t)}$ incurs a constant marginal cost $c_{i(t)} > 0$.

We assume that component battles are carried out completely successively, and
each component battle is modeled as a Tullock contest with complete information. In battle $t$, two matched players compete in a Tullock contest, i.e., player $i(t)$ wins with a probability of $p_{i(t)}(x_{i(t)}, x_{j(t)}) = x_{i(t)}^{r(t)}/(x_{i(t)}^{r(t)} + x_{j(t)}^{r(t)})$, where $i, j \in \{A, B\}$ and $r(t) > 0$ denotes the discriminatory power of battle $t$.

The contest organizer has a fixed budget, which is fully divisible and normalized as 1. The organizer can reward teams contingent on the full path of battle outcomes and fully commit to the prize rule. We assume that the team prize is a public good, and all players on the team equally value it. Meanwhile, there is no private benefit for an individual player from winning his own battle. In other words, a player can only benefit from his team prize. This divisible public good setup is prevalent in team contests in the real world, for instance, gaining points in a rating system for sports contests with team titles and the market share authorized by the government when research alliances compete to develop an innovative product.

The contest organizer aims to maximize the expected total effort of players from both teams by choosing a prize allocation rule. The prize allocation rule can be contingent on the contest outcomes—i.e., the full path of battle outcomes. To better illustrate, we denote the set of winning battles of the concerned team $i$ as subset $W^i(\in 2^N), \forall i \in \{A, B\}$. Apparently, if team $i$ wins battles $W^i$, team $j$ must win the remaining battles $W^j = N \setminus W^i$, and there are $2^N$ possible outcomes in total. We use $v^i(W^i)(\geq 0)$ to denote the prize allocated to team $i$. Our prize allocation rule $(v^A(\cdot), v^B(\cdot))$ can be path-dependent since a team’s prize is contingent on the full path of battle outcomes or the set of battles it wins, i.e., $v^i(\cdot) : 2^N \rightarrow [0, 1]$.

We now turn to the prize allocations we study. Throughout the paper, we restrict our attention to the prize allocations that satisfy nonnegativity, monotonicity, and budget balance conditions.

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6We will show that our analysis and results remain robust under alternative contest technologies and temporal structures in Section 5.

7We assume that the prize to a team under each winning outcome must be non-negative. See Moldovanu, Sela, and Shi (2012) and Liu, Lu, Wang, and Zhang (2018) for negative prizes.
Assumption 1 (i) Nonnegativity. $v^i(W^i) \geq 0, \forall W^i$.

(ii) Monotonicity. $(W'^i) \subseteq W^i \implies v^i((W'^i)) \leq v^i(W^i)$.

(iii) Budget Balance. $v^i(W^i) + v^j(W^j) = 1, \forall W^i$ and $W^j = N \setminus W^i$.

The nonnegativity condition requires that the prizes are non-negative, and the monotonicity condition requires that additional victory is never detrimental. The budget balance condition requires that the prize budget is always exhausted, which plays a crucial role in our analysis. Due to the budget balance condition, team $B$'s prize can be fully determined by team $A$'s prize. Therefore, to search for the effort-maximizing prize allocation rule, it suffices to focus on team $A$'s prize allocation rule $v^A(\cdot)$.

A prize allocation rule $(v^A(\cdot), v^B(\cdot))$ is path-independent if and only if a team’s prize is solely contingent on the number of battles each team wins, i.e., $v^i(\cdot) : N \to [0,1]$. Let $k^A$ be the number of winning battles of team $A$. Therefore, $v^A(W^A)$ and $v^A(k^A)$ respectively denote the prize to team $A$ for path-dependent and path-independent allocation rules. We will study how to design the effort-maximizing prize-allocation rule subject to Assumption 1.\(^8\)

3 Baseline Setting with Homogeneous Battles

In this section, we analyze a baseline setting with homogeneous battles. The contest technology is uniform across all battles and players on each team are homogenous, while the two competing teams can be asymmetric in terms of players’ marginal effort costs. Without loss of generality, we assume that team $A$ is stronger than team $B$—i.e., team-$A$ players have a lower marginal cost of effort denoted by $c_A \in (0,1]$—while we normalize team-$B$ players’ marginal effort cost as $c_B = 1$. The discriminatory power of all battles are fixed at $r \in (0, +\infty]$. This setting allows us to concentrate on the asymmetry between rivalry teams.

\(^8\)When the prize is an indivisible good, such as a trophy, our prize design is about the teams’ winning probabilities of the prize.
We first derive the expected total effort in terms of prize allocation rules and formulate the organizer’s problem in Section 3.1. In particular, we establish that the expected total effort resulting from any path-dependent allocation rule can be duplicated by a path-independent allocation rule. Based on the result, we derive optimal prize design in Section 3.2. In Section 3.3, we provide comparative statics and some other properties of the optimal prize allocation rule.

### 3.1 Optimality of The Path-independent Allocation Rule

In this subsection, we illuminate that the organizer’s effort-maximizing design problem boils down to maximizing the weighted sum of the prize spreads, which are directly determined by prize allocations. In particular, we find that the total expected effort depends on the contest outcomes only through the number of winning battles—i.e., the order of wins does not affect expected total effort, which leads to the optimality of path-independent allocation rules. Based on this, we continue to analyze the optimal prize design problem in Section 3.2.

We begin by analyzing path-dependent allocation rules $v^A(\cdot) : 2^N \to [0, 1]$, which are contingent on the path/history of battle outcomes. The state of the contest before battle $t$ is summarized by a tuple $(\mathcal{N}_t, \mathcal{W}^A_t, \mathcal{W}^B_t)$, where $\mathcal{N}_t = \{1, 2, \cdots, t-1\}$ denotes the set of finished battles and $\mathcal{W}^i_t \subseteq \mathcal{N}_t$ the set of battles that team $i$ wins. Clearly, $\mathcal{W}^A_t \cup \mathcal{W}^B_t = \mathcal{N}_t$ and $\mathcal{W}^A_t \cap \mathcal{W}^B_t = \emptyset$. Because of the budget balance condition, it is without loss of generality to conduct the analysis from the perspective of team $A$. We therefore simply use $(\mathcal{N}_t, \mathcal{W}^A_t)$ to represent the state. We denote $\mathbb{E}x_{i(t)}(\mathcal{W}^A_t)$ as player $i(t)$’s expected effort in battle $t$ when the state is $(\mathcal{N}_t, \mathcal{W}^A_t)$. Therefore, the expected total effort with homogeneous battles can be written as follows.

$$
\text{TE}^{\text{Homo}}(v^A) \triangleq \sum_{t \in \mathcal{N}} \sum_{\mathcal{W}^A_t \subseteq \mathcal{N}_t} \text{Pr}(\mathcal{W}^A_t) [\mathbb{E}x_{A(t)}(\mathcal{W}^A_t) + \mathbb{E}x_{B(t)}(\mathcal{W}^A_t)],
$$

where $\text{Pr}(\mathcal{W}^A_t)$ is the probability that the state is $(\mathcal{N}_t, \mathcal{W}^A_t)$ when finishing the first
To calculate the expected total effort given by Equation (1), we adopt the following scheme: First, we consider a component battle at an arbitrary state \((N_t, W_t^A)\). Second, we analyze the contest dynamics. To proceed, we introduce some useful notations: Let \(U_i^A(W_{i+1}^A)\) denote the expected prize of the player on team \(A\) before battle \(t+1\) with history \(W_{i+1}^A\). Let \(V_i(W_t^A)\) denote player \(A(t)\)’s valuation of winning current battle \(t\) at state \((N_t, W_t^A)\). Since each player in a team contest turns up only once and does not bear any cost in future battles, the prize spread for battle \(t\) with \(W_t^A\) is merely

\[ V_i(W_t^A) = U_i^A(W_t^A \cup \{t\}) - U_i^A(W_t^A). \]

Consider a component battle \(t\) at state \((N_t, W_t^A)\), it is shown in Theorem 1 of Fu, Lu, and Pan (2015) that two matched players have the same valuation of winning the current battle \(t\), that is, \(V_i(W_t^A)\). In our context, we call this common valuation of winning as the effective prize spread of battle \(t\). The common value result leads to the following property.

**Property 1** Given players’ marginal effort costs and the contest technology,

- (i) \(E x_{A(t)}(W_t^A) + E x_{B(t)}(W_t^A) = \alpha V_i(W_t^A)\), regardless of the battle and the state, and \(\alpha\) only depends on \(c_A\) and \(r\);

- (ii) player \(A(t)\) wins an arbitrary battle with the same probability \(p_A\), regardless of the state, and \(p_A\) only depends on \(c_A\) and \(r\).

The analytical derivation of \(\alpha\) and \(p_A\) are provided in Equations (8) and (9) in the Appendix. The property stems directly from Theorem 1 in Fu, Lu, and Pan (2015), which depends crucially on the budget balance condition. The first part of Property 1 means that the sum of players’ expected efforts in each battle must be proportional to the prize spread. Moreover, the proportion \(\alpha\) is invariant to the battles as well as the states, which implies that the expected total effort must also be proportional to the expected total prize spread.
When the effective prize spread is positive, the second part of Property 1 follows directly from the so-called history independence result by Fu, Lu, and Pan (2015). When the effective prize spread is zero, i.e., $V_t(W_t^A) = 0$, we call such a battle $t|W_t^A$ a trivial battle wherein players simply make zero effort. In the following lemma, we establish that it is without loss of generality to assume that player $A(t)$ wins with probability $p_A$, even for trivial battles.

**Lemma 1 (Trivial Battle)** If battle $t|W_t^A$ is trivial, the expected total effort remains the same when the winning probability changes from $(0.5, 0.5)$ to $(p_A, 1 - p_A)$.

**Proof.** See the Appendix. ■

With the second part of Property 1, we can treat the outcome of each battle as independent lotteries, which inherits the merit of history independence in the literature. The probability that $W_t^A$ occurs is $P_{A}^{W_t^A}(1 - p_A)^t - 1 - |W_t^A|$ and the corresponding effective prize spread is $V_t(W_t^A)$. Therefore, the expected total effort in Equation (1) can be written as

$$\text{TE}^{Homo}(v^A) = \alpha \sum_{t \in \mathcal{N}} E_{W_t^A}(V_t(W_t^A)).$$

where $E_{W_t^A}(V_t(W_t^A)) = \sum_{W_t^A \subset \mathcal{N}_t} P_{A}^{W_t^A}(1 - p_A)^t - 1 - |W_t^A| V_t(W_t^A)$ denotes the ex ante expected effective prize spread of battle $t$.

We then move to study the contest dynamics. For that, we track players’ incentives by computing $U_t^A$ backward. At the end of the contest, i.e., $t = N$, the continuation value coincides with the prize, which yields the **boundary condition** for $U^A$: $U_N^A(W_{N+1}^A) = v^A(W_{N+1}^A)$. Given an arbitrary battle $t$ at state $(\mathcal{N}_t, W_t^A)$, if player $A(t)$ wins, the contest reaches state $(\mathcal{N}_{t+1}, W_t^A \cup \{t\})$ and the continuation value for team $A$’s players becomes $U_t^A(W_t^A \cup \{t\})$; if player $A(t)$ loses, the contest reaches state $(\mathcal{N}_{t+1}, W_t^A)$ and the continuation value correspondingly becomes $U_t^A(W_t^A)$. Since player $A(t)$ wins battle $t$ with probability $p_A$ regardless of the state (Property 1(ii)), we obtain the **recursive definition** for $U^A$: $U_{t-1}^A(W_t^A) =$
$p_A U_t^A(\mathcal{W}_t^A \cup \{t\}) + (1 - p_A) U_t^A(\mathcal{W}_t^A)$. Figure 1 illustrates the dynamics of the team contests.

Figure 1: Dynamics of the Team Contest

Based on the boundary condition and recursive definition, we derive the analytical formulas of $U_{t-1}^A(\mathcal{W}_t^A)$ and $V_t(\mathcal{W}_t^A)$ in terms of prizes $\{v^A(\mathcal{W}_t^A)\}_{\mathcal{W}_t^A \in 2^N}$ and further characterize $\sum_{t \in N} \mathbb{E}_{\mathcal{W}_t^A}(V_t(\mathcal{W}_t^A))$ in terms of prizes $\{v^A(\mathcal{W}_t^A)\}_{\mathcal{W}_t^A \in 2^N}$. The result is summarized as follows.

**Lemma 2** The aggregate effective prize spreads over all $N$ battles, $\sum_{t \in N} \mathbb{E}_{\mathcal{W}_t^A}(V_t(\mathcal{W}_t^A))$, is a linear function of $v^A(\mathcal{W}_t^A)$, $\forall \mathcal{W}_t^A \subset N$. Moreover,

$$\sum_{t \in N} \mathbb{E}_{\mathcal{W}_t^A}(V_t(\mathcal{W}_t^A)) = \sum_{\mathcal{W}_t^A} C(|\mathcal{W}_t^A|) v^A(\mathcal{W}_t^A),$$

where $C(|\mathcal{W}_t^A|) \triangleq |\mathcal{W}_t^A| p_A^{|\mathcal{W}_t^A| - 1} (1 - p_A)^{N - |\mathcal{W}_t^A| - (N - |\mathcal{W}_t^A|) p_A^{|\mathcal{W}_t^A|} (1 - p_A)^{N - |\mathcal{W}_t^A| - 1}}.$

**Proof.** See the Online Appendix. ■

By Lemma 2 and Equation (2), we have

$$\text{TE}^{Homo}(v^A) = \alpha \sum_{\mathcal{W}_t^A} C(|\mathcal{W}_t^A|) v^A(\mathcal{W}_t^A). \quad (3)$$
As a result, the coefficient of $v^A(\mathcal{W}^A)$ in the objective function is determined only by the number of winning battles $|\mathcal{W}^A|$, rather than the full path of battle wins. Therefore, it suffices to focus on prize allocations that are contingent on the number of battle wins. In Proposition 1, we formally establish the result.

**Proposition 1 (Optimality of The Path-independent Allocation Rule)** The expected total effort resulting from any path-dependent prize allocation rule can be duplicated by a path-independent allocation rule.

**Proof.** For any prize allocation rule contingent on the full path of battle outcomes, we can always construct an effort-equivalent prize allocation rule, which is solely contingent on the number of winning battles. The details are relegated into the Appendix.

Proposition 1 presents the first main result of this paper, which enables us to focus solely on the prize allocations contingent on the number of battle wins in the follow-up analysis of Section 3. By applying Proposition 1, we rewrite the organizer’s problem as follows.

**Corollary 1** The organizer’s optimization problem can be written as

\[
\max_{v^A(\cdot)} \text{TE}^{Homo}(v^A) = \alpha N \left\{ \sum_{k=0}^{N-1} \binom{N-1}{k} p^k_A (1 - p_A)^{N-k-1} [v^A(k + 1) - v^A(k)] \right\}
\]

s.t. $v^A(k) \in [0, 1]$, $v^A(k + 1) \geq v^A(k)$.  

(4)

**Proof.** The proof follows directly from Equation (3). Details are relegated to the Online Appendix.

We also provide an alternative way to derive the result of Corollary 1 as follows. Fix a player, assuming his teammates win $k$ out of $N - 1$ battles, his effective prize spread from winning the current battle equals $v^A(k + 1) - v^A(k)$. The probability of winning $k$ out of $N - 1$ battles is $\binom{N-1}{k} p^k_A (1 - p_A)^{N-k-1}$. Therefore, the ex ante
expected effective prize spread for each battle equals

\[
\sum_{k=0}^{N-1} \binom{N-1}{k} p_A^k (1 - p_A)^{N-k-1} [v^A(k+1) - v^A(k)].
\]

Since battles are homogeneous, the ex ante expected total prize spread is merely \( N \) times the ex ante expected effective prize spread for each battle.

From Corollary 1, the organizer’s prize allocation problem is reduced to a linear programming problem. In particular, it is without loss of generality to consider the path-independent allocation rules, which are solely contingent on the number of battle wins. We therefore simplify our notations as follows: The state of the contest before battle \( t \) is summarized by a tuple \( (t-1, k_A^t) \), where \( k_i^t \in [0, t-1] \) denotes the number of battles that team \( i \) wins, \( i \in \{A, B\} \). Clearly, \( k_B^t = t - 1 - k_A^t \). \( U_{t}^A(k_{t+1}^A) \) denotes the expected prize of team \( A \) when its players win \( k_{t+1}^A \) of the first \( t \) battles. \( V_t(k_{t}^A) = U_{t}^A(k_{t}^A + 1) - U_{t}^A(k_{t}^A) \) denotes the effective prize spread for battle \( t \) when team \( A \) wins \( k_{t}^A \) of the first \( t - 1 \) battles. In the following analysis of Section 3, we will focus on the optimization problem in (4) to derive the optimal rule.\(^9\)

### 3.2 Optimal Design

We first introduce the following definition, which is helpful in our follow-up analysis.

**Definition 1** (Majority Rule with a Headstart) Team \( A \) will be allocated the entire prize if it wins at least \( K \times (\frac{N}{2}) \) battles; otherwise, the entire prize is allocated to team \( B \). Equivalently, the weaker team is given a headstart in the form of \( 2K - N - 1 \) initial wins, and the entire prize is awarded to the team with the higher number of wins.

Heating up an unbalanced competition through headstart is well documented in the literature. Studies on biased contests often treat headstarts as a part of the design

\(^9\)In Section 4, Proposition 1 fails, we can no longer solely focus on the path-independent rules.
(see Li and Yu (2012); Pastine and Pastine (2012); and Seel and Wasser (2014), among others). We borrow the word *headstart* since the aforementioned rule simply gives the weaker team an additive headstart with size \(2K_S - N - 1\): If team \(A\) wins at least \(K_S\) battles, team \(B\) wins at most \(N - K_S\) battles, and hence team \(B\) gains at most \(K_S - 1\) wins after counting in the headstart; if team \(A\) wins at most \(K_S - 1\) battles, team \(B\) wins at least \(N - K_S + 1\) battles, and hence team \(B\) gains \(K_S\) scores after counting in the additive headstart. Intuitively, the contest organizer levels the playing field by offering a headstart to the weaker.

Note that the prize structures that we study incorporate the possibility that the prize to a team can take any value between 0 and 1. However, under the majority rule with a headstart, the size of the prize is restricted to two options, \(\{0, 1\}\), and therefore we can formally define the winner and loser of the whole contest. A team is called the winner if it acquires the entire prize. In particular, there is no draw under the majority rule with a headstart.

To tackle the organizer’s original optimization problem given by (4), we define the *auxiliary problem* below by letting \(\Delta v^A(k) \triangleq v^A(k + 1) - v^A(k)\) and \(F(k) \triangleq \binom{N - 1}{k} p_A^k (1 - p_A)^{N - 1 - k}\):

\[
\max_{\Delta v^A(\cdot)} \alpha N \left[ \sum_{k=0}^{N-1} F(k) \Delta v^A(k) \right]
\]

**s.t.** \( \Delta v^A(k) \geq 0, \sum_{k=0}^{N-1} \Delta v^A(k) \leq 1. \) (5)

The incremental prize \(\Delta v^A(k)\) is the marginal benefit from an additional victory for team \(A\) with \(k\) wins. If \(\Delta v^A(k) > 0\), the \((k+1)\)-th battle is *active*, or called *critical*, as the incremental prize is strictly positive. For the auxiliary problem, the organizer’s maximization objective is a linear function on incremental prizes, and the domain of decision variables \(\{\Delta v^A(k)\}_{k=0}^{N-1}\) is an \((N - 1)\)-simplex. We establish the equivalence between the original optimization problem in (4) and the auxiliary problem in (5) in
Lemma 3.

**Lemma 3 (Auxiliary Problem)** The organizer’s optimization problem described by (4) can equivalently be transformed to the auxiliary problem described by (5).

**Proof.** See the Appendix. ■

Based on the equivalence result in Lemma 3, we can derive the optimal path-independent allocation rule from the auxiliary problem directly, which is summarized in Theorem 1.

**Theorem 1 (Optimality of Majority Rule with a Headstart)** When players on each team are homogeneous and discriminatory powers across battles are identical, the optimal allocation rule is a **majority rule with a headstart** (in terms of an initial number of winning battles) to the weaker team, in which the headstart is $H = 2K_s - (N + 1)$, where $K_s = \lfloor p_A N \rfloor + 1$. The team collects the higher (adjusted) number of wins is rewarded the entire prize budget.

**Proof.** Since the domain of decision variables $\{\Delta v^A(k)\}_{k=0}^{N-1}$ is an $(N-1)$-simplex, the maximum value of the objective function is $\alpha N \max_{k \in \{0, ..., N-1\}} F(k)$ with optimal solution $\Delta v^A(k) = \{1, \text{if } k = k^*, 0, \text{if } k \neq k^*\}$, where $k^*$ maximizes $F(k)$. By recovering $v^A(k)$ from $\Delta v^A(k)$, we have $v^A(k) = \{1, \text{if } k \geq k^* + 1, 0, \text{if } k < k^* + 1\}$. It follows from the property of binomial coefficients that $k^* = \lfloor p_A N \rfloor$. Note that when $p_A N$ is an integer, both $p_A N$ and $p_A N - 1$ maximize $F(k)$. In this case, the optimal path-independent allocation rule is not unique. ■

Theorem 1 says that in terms of the number of actual winning battles, the $K_s$-th (resp. $(N - K_s + 1)$-th) win is critical for team $A$ (resp. team $B$). Note that $K_s \geq N - K_s + 1$. Compared to the conventional majority rule, the majority rule with a headstart favors the weaker team, which in turn stimulates the stronger team.
3.3 Properties of Optimal Design

Comparative Statics

Our main results in Section 3.2 show that the optimal allocation rule compensates the underdog and thus disciplines the favorite to intensify the competition. The contest designer can mitigate the asymmetry between the two teams by awarding the weaker team a headstart to heat up the competition. Since a headstart to the weaker team encourages the weaker team but discourages the stronger, the headstart should be set at a moderate level at the optimum. This brings the question that how the level of headstart varies as the team contest grows increasingly uneven.

Recall that Theorem 1 states that the minimum winning requirement for team $A$ is $K_S = \lfloor p_A N \rfloor + 1$. Clearly, $\lfloor p_A N \rfloor$ (weakly) increases with winning probability $p_A$, so does $K_S$.

**Corollary 2** In the optimal rule (i.e., majority rule with a headstart), the minimum number of winning battles $K_S$ (weakly) increases with $p_A$.

Corollary 2 demonstrates that when $p_A$ increases, a higher $K_S$ should be set to induce more effort. Recall that the contest organizer uses the majority rule with a headstart with $K_S$ to favor the weaker team, in order to balance the contest with two asymmetric teams. When the disparity in capabilities (measured by $p_A$) between the two teams increases, more favoritism should be offered to the weaker team to mitigate the asymmetry so that the overall effort would increase.

Recall that $p_A$ (see the Appendix for the formula) is determined by the marginal effort costs of the two teams and the discriminatory power of Tullock contests, whenever battle $t$ is not trivial. Note that $p_A$ decreases with $c_A$ and (weakly) increases with $r$. When the two teams become more asymmetric (i.e., a lower $c_A$) or the contest becomes discriminatory (i.e., a higher $r$), Corollary 2 implies that a higher $K_S$ should be set to induce more effort.
Convergence to Majority Rule

When two teams are close to symmetry, i.e., \( p_A \) is close to 0.5, the majority rule with a headstart with \( K_S = \left\lfloor \frac{N}{2} \right\rfloor + 1 = \left\lfloor \frac{N}{2} + 1 \right\rfloor \) is optimal.

**Corollary 3** When two teams are close to symmetry, i.e., \( p_A \in \left[ \frac{1}{2}, \frac{N+1}{2N} \right) \), the minimum winning requirement \( K_S = \left\lfloor \frac{N}{2} + 1 \right\rfloor \).

The majority rule with a headstart with \( K_S = \left\lfloor \frac{N}{2} + 1 \right\rfloor \) coincides with the conventional best-of-\( N \) allocation rule when the number of battles is odd. Consider the most frequently used 3-battle (5-battle) contests between two teams, best-of-three (best-of-five) is the unique optimal prize allocation rule when \( p_A \) is lower than 66.67% (60%). If the winning probability of the stronger team in each battle is higher than 66.67% (60%) with 3 (5) battles, the optimal allocation rule is no longer majority rule. Nevertheless, such extremely lopsided contests with an exceeding disparity in capabilities seldom occur in reality.

Over-corrections

We further study the outcome of the team contest under the optimal rule in terms of the ex ante expected prize received by each team. For this, we first introduce the following definition.

**Definition 2** An allocation rule is over-corrected (resp. under-corrected) if the ex ante expected prize received by team \( A \) (the stronger team) is less (resp. greater) than \( \frac{1}{2} \).

Recall that the optimal allocation rule tends to mitigate the asymmetry between two teams by favoring the weaker one. If an allocation rule is under-corrected, the ex ante expected prize of the weaker team is still lower than the expected prize of the stronger team, even though the gap is narrowed. However, when compensation to the weaker team is excessive, over-correction may occur. This naturally raises
the question: Whether the optimal allocation rule is over-corrected or not? In the following proposition, we find that the over-correction may occur. To avoid the trivial case, we assume that $p_A > \frac{1}{2}$ such that the two teams are asymmetric.

**Proposition 2** When $p_A > \frac{1}{2}$, the optimal rule may be over-corrected.

**Proof.** See the Online Appendix. ■

Proposition 2 shows that over-correction may occur under the optimal rule. The possibility of over-correction demonstrates that the optimal rule causes the two teams to have equal chances of winning (approximately). In other words, the optimal allocation rule (almost) completely offsets the imbalance.

### 3.4 Examples

Consider a 3-battle team contest and each battle is modeled as a lottery contest, i.e., Tullock contest with discriminatory power $r = 1$. We plot the expected total efforts that result from majority rule (MR) and majority rule with a headstart $K^*$ (MRH) in Figure 2 by varying $c_A$.

![Figure 2: Comparisons](image1)

![Figure 3: Over-corrections](image2)
By Corollary 3, when the two teams are close to symmetry, the optimal allocation rule converges to the majority rule. In Figure 2, the two curves coincide when $c_A \rightarrow c_B = 1$. On the contrary, when the two teams become more asymmetric, the optimal rule outperforms the majority rule. In particular, when $c_A$ approaches to 0, the total effort induced by the majority rule is approximately zero while the efforts induced by the optimal rule remains at high level.

Figure 3 plots the expected prizes received by team $A$ under majority rule with a head start $K^*$ (MRH) with different $c_A$. When $c_A$ gradually decreases below 0.5, Figure 2 shows that the majority rule with a head start $K^* = 3$ is the optimal rule since it generates higher expected total effort than majority rule. In the meanwhile, as can be seen in Figure 3, this optimal rule is overcorrected when $c_A$ is slightly below 0.5; while when $c_A$ approaches to 0, overcorrection disappears again. This is consistent with the argument in the proof of Proposition 2.

4 Analysis with Full-fledged Heterogeneity

Heretofore, we have assumed that players are homogeneous within each team and discriminatory powers are identical across battles. In this section, we accommodate full-fledged heterogeneity, i.e., players are fully heterogeneous within and across teams (in terms of marginal costs) and contest technologies (in terms of discriminatory powers) can differ across battles.\(^{10}\)

4.1 Preliminaries

As before, we first formulate the ex ante expected total effort in a team contest with full-fledged heterogeneity. In particular, we find that the outcomes of component battles can be still viewed as independent draws while probabilities can differ across

\(^{10}\)In real-life contests, beating a stronger opponent is often more rewarding than beating a weaker one, including both visible gain (e.g., money awarded by sponsors) and invisible gain (e.g., pride or reputation). This can be captured by battle heterogeneity as well.
the battles. Unlike Property 1, both the proportions and the winning probabilities vary with the battles, we therefore denote \( p_i(t) \) the probability that team \( i \in \{A, B\} \) wins for battle \( t \), provided that the battle is nontrivial and denote \( \alpha_t \) the proportion of total effort in battle \( t \) to the effective prize spread of battle \( t \). Note that \( p_A(t) + p_B(t) = 1 \), the analytical formula of \( p_A(t) \) and \( \alpha_t \) are provided in the Appendix.

Although both \( p_A(t) \) and \( \alpha_t \) can differ across battles, we compute the ex ante expected prize spread for battle \( t \) in the following.

\[
\text{PS}_t(v^A) \triangleq \sum_{W^A \subseteq \mathcal{N} \setminus \{t\}} \left[ \prod_{j \in W^A, j \neq t} p_A(j) \prod_{j \notin W^A, j \neq t} (1 - p_A(j)) \left[ v^A(W^A \cup \{t\}) - v^A(W^A) \right] \right] = \sum_{W^A \subseteq \mathcal{N}} (-1)^{1(t \notin W^A)} \prod_{j \in W^A, j \neq t} p_A(j) \prod_{j \notin W^A, j \neq t} (1 - p_A(j)) v^A(W^A). \tag{6}
\]

The ex ante expected total effort with heterogeneous battles can be rewritten as

\[
\text{TE}^\text{Hete}(v^A) \triangleq \sum_{t \in \mathcal{N}} \alpha_t \text{PS}_t(v^A), \tag{7}
\]

which coincides to Equation (3) in our baseline model.

To formally characterize the optimal rule, we first demonstrate that the optimal allocation rule must be in form of \( \{0, 1\} \), i.e., each team obtains either the whole prize or nothing.

**Lemma 4** With full-fledged heterogeneity, there must exist an optimal prize allocation rule \( v^A(\cdot) \) such that \( v^A(W^A) \in \{0, 1\} \).

**Proof.** See the Appendix. ■

Lemma 4 facilitates our search for the optimal prize rules tremendously since it reduces the number of candidates from infinite to finite, which makes enumeration possible.

\[^{11} \text{The argument for trivial battles (Lemma 1) still applies.} \]
4.2 Optimal Design

Due to within-team player heterogeneity, it becomes less straightforward to tell which team is stronger as a whole, since a team may contain both the weaker and the stronger players, relative to his rival team. Nevertheless, we propose a measurement of team strength that aggregates all relevant information to determine which team is stronger and by how much. This step plays a key role in characterizing and describing the optimal rule.

In battle $t$, the ratio of the two players’ equilibrium winning probabilities, i.e., $p_A(t)/p_B(t)$, constitutes a natural measure for their relative strength. Recall that $\alpha_t$ is the proportion of total effort in battle $t$ to the effective prize spread of battle $t$. Thus, $\alpha_t$ can be viewed as a common factor in the player’s absolute strengths in battle $t$. Note that $p_A(t) + p_B(t) = 1$. We naturally define $s_i(t) \triangleq \frac{\alpha_t}{1 - p_i(t)}$ as a measure of the player $i(t)$’s (absolute) strength, which increases with both $\alpha_t$ and $p_i(t)$.

We further define $S_i \triangleq \sum_{t \in N} s_i(t) = \sum_{t \in N} \frac{\alpha_t}{1 - p_i(t)}$ as the team strength of team $i$, which is simply the sum of all players’ strengths in the team. Note that $S_i$ is solely determined by the model primitives $\{c_{A(t)}, c_{B(t)}, r(t)\}_{t=1}^N$. Relying on this definition, a stronger team is thus the team with higher team strength. Note that $S_i > S_j$ does not imply that $p_i(t) > p_j(t)$ holds uniformly across all battles. This definition thus provides a way to aggregate a team’s players’ strengths into a single-dimensional measure. Without loss of generality, we assume in the subsequent analysis that team $A$ is the stronger team, i.e., $S_A \geq S_B$.

Moreover, we define $w_t \triangleq \frac{\alpha_t}{p_A(t)p_{B(t)}}$ as the score assigned to battle $t$. One can verify that $w_t = s_{A(t)} + s_{B(t)}$. Let $w^A = \sum_{t \in W^A} w_t$ and $w^B = \sum_{t \in W^B} w_t$ denote the sum of scores won by teams $A$ and $B$, respectively. The following theorem fully characterizes the optimal prize allocation rule.

**Theorem 2** (Optimality of Majority-score Rule with a Headstart) With full-fledged heterogeneity, the optimal prize allocation rule is a majority-score rule with a headstart: Team $A$ (the stronger team) collects the whole prize budget if $w^A > w^B +$
H; and team B collects the whole prize budget if \( w^A < w^B + H \); when \( w^A = w^B + H \), tie is broken randomly. Here, \( H = S_A - S_B \) denotes the headstart score allocated to the weaker team B.

Equivalently, \( v^A(W^A) = \begin{cases} 1, & \text{if } w^A > S_A, \\ 0, & \text{if } w^A < S_A, \text{ and } v^B(W^B) = 1 - v^A(W^A), \\ 0 \text{ or } 1, & \text{if } w^A = S_A. \end{cases} \)

since \( S_A + S_B = \sum_{t \in \mathcal{N}} w_t \) by the definition. The entire reward can be allocated to either team A or team B when \( w^A = w^B + H \).

Unlike Section 3, battles are no longer identical and the optimality of path-independent rules (Proposition 1) fails due to the full-fledged heterogeneity. Thus, it is a demanding task to pin down the optimal prize allocation rule completely. Nevertheless, our analysis yields a surprisingly simple solution, which elegantly extends the optimal design with homogenous battles. The contest organizer assigns different scores to battles and both teams earn the same score from winning a battle. The weaker team is favored by a headstart score. The team with the higher (adjusted by the headstart score) total scores wins the entire prize.

The optimal rule works through two instruments: the score of each battle \( w_t \) and the winning threshold for each team \( S_i \) (in terms of unadjusted total scores). Under the optimal prize structure, a team’s strength coincides with its winning threshold, which implies that our optimal rule fully balances the competition. Recall that \( S_i \) measures the team \( i \)'s strength, which is simply the sum of all players’ strengths within team \( i \). For a given battle, the weaker player contributes less to his team’s winning threshold than his stronger opponent does.

To see how the optimal rule in Theorem 2 reacts to the heterogeneity within a battle through the two instruments, consider an unbalanced battle \( t \) in which player \( i(t) \) is stronger than his opponent \( j(t) \), i.e., \( c_{i(t)} < c_{j(t)} \) or \( p_{i(t)} > p_{j(t)} \). When \( \alpha_t \) increases, the optimal prize rule raises score \( w_t \) to battle \( t \) because a higher score should be set to provide higher incentive in more effective battles.
When \( p_i(t) \) increases, the degree of imbalance measured by \( \frac{1}{p_A(t)p_B(t)} \) also increases, and the optimal prize rule also assigns a greater score \( w_t \) to battle \( t \). Moreover, the higher \( p_i(t) \) is, the higher player strength \( s_i(t) \) and also the higher winning threshold \( S_i \) would be. In the meanwhile, since \( p_j(t) = 1 - p_i(t) \), the weaker player’s strength \( s_j(t) \) and the winning threshold \( S_j \) both decrease. Our result reveals that if team \( i \) is the stronger team (i.e., team \( j \) is the weaker team), headstart to the weaker team \( j \) should increase with \( p_i(t) \); otherwise, team \( i \) is the weaker team and headstart to the weaker team \( i \) should decrease with \( p_i(t) \) instead. In addition, \( w_t \) should always increase whenever battle \( t \) becomes more unbalanced.

The intuition is as follows. When a component battle gets more unbalanced, the optimal rule suggests raising battle score to motivate the matched pair. With a more unbalanced battle, the team of the weaker player needs to win lower total scores to win the contest, and winning the battle earns a higher score. Thus, the weaker player is better incentivized, which further disciplines the stronger player. Meanwhile, to balance the overall contest, the optimal rule also suggests increasing the winning threshold of the team that contains a stronger player relative to his opponent in the concerned battle. It is worth noting that a team that contains a stronger player is not necessarily the stronger team.

To establish the connection between Theorems 1 and 2, we consider the case where battles are homogeneous as in Section 3, i.e., \( \alpha_t = \alpha \) and \( p_A(t) = p_A \) for all \( t \). By definitions, the score of each battle equals \( w_t = \frac{\alpha}{p_A(1-p_A)} \), which are identical across the battles, and the threshold \( S_A = \frac{N\alpha}{1-p_A} \). By applying Theorem 2, \( w^A > S_A \) if and only if \( \frac{\alpha}{p_A(1-p_A)}|W^A| > \frac{N\alpha}{1-p_A} \), i.e., \( |W^A| > pN \); \( w^A < S_A \) if and only if \( |W^A| < pN \); \( w^A = S_A \) if and only if \( |W^A| = pN \), by which we recover the optimal prize allocation rule characterized in Theorem 1.
5 Discussion

In this section, we first discuss the main implications of our results, and then check the robustness of baseline model proposed in Section 2.

5.1 Implications

Our results on optimal prize design shed light on contest design in various contexts. The majority rule is pervasive in sports contests with team titles, and our analysis shows that the majority rule is optimal when two teams are more or less evenly matched. In these team contests, prizes are usually awarded following the best-of-5 rule—for instance, the Davis Cup and the Billie Jean King Cup (the so-called Fed Cup) in tennis; the Thomas Cup, the Uber Cup, and the Sudirman Cup in badminton; and the Swaythling Cup and the Corbillon Cup in table tennis. Our results provide a theoretical foundation and validate the rationality of conventional majority rules in team competitions.

Our result also provides a novel perspective on the design of legislative elections. In a legislative election, candidates representing opponent parties compete for legislative seats in each constituency. Commonly, a party can form a government or set political agendas in the legislature if it acquires majority status. However, under this prevailing election rule, the election per se may not be fair due to incumbency advantage. The ruling party can be less productive in serving the public interest if it has little chance to be defeated. Pastine and Pastine (2012) identify various channels through which incumbency advantage may generate deleterious effects on social welfare. Our result sheds light on this issue. Granting the opposition party a

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12Some empirical studies show that obtaining a majority of seats in the legislature is lucrative to the party in power and its politicians. For example, Cox and Magar (1999) evaluate the majority status in terms of contributions from political action committees by investigating changes in party control of the House and Senate. Snyder (1989) takes maximizing the probability of obtaining a majority status as a political party’s objective.

13Incumbency advantage is one of the most frequently studied issues in congressional elections research, see, e.g., Levitt (1994) and Jamie, Engstrom, and Roberts (2007).
headstart in parliamentary elections can better incentivize both parties in this regard.

In many patent races and government procurements of enormous magnitude, competitions are often held between research alliances. Member entities within an alliance execute different tasks in order to deliver innovation or products. For example, a research alliance on vaccine development typically consists of organizations such as virus research institutes, medical schools, and pharmaceutical companies. It takes multiple stages to develop a final product, including screening vaccine strains, preclinical vaccine development, and clinical trials. Each member competes with its counterparts of the rivaling alliance in a particular stage, and its stage performance would determine the overall performance of the alliance. Our analysis implies that the designer should encapsulate multi-dimensional heterogeneity of research alliances into a single-dimensional strength measure and formulate corresponding procurement rules according to the team strengths beforehand.

It is worth noting that stage heterogeneity is an essential feature in most multi-stage R&D races. On one hand, stages can be different, which can be captured by the ratio of total effort to effective prize spread. On the other hand, the relative strength of rival teams can also differ across battles. For instance, two alliances compete for vaccine development; one research alliance may specialize in preclinical vaccine development while the other alliance is skilled in clinical trials. Our results suggest that the government in charge of vaccine R&D should assign the highest score to Clinical Phase III. The reason is two-fold: first, Clinical Phase III is the most critical stage prior to marketing approval; second, the disparity between opponent alliances in terms of research productivity in Clinical Phase III is vast compared to other stages.

\footnote{For example, BioNTech and Pfizer ally to develop the mRNA vaccine BNT162b2. BioNTech is responsible for preclinical development, and Pfizer takes charge of clinical trials, marketing, and manufacturing.}
5.2 Robustness Check

Our analysis can be applied to a more general setting. In this section, we discuss several extensions that validate the generality and robustness of our main insights.

Contest Technologies

Previously, we assume the marginal cost $c_{i(t)}$ is public information. To incorporate incomplete information, we assume that $c_{i(t)}$ is drawn from a cumulative distribution function $F_{i(t)}(\cdot)$. The marginal costs of all players from both teams are independently distributed. The distribution of cost is common knowledge and the realization of $c_{i(t)}$ is privately observed by player $i(t)$ only. Following Fu, Lu, and Pan (2015), we can assume homogeneity-of-degree-zero contest technologies, which covers the generalized Tullock contest framework.

Assumption 2 (Homogeneity-of-Degree-Zero Contest Technologies) \( \forall x_{A(t)}, x_{B(t)} \geq 0, \theta > 0 \) and \( t \), \( p_{i(t)}(\theta x_{A(t)}, \theta x_{B(t)}) = p_{i(t)}(x_{A(t)}, x_{B(t)}) \).

With homogeneity-of-degree-zero contest technologies, two matched players must be equally motivated (See Fu, Lu, and Pan (2015)) across battles and states. As a result, we can easily prove the following properties as before: (1) the \textit{ex ante winning probability} of each nontrivial battle \( t \) for team \( A \) are unrelated to its effective prize spread \( V_t \); and (2) the \textit{ex ante expected total effort} for each battle are proportional to its effective prize spread \( V_t \). All the subsequent analysis in Sections 3 and 4 follows.

Temporal-structure Independence

We have assumed that component battles are carried out completely successively and battle \( t \) refers to the battle played in the \( t \)-th round. We now consider that \( N \) battles can be carried out (partially) simultaneously as in Fu, Lu, and Pan (2015), which show the temporal-structure independence result under majority rule with an odd number of battles.
**Definition 3** *(Temporal-Structure Independence, by Fu, Lu, and Pan (2015)) The allocation rule exhibits temporal-structure independence if the temporal structure does not influence the ex ante expected effective prize spread for each battle.*

We generalize the temporal-structure independence result by allowing arbitrary path-dependent prize allocation rules with an arbitrary number of battles. Since the ex ante expected effective prize spread for each battle remains the same, the *ex ante expected* winning outcome, total effort, and players’ payoff are all independent of the prevailing temporal structure of the contest.

6 Concluding Remarks

This paper studies the optimal prize design problem in team contests with an arbitrary number of battles, in which rivaling players compete head-to-head on their own battlefields. The organizer can reward teams according to the full path of battle outcomes subject to budget balance constraints. Our main results in this paper lie in the history independence result, i.e., the outcomes of early battles do not distort the winning probability in future battles. Fu, Lu, and Pan (2015) establish this history independence, and we generalize it to settings with an arbitrary number of battles and an arbitrary prize allocation rule that satisfies budget balance constraints. This history independence result ensures that players’ winning odds in each battle can be viewed as independently and identically distributed lotteries at any state, which plays a crucial role in our analysis.

We begin our analysis on *partial heterogeneity* by assuming that players are homogeneous within teams and contest technologies are identical across battles. In analyzing optimal prize allocations, we first demonstrate that it is without loss of generality to focus on allocation rules that are solely contingent on the number of battle wins, i.e., teams are awarded based on their numbers of battle wins, rather than the order of wins. By restricting our attention to path-independent rules, the
original problems of prize design can be equivalently transformed into auxiliary problems, which are solvable using a linear programming technique. By recovering the optimal prize-allocation rule, we find that the optimum is a majority rule with a headstart for the weaker team in the form of an initial number of wins. Equivalently, the optimal prize allocation rule favors the weaker team by requiring fewer wins. This result confirms the conventional wisdom in contest design that an effort-maximizing designer should handicap the stronger party. In particular, when the two teams are sufficiently homogeneous, the optimum reduces to a conventional best-of-$N$ prize allocation rule. We, therefore, justify the widely adopted majority rules and shed new light on incentive provision in team contests.

When players on each team are heterogeneous and contest technologies across battles are different, a series of distinct results emerge. With full-fledged heterogeneity, the optimality of path-independence fails, which nullifies the previous method. Nevertheless, we measure teams’ strengths in an intuitive way and develop another technique called iterative adjustment method to derive the closed-form optimal rule, which is a majority-score rule with a headstart score for the weaker team. The optimal rule can be implemented in the following way. Two teams collect scores by winning component battles and they obtain the same score from winning the same battle. The scores can be different across the battles. The rule favors the weaker team by awarding it a headstart score and the team that collects higher (adjusted) total scores wins the contest. When battles become homogeneous, all battles carry the same score, the optimal rule is just the majority rule with a headstart.

The optimal designs are fully robust to homogeneity-of-degree-zero contest technologies and contest temporal structures. However, problematic issues may arise when the budget balance condition is slack, players are no longer risk-neutral, or have nonlinear effort costs. We leave these to future work.
Appendix

This appendix covers analytical formulas for $p_{A(t)}$ and $\alpha_t$, and the proofs of Lemma 1, Proposition 1, Lemma 3, Lemma 4 and Theorem 2.

Analytical Formulas for $p_{A(t)}$ and $\alpha_t$

Under generalized Tullock contest technology, the success function is given by $p_{A(t)} = \frac{x_{A(t)}^r}{x_{A(t)}^r + x_{B(t)}^r}$. Player $i(t)$ chooses effort $x_{i(t)}$ to maximize $\frac{x_{A(t)}^r}{x_{A(t)}^r + x_{B(t)}^r} V_i(W_i^A) - c_{i(t)}x_{i(t)}$, i.e., $\frac{x_{A(t)}^r}{x_{A(t)}^r + x_{B(t)}^r} \frac{V_i(W_i^A)}{c_{i(t)}} - x_{i(t)}$. In a two-player one-shot Tullock contest with an arbitrary $r(t)$, the equilibrium strategies are summarized in Lemma 1 in Feng and Lu (2018). Both $p_{A(t)}$ and $\alpha_t$ can be calculated accordingly based on the existing equilibrium characterization. Let $\hat{r}(z) \in (1, 2)$ represent the unique solution to $r = 1 + z^r$ with $z \in (0, 1]$. The results are summarized as follows.

Heterogeneous Battles

Without loss of generality, we assume that $c_{A(t)} \geq c_{B(t)}$, the analytical formulas of $\alpha_t$ and $p_{A(t)}$ are

$$p_{A(t)} = \begin{cases} 
\frac{c_{B(t)}^r}{(r(t)-1)^{1-1/r(t)} c_{B(t)} / (r(t)c_{A(t)})}, & \text{if } r(t) \leq \hat{r}(c_{B(t)}/c_{A(t)}) \\
\frac{c_{B(t)}}{2c_{A(t)}}, & \text{if } r(t) > 2.
\end{cases}$$

and

$$\alpha_t = \begin{cases} 
\frac{r(t)c_{A(t)}^{r(t)-1}}{(r(t)-1)^{1-1/r(t)} (c_{A(t)} + c_{B(t)})} \left(\frac{r(t)}{2c_{A(t)}}\right)^{-2}, & \text{if } r(t) \leq \hat{r}(c_{B(t)}/c_{A(t)}) \\
\frac{(r(t)-1)^{1-1/r(t)} (c_{A(t)} + c_{B(t)}) / (r(t)c_{A(t)}^2)}{2c_{A(t)}}, & \text{if } r(t) \in (\hat{r}(c_{B(t)}/c_{A(t)}), 2], \\
\frac{(c_{A(t)} + c_{B(t)}) / (2c_{A(t)}^2)}{2}, & \text{if } r(t) > 2.
\end{cases}$$

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Homogeneous Battles

With homogeneous battles, \( c_{A(t)} = c_A \in (0, 1], c_{B(t)} = 1 \) and \( r(t) = r \) for all \( t \), the ratio of expected total effort to effective prize spread is identical among all battles. Hence, the analytical formula of \( \alpha \) defined in Property 1 is

\[
\alpha = \begin{cases} 
rc_A^{-1} (1 + c_A) (1 + c_A')^{-2}, & \text{if } r \leq \hat{r}(c_A), \\
(r - 1)^{1 - 1/r} (1 + c_A) / r, & \text{if } r \in (\hat{r}(c_A), 2], \\
(1 + c_A) / 2, & \text{if } r > 2.
\end{cases}
\]  

(8)

Meanwhile, player \( A(t) \) wins an arbitrary battle with the same probability \( p_A \) and the analytical formula of \( p_A \) defined in Property 1 is

\[
p_A = \begin{cases} 
1 / (1 + c_A'), & \text{if } r \leq \hat{r}(c_A), \\
1 - (r - 1)^{1 - 1/r} c_A / r, & \text{if } r \in (\hat{r}(c_A), 2], \\
1 - c_A / 2, & \text{if } r > 2.
\end{cases}
\]  

(9)

Proof of Lemma 1

If battle \( t \) is trivial, i.e., \( U_{A}^A(W_{A}^A \cup \{t\}) = U_{A}^A(W_{A}^A) \), it elicits zero effort and its winning outcome is determined by the default tie-breaking rule; hence \( U_{A}^A(W_{A}^A \cup \{t\}) = U_{A}^A(W_{A}^A) \). Therefore, the recursive definition for \( U^A \), \( U_{A}^A(W_{A}^A) = p_AU_{A}^A(W_{A}^A \cup \{t\}) + (1 - p_A)U_{A}^A(W_{A}^A_{t+1}) \), holds for any \( p_A \in [0, 1] \).

We next show that the outcome of a trivial battle does not affect the boundary conditions, which ensures that our formula for the effective prize spread remains valid when trivial battles are taken into account.

Lemma A.1: (Outcome Equivalence) If battle \( t \) is trivial, then for all possible sets of winning battles of team \( A \) for the remaining battles, \( Q \subseteq N \setminus N_{t+1}, v^A(W_{A}^A \cup \{t\} \cup Q) = v^A(W_{A}^A \cup Q) \).

We now explain why the above result must hold. By monotonicity conditions, \( v^A(W_{A}^A \cup \{t\} \cup Q) \geq v^A(W_{A}^A \cup Q) \) for all \( Q \subseteq N \setminus N_{t+1} \) and hence \( V_t(W_{A}^A) \geq 0 \). It
is worth noting that $V_t(W_{t+1}^A) = 0$ implies that $v^A(W_{t+1}^A \cup \{t\} \cup Q) = v^A(W_{t+1}^A \cup Q)$, $\forall Q \subseteq \mathcal{N} \setminus \mathcal{N}_{t+1}$. In words, the subgames of a team contest are exactly the same, regardless of the outcome of the trivial battle. As a result, the trivial battle is inconsequential in determining the prize.

Since the outcome of a trivial battle does not affect the boundary conditions or recursive definitions of $U^A$, we have the following two remarks.

**Remark A.1:** (State Equivalence) If battle $t \mid W_t^A$ is trivial, for $\tilde{t} \geq t$ and $Q \subseteq \mathcal{N}_{\tilde{t}+1}\setminus \mathcal{N}_{t+1}$, $(\mathcal{N}_{\tilde{t}+1}, W_{\tilde{t}+1}^A \cup \{t\} \cup Q)$ and $(\mathcal{N}_{t+1}, W_t^A \cup Q)$ are equivalent states. In other words, (i) the expected prize is identical, $U_{\tilde{t}}^A(W_{\tilde{t}+1}^A \cup \{t\} \cup Q) = U_t^A(W_t^A \cup Q)$; (ii) the effective prize spread is identical, $V_{\tilde{t}+1}(W_{\tilde{t}+1}^A \cup \{t\} \cup Q) = V_{t+1}(W_t^A \cup Q)$.

**Remark A.2:** (Transition Probability Irrelevance) If battle $t \mid W_t^A$ is trivial, both the expected prize and effective prize spread for all battles will not change if the transition probability for these two subgames changes.

Remark A.2 illustrates the fact that if a battle is trivial, then for two decision points representing two outcomes of this trivial battle, the total effort generated until the contest ends does not depend on which decision point to go to. All subsequent processes are exactly the same for these two decision points. Starting from these two points, two subgames are identical and the expected prize and effective prize spread remain the same when the transition probability of these two subgames changes. As a result, we can freely adjust the winning probability of trivial battles.

If all previous battles before battle $t$ are nontrivial, the probability that history $W_t^A$ occurs can be calculated by the multiplicative law of probability and hence given by $p_A^{\mid W_t^A\mid} (1 - p_A)^{t-1-\mid W_t^A\mid}$. If some of the battles are trivial, we can adjust the probabilities such that history $W_t^A$ occurs with probability $p_A^{\mid W_t^A\mid} (1 - p_A)^{t-1-\mid W_t^A\mid}$.

**Proof of Proposition 1**

Consider a prize allocation rule $v^A(\cdot)$, which depends on the full path of battle outcomes. We will show how to construct an effort-equivalent allocation rule $\overline{v}^A(\cdot)$,
which solely depends on the number of battle wins. We show the result in two steps: In step 1, we present the average-level monotonicity condition to ensure that the monotonicity condition is satisfied under our construction; in step 2, we construct an effort-equivalent prize allocation rule.

**Step 1:** We show that
\[
\frac{\sum_{|W^A|=k^A+1} v^A(W^A)}{\binom{k^A+1}{k^A}} \geq \frac{\sum_{|W^A|=k^A} v^A((W^A)')}{\binom{k^A}{k^A}},
\]
which is called the average-level monotonicity condition, since the average prize with \(k^A\) battle wins is computed as \(\frac{\sum_{|W^A|=k^A} v^A(W^A)}{\binom{k^A}{k^A}}\). The average-level monotonicity condition is intuitively obvious by outcome-level monotonicity conditions.

To show that, we enumerate all monotonicity conditions such that \(|(W^A)'| = k^A, |W^A| = k^A + 1\) and \((W^A)' \subset W^A\). Note that there are \(\binom{N}{k^A+1}\) different \(W^A\), and for each \(W^A\) there are \(k^A + 1\) inequalities according to monotonicity conditions. In the meantime, there are \(\binom{N}{k^A}\) different \((W^A)'\), and for each \((W^A)'\), there are \(N - k^A\) inequalities. Hence, there are \(\binom{N}{k^A+1}(k^A + 1) = \frac{N!}{k^A!(N-k^A-1)!} = \binom{N}{k^A}(N-k^A)\) inequalities in total. Adding all of these inequalities together, we have
\[
(k^A + 1) \sum_{|W^A|=k^A+1} v^A(W^A) \geq (N - k^A) \sum_{|W^A|=k^A} v^A((W^A)') .
\]

Note that each \(v^A(W^A)\) occurs in \(k^A + 1\) inequalities and each \(v^A((W^A)')\) occurs in \(N - k^A\) inequalities, by dividing both sides of the inequalities by \(\frac{N!}{k^A!(N-k^A-1)!}\), we have
\[
\frac{\sum_{|W^A|=k^A+1} v^A(W^A)}{\binom{k^A+1}{k^A}} \geq \frac{\sum_{|W^A|=k^A} v^A((W^A)')}{\binom{k^A}{k^A}}.
\]

From the arguments above, the average prize of winning \(k^A\) battles is weakly increasing in \(k^A\) for all feasible path-dependent allocation rules \(v^A(\cdot)\), which guarantees that monotonicity conditions hold when constructing the path-independent allocation rule.

**Step 2:** Let \(\widehat{W}^A\) denote the representative contest outcome. Consider the path-dependent allocation rule \(v^A(\cdot)\). We can construct the following effort-equivalent path-independent allocation rule \(\overline{v}^A\): \(\overline{v}^A(W^A) = \frac{\sum_{|W^A|=|\widehat{W}^A|} v^A(W^A)}{|\widehat{W}^A|}\), i.e., the prize received under \(\overline{v}^A(\cdot)\) with winning \(|W^A|\) battles equals the average prize under \(v^A(\cdot)\).
for winning $|W^A|$ battles. Therefore, the average prize for winning $|W^A|$ battles is the same under allocation rules $v^A(\cdot)$ and $\pi^A(\cdot)$. Obviously, the objective function remains the same.

Next, we need to verify that all constraints still hold. Path-dependent allocation rule $v^A(\cdot)$ is feasible. We need to show that $\pi^A(\cdot)$ is feasible.

- Nonnegativity conditions:

$$\pi^A(W^A) = \frac{\sum_{|\bar{W}^A| = |W^A|} v^A(\bar{W}^A)}{\binom{N}{|W^A|}} \in \left[ \frac{\sum_{|\bar{W}^A| = |W^A|} 0}{\binom{N}{|W^A|}}, \frac{\sum_{|\bar{W}^A| = |W^A|} 1}{\binom{N}{|W^A|}} \right] = [0, 1].$$

- Monotonicity conditions: If $(W^A)' \subseteq W^A$, then $|W^A)'| \leq |W|$ and

$$\pi^A((W^A)') = \frac{\sum_{|\bar{W}^A| = |W^A|} v^A(\bar{W}^A)}{\binom{N}{|W^A|}} \leq \frac{\sum_{|\bar{W}^A| = |W^A|} v^A(\bar{W}^A)}{\binom{N}{|W^A|}} \pi^A(W^A),$$

where the inequality holds by Step 1.

By combining steps 1 and 2, we show that for any path-dependent allocation rule, we can always construct a path-independent allocation rule that achieves the same effort level.

**Proof of Lemma 3**

We transform the original problem to the auxiliary problem by letting $\Delta v^A(k) = v^A(k + 1) - v^A(k)$. To show the equivalence, it suffices to show that both the objective functions and constraints are identical after the variable substitutions. It follows from direct substitution that the two objective functions are identical. To show the equivalence between the constraints, we prove that (i) $\Delta v^A(k)$ is feasible if $v^A(k)$ is feasible in the original optimization problem, and (ii) we can always recover a feasible $v^A(k)$ whenever $\Delta v^A(k)$ is feasible in the equivalent auxiliary problem.
(i) Consider a feasible allocation rule $v^A(\cdot)$ that satisfies nonnegativity, monotonicity, and budget balance conditions. Then $\Delta v^A(\cdot)$ is uniquely determined. In particular, it follows from monotonicity conditions that $\Delta v^A(k) = v^A(k+1) - v^A(k) \geq 0$ holds for any $k \in \{0, 1, ..., N - 1\}$, and $\sum_{k=0}^{N-1} \Delta v^A(k) \leq 1$ holds because $\sum_{k=0}^{N-1} \Delta v^A(k) = v^A(N) - v^A(0) \leq v^A(N) \leq 1$.

(ii) Consider a feasible $\Delta v^A(\cdot)$ that satisfies $\Delta v^A(k) \geq 0$ and $\sum_{k=0}^{N-1} \Delta v^A(k) \leq 1$, $\forall k \in \{0, 1, ..., N - 1\}$. We can recover the corresponding $v^A(\cdot)$ as the following: $v^A(0) = \varepsilon \in \left[0, 1 - \sum_{k=0}^{N-1} \Delta v^A(k)\right]$, and $v^A(k+1) = v^A(k) + \Delta v^A(k), k \in [0, N - 1]$. $v^A(\cdot)$ is feasible and meets nonnegativity, monotonicity and budget balance conditions.

Proof of Lemma 4

To show Lemma 4, we first establish the linearity of $\text{TE}^{\text{Hete}}(v^A)$ in the following lemma.

**Lemma A.2:** Given an allocation rule $v^A(\cdot)$, if there exist two different allocation rules, $\tilde{v}^A(\cdot) \neq \tilde{\tilde{v}}^A(\cdot)$, such that

$$v^A(\cdot) = \theta \tilde{v}^A(\cdot) + (1 - \theta) \tilde{\tilde{v}}^A(\cdot),$$

for some $\theta \in (0, 1)$, then $\text{TE}^{\text{Hete}}(v^A) = \theta \text{TE}^{\text{Hete}}(\tilde{v}^A) + (1 - \theta) \text{TE}^{\text{Hete}}(\tilde{\tilde{v}}^A)$.

**Proof.** Combining Equation (6) and Equation (7), $\text{TE}^{\text{Hete}}(v^A)$ can be viewed as a linear combination of $\{v^A(W^A)\}$. Compared to Equation (2) in our baseline model, the weight of $v^A(W^A)$ in $\text{TE}^{\text{Hete}}(v^A)$ is different, since players are heterogeneous and contest technologies can differ across battles. Analogous to the proof of Lemma 2, we could successively show the linearity of (i) contingent expected prize given history $W^A_{t+1}$; (ii) contingent prize spread given history $W^A_t$; (iii) ex ante expected prize spread for battle $t$; (iv) ex ante expected total effort for battle $t$; and (v) ex ante expected total effort. The linearity of $\text{TE}^{\text{Hete}}(v^A)$ follows. Details are omitted for brevity. ■
By Lemma A.2, we have $\text{TE}_{\text{Hete}}(v^A) = \theta \text{TE}_{\text{Hete}}(\tilde{v}^A) + (1 - \theta) \text{TE}_{\text{Hete}}(\bar{v}^A) \leq \max(\text{TE}_{\text{Hete}}(\tilde{v}^A), \text{TE}_{\text{Hete}}(\bar{v}^A))$, which implies that at least one of $\tilde{v}^A(\cdot)$ and $\bar{v}^A(\cdot)$ weakly dominates $v^A(\cdot)$. We call $v^A(\cdot)$ a weakly dominated allocation rule. To prove Lemma 4, we apply the technique called iterative elimination of weakly dominated allocation rules. When $v^A(\cdot)$ is eliminated, then there exists at least one weakly dominant rule. Applying the elimination procedures repeatedly, we can reduce the set of allocations while maintain the maximum total effort level. Therefore, to find the optimal prize allocation rule, it is without loss of generality to search among those rules surviving from iterative elimination of weakly dominated allocation rules.

**Corollary A.1:** Consider an iterative process of eliminating candidates on optimal rules. If for one allocation rule $v^A(\cdot)$, there exist two different allocation rules, $\tilde{v}^A(\cdot) \neq \bar{v}^A(\cdot)$ such that

$$v^A(\cdot) = \theta \tilde{v}^A(\cdot) + (1 - \theta) \bar{v}^A(\cdot),$$

for some $\theta \in (0, 1)$ and $\tilde{v}^A(\cdot), \bar{v}^A(\cdot)$ have not been eliminated yet, then the elimination of $v^A(\cdot)$ would not affect the maximum expected total effort level.

For an arbitrary $v^A(\cdot)$, let $\text{rank}(v^A)$ denote the number of different values that $v^A(\cdot)$ takes excluding 0 and 1. For example, when $v^A(\cdot)$ takes values 0, 0.5, and 1, only 0.5 counts and therefore $\text{rank}(v^A) = 1$. Note that $\text{rank}(v^A)$ must be finite, since $v^A(\cdot)$ takes at most $2^N$ different values. If $\text{rank}(v^A) > 0$, we will show that there must exist $\theta \in (0, 1), \tilde{v}^A(\cdot)$, and $\bar{v}^A(\cdot)$, such that

$$v^A(\cdot) = \theta \tilde{v}^A(\cdot) + (1 - \theta) \bar{v}^A(\cdot),$$

where $\max\{\text{rank}(\tilde{v}^A), \text{rank}(\bar{v}^A)\} = \text{rank}(v^A) - 1$.

Let $\gamma_1(v^A) = \max\{v^A(\mathcal{W}^A): v^A(\mathcal{W}^A) < 1\}$ denote the maximum value of $v^A(\cdot)$ excluding 1; and $\gamma_2(v^A) = \max\{v^A(\mathcal{W}^A): v^A(\mathcal{W}^A) < \gamma_1(v^A)\}$ denote the maximum value that $v^A(\cdot)$ takes excluding 1 and $\gamma_1(v^A)$. Note that $\gamma_1(v^A) \in (0, 1)$ since $\text{rank}(v^A) > 0$, while it is possible that $\gamma_2(v^A) = 0$. Let $\mathcal{H}(v^A) = \{\mathcal{W}^A:
\(\mathcal{W}^A = \gamma_1(v^A)\) denote the set of final outcomes \(\mathcal{W}^A\) such that \(v^A(\mathcal{W}^A) = \gamma_1(v^A)\).

We define two allocation rules in the following:

\[
\tilde{v}^A(\mathcal{W}^A) = \begin{cases} 
1, & \text{if } \mathcal{W}^A \in \mathcal{H}(v^A), \\
v^A(\mathcal{W}^A), & \text{otherwise,}
\end{cases}
\]

and

\[
\check{v}^A(\mathcal{W}^A) = \begin{cases} 
\gamma_2(v^A), & \text{if } \mathcal{W}^A \in \mathcal{H}(v^A), \\
v^A(\mathcal{W}^A), & \text{otherwise.}
\end{cases}
\]

Besides 0 and 1, both \(\tilde{v}^A(\cdot)\) and \(\check{v}^A(\cdot)\) only take the values that \(v^A(\cdot)\) take excluding \(\gamma_1(v^A)\), which implies that \(\max\{\text{rank}(\tilde{v}^A), \text{rank}(\check{v}^A)\} = \text{rank}(v^A) - 1\). In addition, both \(\tilde{v}^A(\cdot)\) and \(\check{v}^A(\cdot)\) satisfy nonnegativity, monotonicity and budget balance conditions. Therefore, we can construct \(\theta = \frac{\gamma_1(v^A) - \gamma_2(v^A)}{1 - \gamma_2(v^A)} \in (0, 1)\) such that

\[v^A(\cdot) = \theta \tilde{v}^A(\cdot) + (1 - \theta) \check{v}^A(\cdot).\]

In combination with Corollary A.1 and the above result, we now conduct the iterative elimination process as follows: firstly, we eliminate all allocation rules with \(\text{rank}(\cdot) = 2^N\); secondly, we eliminate rules with \(\text{rank}(\cdot) = 2^N - 1\); and so on. Lastly, we eliminate all allocation rules with \(\text{rank}(\cdot) = 1\). With \(2^N\) rounds of elimination, we eliminate all the allocation rules with positive rank while keep the maximum total effort level unchanged.

**Proof of Theorem 2**

Our purpose is to prove the optimal \(v^A\) is given by

\[
v^A(\mathcal{W}^A) = \begin{cases} 
1, & \text{if } w^A > S_A, \\
0, & \text{if } w^A < S_A, \\
0 \text{ or } 1, & \text{if } w^A = S_A.
\end{cases}
\]
We first write down the ex ante expected total effort by substituting Equation (6) into Equation (7), which is given by

\[ \text{TE}^{Hete}(v^A) = \sum_{t \in N} \alpha_t \text{PS}_t(v^A) \]

\[ = \sum_{t \in N} \alpha_t \sum_{W^A \subseteq N} \left[ (-1)^1(t \not\in W^A) \prod_{j \in W^A, j \neq t} p_{A(j)} \prod_{j \not\in W^A, j \neq t} (1 - p_{A(j)}) v^A(W^A) \right] \]

\[ = \prod_{t \in N} (1 - p_{A(t)}) \sum_{t \in N} \frac{\alpha_t}{1 - p_{A(t)}} \cdot \sum_{W^A \subseteq N} \left[ (-1)^1(t \not\in W^A) \prod_{j \in W^A, j \neq t} \frac{p_{A(j)}}{1 - p_{A(j)}} v^A(W^A) \right] \]

\[ = \beta \sum_{t \in N} \hat{\alpha}_t \hat{\text{PS}}_t(v^A), \]

where \( \hat{\text{PS}}_t(v^A) \triangleq \sum_{W^A \subseteq N} \left[ (-1)^1(t \not\in W^A) \prod_{j \in W^A, j \neq t} \frac{p_{A(j)}}{1 - p_{A(j)}} v^A(W^A) \right] \), \( \hat{\alpha}_t \triangleq \frac{\alpha_t}{1 - p_{A(t)}} \), and \( \beta \triangleq \prod_{t \in N} (1 - p_{A(t)}) \).

By Lemma 4, it suffices to focus on the allocations, which only take 0 or 1. Consider such a rule \( v^A \) that satisfies \( v^A(W^A) = 1 \) and \( \sum_{t \in W^A} w_t < S_A \) for some \( W^A \), where \( w_t = \hat{\alpha}_t/p_{A(t)} \) and \( S_A = \sum_{t \in N} \hat{\alpha}_t \). Since \( v^A \) only take 0 or 1, there must exist a \( W^A \subseteq \hat{W}^A \) such that \( v^A(W^A) = 1 \) and \( v^A(W^A) = 0 \) for any \( W^A \not\subseteq W^A \). We then construct a feasible rule \( \hat{v}^A \) in the following way:

\[ \hat{v}^A(W^A) = \begin{cases} 0, & \text{if } W^A = W^A, \\ v^A(W^A), & \text{otherwise}. \end{cases} \]

We will claim that \( \hat{v}^A \) dominates \( v^A \) in terms of total effort induced.

When \( v^A(W^A) \) changes from 1 to 0, the change in \( \hat{\text{PS}}_t(v^A) \) equals \( \Delta \hat{\text{PS}}_t(v^A) = (-1)^1(t \in W^A) \prod_{j \in W^A, j \neq t} \frac{p_{A(j)}}{1 - p_{A(j)}} \), and the change in \( \text{TE}^{Hete}(v^A) \) thus equals

\[ \Delta \text{TE}^{Hete}(v^A) = \beta \sum_{t \in N} \hat{\alpha}_t \Delta \hat{\text{PS}}_t(v^A) \]

\[ = \beta \left[ \sum_{t \in W^A} \hat{\alpha}_t \Delta \hat{\text{PS}}_t(v^A) + \sum_{t \not\in W^A} \hat{\alpha}_t \Delta \hat{\text{PS}}_t(v^A) \right] \]

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\[
\begin{align*}
&= \beta \left[ -\sum_{t \in \mathcal{W}^A} \hat{\alpha}_t \prod_{j \in \mathcal{W}^A, j \neq t} \frac{p_{A(j)}}{1 - p_{A(j)}} + \sum_{t \in \mathcal{W}^A} \hat{\alpha}_t \prod_{j \in \mathcal{W}^A, j \neq t} \frac{p_{A(j)}}{1 - p_{A(j)}} \right] \\
&= \beta \left( \prod_{j \in \mathcal{W}^A} \frac{p_{A(j)}}{1 - p_{A(j)}} \right) \left[ -\sum_{t \in \mathcal{W}^A} \hat{\alpha}_t \frac{1 - p_{A(t)}}{p_{A(t)}} + \sum_{t \notin \mathcal{W}^A} \hat{\alpha}_t \right] \\
&= \beta \left( \prod_{j \in \mathcal{W}^A} \frac{p_{A(j)}}{1 - p_{A(j)}} \right) \left[ \sum_{t \in \mathcal{W}^A} \hat{\alpha}_t + \sum_{t \notin \mathcal{W}^A} \hat{\alpha}_t - \sum_{t \in \mathcal{W}^A} \hat{\alpha}_t \right] \\
&\geq \beta \left( \prod_{j \in \mathcal{W}^A} \frac{p_{A(j)}}{1 - p_{A(j)}} \right) \left[ S_A - \sum_{t \in \mathcal{W}^A} w_t \right] > 0,
\end{align*}
\]

where \(w_t = \frac{\alpha_t}{p_{A(t)} p_{B(t)}}\) and \(S_A = \sum_{t \in \mathcal{N}} \frac{\alpha_t}{1 - p_{A(t)}}\). This implies that for an allocation rule \(v^A\), if there exists a \(\mathcal{W}^A\) such that \(v^A(\mathcal{W}^A) = 1\) and \(\sum_{t \in \mathcal{W}^A} w_t < S_A\), we can always construct a feasible rule \(v^A\) that yields strictly greater expected total effort than \(v^A\) does, as a result, \(v^A\) is not optimal. Therefore, the optimal rule must satisfy that

\[v^A(\mathcal{W}^A) = 0 \text{ whenever } w^A < S_A.\]

Analogously, consider a rule \(v^A\) that takes 0 and 1. If there exists a \(\mathcal{W}^A\) such that \(v^A(\mathcal{W}^A) = 0\) and \(\sum_{t \in \mathcal{W}^A} w_t > S_A\). As before, we can always find a \(\mathcal{W}^A \supseteq \mathcal{W}^A\) such that \(v^A(\mathcal{W}^A) = 0\) and \(v^A(\mathcal{W}^A) = 1\) for all \(\mathcal{W}^A \supsetneq \mathcal{W}^A\). We construct a feasible rule \(\tilde{v}^A\) such that \(\tilde{v}^A(\mathcal{W}^A) = \begin{cases} 1 & \text{if } \mathcal{W}^A = \mathcal{W}^A, \\ v^A(\mathcal{W}^A) & \text{otherwise.} \end{cases}\) We will claim that \(\tilde{v}^A\) dominates \(v^A\) in terms of the expected total effort induced.

When \(v^A(\mathcal{W}^A)\) changes from 0 to 1, the change in \(\tilde{\mathbf{PS}}_t(v^A)\) equals \(\Delta \tilde{\mathbf{PS}}_t(v^A) = (-1)^{1(t \notin \mathcal{W}^A)} \prod_{j \in \mathcal{W}^A, j \neq t} \frac{p_{A(j)}}{1 - p_{A(j)}}\), and the change in \(\mathbf{TE}^{Hete}(v^A)\) thus equals

\[
\Delta \mathbf{TE}^{Hete}(v^A) = \beta \sum_{t \in \mathcal{N}} \hat{\alpha}_t \Delta \tilde{\mathbf{PS}}_t(v^A)
\]
\[
\begin{align*}
&= \beta \left( \prod_{j \in W^A} \frac{p_{A(j)}}{1 - p_{A(j)}} \right) \left[ \sum_{t \in W^A} w_t - S_A \right] \\
&\geq \beta \left( \prod_{j \in W^A} \frac{p_{A(j)}}{1 - p_{A(j)}} \right) \left[ \sum_{t \in W^A} w_t - S_A \right] > 0,
\end{align*}
\]

where \( w_t = \frac{\alpha_t}{p_{A(t)} p_{B(t)}} \) and \( T_A = \sum_{t \in N} \frac{\alpha_t}{1 - p_{A(t)}} \). This implies that for some allocation rule \( v^A \), if there exists a \( W^A \) such that \( v^A(W^A) = 0 \) and \( \sum_{t \in W^A} w_t > S_A \), we can always construct a feasible rule \( \tilde{v}^A \) that gives strictly greater expected total effort than \( v^A \), as a result, \( v^A \) is not optimal. Therefore, the optimal rule must satisfy that

\[
v^A(W^A) = 1 \text{ whenever } w^A > S_A.
\]

Up to now, it is shown that \( v^A(W^A) = \begin{cases} 
1, & \text{if } w^A > S_A, \\
0, & \text{if } w^A < S_A.
\end{cases} \) It remains to investigate the case where \( w^A = S_A \). Consider a \( W^A \) such that \( w^A = S_A \). Clearly, both \( v^A(W^A) = 0 \) and \( v^A(W^A) = 1 \) are feasible, since monotonicity conditions hold. By previous analysis, the expected total effort remains unchanged when \( v^A(W^A) \) switches from 0 to 1 or from 1 to 0. The argument holds for all \( W^A \) such that \( w^A = S_A \). We therefore complete our analysis by discussing all the three cases.

In particular, when there does not exist a \( W^A \) such that \( w^A = S_A \), the optimal prize allocation rule is unique. Otherwise, the optimal prize allocation rule is not unique, since \( v^A(W^A) \) can take either 0 or 1 for any \( W^A \) such that \( w^A = S_A \).
References


Online Appendix for “Optimal Prize Design in Team Contests” by X. Feng, Q. Jiao, Z. Kuang, and J. Lu

This online appendix covers the proofs of Lemma 2, Corollary 1, and Proposition 2.

Proof of Lemma 2

We proceed through our proof in four steps, in accordance with the calculation procedures for the objective function.

**Step 1**: Determine the coefficient of $v^A(W^A)$ in $U^A_t(W^A_{t+1})$. Suppose the first $t$ battles are finished with history $W^A_{t+1}$. Consider an outcome $W^A$ that is possible to achieve after history $W^A_{t+1}$, i.e., $W^A \cap N_{t+1} = W^A_{t+1}$; it follows from the multiplicative law of probability that the coefficient of $v^A(W^A)$ in $U^A_t(W^A_{t+1})$ is

$$p_A^{|W^A| - |W^A_{t+1}|} (1 - p_A)^{(N-t) - (|W^A| - |W^A_{t+1}|)},$$

where $|W^A| - |W^A_{t+1}|$ is the number of winning battles after the first $t$ battles for team $A$. Therefore, $U^A_t(W^A_{t+1})$ is a linear function of $v^A(W^A)$, for any $W^A \subseteq N$,

$$U^A_t(W^A_{t+1}) = \sum_{W^A: W^A \cap N_{t+1} = W^A_{t+1}} p_A^{|W^A| - |W^A_{t+1}|} (1 - p_A)^{(N-t) - (|W^A| - |W^A_{t+1}|)} v^A(W^A).$$

In addition, the coefficient of $v^A(W^A)$ in $U^A_t(W^A_{t+1})$ is zero when $W^A \cap N_{t+1} \neq W^A_{t+1}$, i.e., $W^A$ is impossible to achieve after history $W^A_{t+1}$.

In sum, given $W^A_{t+1}$, if $W^A \cap N_{t+1} \neq W^A_{t+1}$, the coefficient is zero; if $W^A \cap N_{t+1} = W^A_{t+1}$, the coefficient of $v^A(W^A)$ in $U^A_t(W^A_{t+1})$ is $p_A^{|W^A| - |W^A_{t+1}|} (1 - p_A)^{(N-t) - (|W^A| - |W^A_{t+1}|)}$.

**Step 2**: Determine the coefficient of $v^A(W^A)$ in $V_t(W^A_t)$.

Note that the values of $V_t$ are determined by $U^A_t$, $V_t(W^A_t) = U^A_t(W^A_t \cup \{t\}) - U^A_t(W^A_t)$. Let $Q \subseteq N \setminus N_{t+1}$ denote the set of winning battles of team $A$ within the
remaining $N - t$ battles. Rearranging the recursive definition for $V$, we can get

$$V_t(W_t^A) = \sum_{Q: Q \subseteq N \setminus N_{t+1}} p_A^{|Q|} (1 - p_A)^{N - t - |Q|} [v^A(W_t^A \cup \{t\} \cup Q) - v^A(W_t^A \cup Q)].$$

In particular, if $W^A \cap N_t \neq W_t^A$, then there exists no $Q$ such that $W^A = W_t^A \cup \{t\} \cup Q$ or $W^A = W_t^A \cup Q$, and hence the coefficient of $v^A(W^A)$ in $V_t(W_t^A)$ is zero; otherwise, $W^A \cap N_t = W_t^A$, and the coefficient is nonzero, which depends on the outcome of battle $t$ as follows.

If winning battle $t$, i.e., $t \in W^A$, the coefficient of $v^A(W^A)$ in $V_t(W_t^A)$ is

$$p_A^{|W^A| - |W_{t+1}^A|} (1 - p_A)^{(N-t)-(|W^A| - |W_{t+1}^A|)}, \text{ where } W_{t+1}^A = W_t^A \cup \{t\}.$$

If losing battle $t$, i.e., $t \notin W^A$, the coefficient of $v^A(W^A)$ in $V_t(W_t^A)$ is

$$-p_A^{|W^A| - |W_{t+1}^A|} (1 - p_A)^{(N-t)-(|W^A| - |W_{t+1}^A|)}, \text{ where } W_{t+1}^A = W_t^A.$$

**Step 3:** Determine the coefficient of $v^A(W^A)$ in $\mathbb{E}_{W_t^A}(V_t(W_t^A))$.

Recall that $\mathbb{E}_{W_t^A}(V_t(W_t^A)) = \sum_{W_t^A \subseteq N_t} p_A^{|W_t^A|} (1 - p_A)^{|W_t^A| - 1} V_t(W_t^A)$.

If $t \in W^A$, the coefficient of $v^A(W^A)$ in $\mathbb{E}_{W_t^A}(V_t(W_t^A))$ is

$$p_A^{|W_t^A|} (1 - p_A)^{|W_t^A| - 1} p_A^{|W^A| - |W_{t+1}^A|} (1 - p_A)^{(N-t)-(|W^A| - |W_{t+1}^A|)}$$

$$= p_A^{|W^A| - 1} (1 - p_A)^{N - |W^A|},$$

since $|W_{t+1}^A| = |W_t^A| + 1$.

If $t \notin W^A$, the coefficient of $v^A(W^A)$ in $\mathbb{E}_{W_t^A}(V_t(W_t^A))$ is

$$-p_A^{|W_t^A|} (1 - p_A)^{|W_t^A| - 1} p_A^{|W^A| - |W_{t+1}^A|} (1 - p_A)^{(N-t)-(|W^A| - |W_{t+1}^A|)}$$

$$= -p_A^{|W^A|} (1 - p_A)^{N - |W^A| - 1},$$

since $|W_{t+1}^A| = |W_t^A|$.
Step 4: Determine the coefficient of $v^A(W^A)$ in $\sum_{t \in \mathcal{N}} \mathbb{E}_{W_t^A}(V_t(W_t^A))$, which is

$$|W^A|p_A^{\gamma - 1} (1 - p_A)^{N - |W^A|} - (N - |W^A|)p_A^{\gamma} (1 - p_A)^{N - |W^A| - 1},$$

by aggregating all coefficients of $v^A(W^A)$ in $\mathbb{E}_{W_t^A}(V_t(W_t^A))$ for all $t$.

Proof of Corollary 1

By Proposition 1, we consider path-independent allocation rules, under which team $i$ gets the same prize whenever its players win the same number of battles. In other words, for any $k^A$, $v^A(k^A) = v^A(W^A)$, $\forall W^A$ such that $|W^A| = k^A$. By Lemma 2, the organizer’s objective is to maximize

$$\text{TE}^{\text{Homo}}(v^A) = \alpha \sum_{W^A} C(|W^A|)v^A(W^A) = \alpha \sum_{k^A=0}^{N} \sum_{W^A:|W^A|=k^A} C(|W^A|)v^A(W^A)$$

$$= \alpha \sum_{k^A=0}^{N} \binom{N}{k^A} [k^A p_A^{k^A - 1} (1 - p_A)^{N - k^A} - (N - k^A) p_A^{k^A} (1 - p_A)^{N - k^A - 1}] v^A(k^A)$$

$$= \alpha \sum_{k^A=0}^{N} \binom{N}{k^A} k^A p_A^{k^A - 1} (1 - p_A)^{N - k^A} v^A(k^A)$$

$$- \alpha \sum_{k^A=0}^{N} \binom{N}{k^A} (N - k^A) p_A^{k^A} (1 - p_A)^{N - k^A - 1} v^A(k^A)$$

$$= \alpha N \sum_{k^A=1}^{N} \binom{N - 1}{k^A - 1} p_A^{k^A - 1} (1 - p_A)^{N - k^A} v^A(k^A)$$

$$- \alpha N \sum_{k^A=0}^{N} \binom{N - 1}{k^A} p_A^{k^A} (1 - p_A)^{N - k^A - 1} v^A(k^A)$$

$$= \alpha N \sum_{k=0}^{N-1} \binom{N - 1}{k} p_A^k (1 - p_A)^{N - k - 1} [v^A(k + 1) - v^A(k)].$$

An alternative way to show Corollary 1 is to start with Proposition 1 and directly calculate effective prize spread. In an arbitrary component battle $t$, two matched
players have the same prize spread that equals
\[
\sum_{k=0}^{N-1} \binom{N-1}{k} p_A^k (1-p_A)^{N-k-1} [v^A(k+1) - v^A(k)],
\]
where \( \binom{N-1}{k} p_A^k (1-p_A)^{N-k-1} \) is the probability that other members of team A wins \( k \) out of all battles excluding battle \( t \), and \( v^A(k+1) - v^A(k) \) is the prize increment when other members of team A win \( k \) battles.

**Proof of Proposition 2**

We provide the following example, which serves our purpose.

**Example B.1:** When \( p_A \in (\frac{k}{N}, \frac{k+1}{N}) \) and \( k \geq \left\{ \begin{array}{ll} \frac{N+1}{2} & \text{when } N \text{ is odd} \\ \frac{N}{2} & \text{when } N \text{ is even} \end{array} \right. \), there exists a unique \( p^*(k) \) such that (1) over-correction happens if \( p_A \in (\frac{k}{N}, p^*(k)) \), (2) under-correction happens if \( p_A \in (p^*(k), \frac{k+1}{N}) \), and (3) just-correction happens if \( p_A = p^*(k) \).

To see that, we consider the following. From Theorem 1, when \( p_A \in (\frac{k}{N}, \frac{k+1}{N}) \) and \( k \geq \left\{ \begin{array}{ll} \frac{N+1}{2} & \text{when } N \text{ is odd} \\ \frac{N}{2} & \text{when } N \text{ is even} \end{array} \right. \), majority rule with a headstart with \( K_S^* = k^* + 1 \) is the optimal, where \( k^* = \lfloor p_A N \rfloor \). We turn to the binomial distribution \( B(N, p_A) \), which is the distribution of the number of battle wins for team A. Let \( m \) denote the median of \( B(N, p_A) \). Note that \( m \) must satisfy \((N-1)p_A < m < (N+1)p_A\) when \( p_A > 0.5 \).

Under majority rule with a headstart with \( K_S^* = k^* + 1 \), when team A wins at most \( k^* \) battles, team B wins the whole prize; when team A wins at least \( K_S^* \) battles, team A wins. In other words, over-correction happens if the median \( m \leq k^* \) and under-correction happens when \( m > k^* \), unless \( \Pr(X \leq m) = 0.5 \).\(^{15}\)

- When \( \{p_A N\} \leq 1 - p_A \), i.e., \( \{p_A N\} \) is relatively small, \( \lfloor p_A N \rfloor \) is the unique

\(^{15}\)In case that \( \Pr(X \leq m) = 0.5 \), over-correction happens if the median \( m < k^* \), just-correction happens if the median \( m = k^* \), and under-correction happens when \( m > k^* \). The argument is analogous.
integer that satisfies $(N - 1)p_A < m < (N + 1)p_A$, and therefore $m = k^*$ and over-correction happens.

- When $\{p_A N\} \geq p_A$, i.e., $\{p_A N\} + 1$ is the unique integer that satisfies $(N - 1)p_A < m < (N + 1)p_A$, and therefore $m = k^* + 1$ and under-correction happens.

Until now, we have shown that within interval $p_A \in (\frac{k}{N}, \frac{k+1}{N})$, over-correction happens when $p_A \in (\frac{k}{N}, \frac{k+1}{N+1})$ and under-correction happens when $p_A \in [\frac{k}{N-1}, \frac{k+1}{N}]$. It remains unclear which situation happens when $p_A \in (\frac{k+1}{N+1}, \frac{k}{N-1})$. Let $F(k, p_A|N)$ denote the cumulative distribution function of binomial distribution $B(N, p_A)$, i.e.,

$$F(k, p_A|N) = \sum_{i=0}^{k} \binom{N}{i} p_A^i (1 - p_A)^{N-i}.$$ 

We have shown that $F(k, p_A|N) > \frac{1}{2}$ when $p_A \in (\frac{k}{N}, \frac{k+1}{N+1})$ and $F(k, p_A|N) < \frac{1}{2}$ when $p_A \in (\frac{k}{N-1}, \frac{k+1}{N})$.

Fixing $k$, $F(k, p_A|N)$ is decreasing in $p_A$, since

$$\frac{\partial F(k, p_A|N)}{\partial p_A} = -N \binom{N-1}{k-1} p_A^k (1 - p_A)^{N-k-1} < 0.$$ 

As a result, there exists a $p^*(k) \in (\frac{k+1}{N+1}, \frac{k}{N-1})$ such that $F(k, p^*(k)|N) = \frac{1}{2}$, which leads to just-correction.