All Pay Quality-Bids in Score Procurement Auctions*

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Abstract

We study score procurement auctions with all-pay quality bids, in which a supplier's score is the difference between his quality and price bids. Equilibrium quality and price bids are solved without first obtaining the corresponding equilibrium scores. In particular, our approach accommodates the case with a minimum score or quality requirement. When the convex effort cost function takes a power form, a higher all-pay component of the quality bid reduces quality provision, total surplus, and suppliers' payoffs, but may increase or decrease the procurer's payoff. If the procurer reimburses the all-pay components of losing suppliers or all suppliers, this would increase quality provision and suppliers' payoffs, but reduce total surplus and the procurer's payoff. Finally, we rely on our approach to identify the procurer-optimal score rule, which is quasi-linear in quality and price. Relative to the procurer's payoff function, the optimal score rule values quality less, and the score rule function increases in quality at a lower rate than the procurer's payoff function. When suppliers' type distribution has a weakly convex reverse hazard rate, the optimal score rule is more responsive to quality or values the quality more when placing the quality bid gets more costly or there are more suppliers.

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1 Introduction

Procurement is widely adopted to acquire goods, services, or work. On average, about 15% of the yearly global domestic product is spent on public procurement alone, including military acquisitions.[1] Procurements involving multi-dimensional bids are ubiquitous. Typically, suppliers are required to bid on both quality and price, which jointly form single-dimensional scores that are used by the procurer to determine the winner, whose offer turns out to be the most economically advantageous.

Since the seminal work of Che [7], score auctions with winner-pay quality bids have been studied in the literature in many contexts, including Branco [6], Asker and Cantillon [1, 2], Bajari, Houghton and Tadelis [3], Wang and Liu [15], Hanazono, Nakabayashi and Tsuruoka [10] and Nishimura [13]. In this literature, the central idea in identifying equilibrium bidding strategies is to transform the multi-dimensional bidding problem into a single-dimensional problem of bidding on the score. This idea is based on the insightful observation that a supplier should choose quality and price bids optimally to maximize his payoff conditional on winning with *any* score s. The optimal quality and price choice for any score s would allow the definition of a *value function* for each supplier, which only depends on his/her own type and the score bid s. With this value function, the original problem with multi-dimensional bids is transformed into a standard single-dimensional problem with a score bid.[2]

In many situations, the quality bids are more realistically interpreted as a partially all-pay component rather than a winner-pay component.[3] For example, all-pay quality bids are often features of procurements in defense contracting and other military design competitions, architectural designs, government construction projects with a design component, and business-to-business customized sales. To the best of our knowledge, Che and Gale [8] are the first to study score procurement with all-pay quality bids in a complete information environment. Their focus is to demonstrate the optimality of shortlisting and handicapping in such environments.

Our study is the first to introduce partially all-pay quality bids in score procurement auctions with incomplete information. Specifically, we adopt an N-supplier environment, in which a supplier's score is the difference between his quality and price bids. The supplier

[1] See "Supplement to the 2013 Annual Statistical Report on United Nations Procurement: Procurement and Innovation," https://www.unsystem.org/content/supplement-2013-annual-statistical-report-united-nations-procurement (accessed Jan. 15, 2019).

[2] For a recent general treatment of winner-pay quality-bid score auctions, please refer to Hanazono, Nakabayashi and Tsuruoka [10] for details.

[3] See Baye, Kovenock and de Vries [4] and Kaplan, Luski, Sela and Wettstein [11] among many others. Gershkov, Moldovanu, Strack and Zhang [9] study an environment where bidders incur costs to increase their values.

with the higher score wins and gets paid his own price bid. The procurer's payoff equals the difference between the winner's quality and her payments to the suppliers.

We provide an alternative procedure to identify the unique symmetric pure strategy equilibrium with scores monotonic in types if such an equilibrium exists.[4] The merit of our alternative procedure lies in that we directly identify the quality bid and price bid without relying on an intermediate step of solving for the score bid. This procedure is made possible because of another observation on the relationship between equilibrium price and quality bids in score auctions, which has not been utilized so far in the score auction literature: We can view the pair of equilibrium quality and price bids as an incentive-compatible direct mechanism, with the quality bid being the allocation rule and the price bid the payment rule. Then, as well understood from Myerson [12], the price bidding function can be fully pinned down as a function of the quality bidding function. This observation together with the well-utilized insight that a supplier chooses quality and price bids optimally to maximize his payoff conditional on bidding the equilibrium score would allow us to obtain a condition on the equilibrium quality bidding function which, together with the appropriate boundary condition identifies that function. It thus follows that we can further identify the equilibrium price bidding function based on the above mentioned relationship.

Our procedure can conveniently accommodate a requirement of minimum score or quality imposed by the procurer. For any given minimum score or quality, we explicitly pin down the symmetric entry cutoff type of suppliers and their symmetric equilibrium bidding strategy (quality and price) of participating types. We find that for a given score rule, it is typically possible to improve the procurer's payoff by imposing a minimum score or quality.

The equilibrium analysis enables us to study the effect of the partially all-pay component of quality bid on quality provision, supplier payoffs, procurer payoffs, and total surplus. When the effort cost function is convex and takes a power form, we find that a higher proportion of all pay components would reduce quality provision, the total surplus, and suppliers' payoff. However, it does not necessarily always increase or decrease the procurer's payoff. In particular, we illustrate that all-pay quality bids can indeed generate either higher or lower procurer payoff than winner-pay quality bids. A higher proportion of all pay components lowers quality bids, which induces the suppliers to lower their price bids in order to make their offers attractive to the procurer. It is thus possible that the drop in price bids dominates the drop in quality bids, such that a higher proportion of all pay components leads to a higher procurer payoff. When the marginal cost curve is less steep, the suppliers would lower less their quality bids when the proportion of all pay components gets higher. As a result, it is more likely that a higher proportion of all pay components may benefit the procurer.

[4] The procedure applies to both cases of all-pay and winner-pay quality bids. Our procedure also applies to intermediate cases in Siegel [14].

We further study the effect of the procurer's reimbursing the losing suppliers' or all suppliers' all-pay component of quality bid on quality provision, supplier payoffs, procurer payoffs and total surplus.[5] When the effort cost function is convex and takes a power form, we find that such reimbursements would increase the quality provision and the suppliers' payoffs, but they would reduce the total surplus and procurer's payoff. It is pretty intuitive that such reimbursements would increase quality provision and supplier payoff as these measures reduce the suppliers' marginal provision costs. However, since social marginal quality provision costs are not affected by these reimbursements and thus are higher than suppliers' individual quality provision costs, the enhanced quality provision can be overly costly to society. As a result, suppliers' optimal decision on quality provision under such subsidies of reimbursements tends to be less efficient. This further means lower procurer payoff under such reimbursements as supplier payoff gets higher.

With an all-pay component of quality bid, the Myerson upper bound on procurer payoff is no longer achievable. One needs to identify a tight upper bound. Our perspective of viewing the symmetric equilibrium quality bid as the allocation rule allows us to establish a version of payoff equivalence in our setting with all pay features. This payoff equivalence result says that the procurer and suppliers' expected payoffs are solely determined by the quality bid and the least efficient type's payoff. A pointwise maximization of the procurer's payoff generates the optimal allocation rule (i.e. the quality bidding function), and we identify explicitly a quasi-linear score rule that induces the optimal allocation rule at equilibrium. The equilibrium price bidding function is also explicitly identified. The optimal score rule leads to a monotone score in types. We find that the optimal score rule must be less responsive to quality or values less the quality than the procurer's payoff function does. Moreover, when supplies' type distribution has a convex reverse hazard rate, the optimal score rule is more responsive to quality or values more the quality when placing the quality bid gets more costly or there are more suppliers.

The rest of the paper is organized as follows. Section 2 sets up the model with partially all-pay quality bids. Section 3 provides a procedure for equilibrium analysis, which applies to a continuum of scenarios from all-pay to winner-pay quality bids. We further demonstrate in this section that our procedure can easily accommodate the requirement of a minimum score imposed by the procurer. This section also covers a comparative static analysis with respect to the proportion of all pay components. Section 4 examines the impact of the procurer's reimbursement of suppliers' all-pay quality bidding costs on quality provision, supplier payoffs, the procurer's payoff, and total surplus. Section 5 derives the procurer-optimal score rule and the associated bidding equilibrium. We also investigated the properties of the optimal

[5] These exercises thus resemble those of Baye, Kovenock and de Vries [5] and Siegel [14] who study equilibria in auctions with contingent investment. Baye, Kovenock and de Vries [5] study auctions with incomplete information, and Siegel [14] studies settings with complete information.

score rule. Section 6 provides some concluding remarks. Technical proofs are relegated to the appendices.

2 Model

We consider a score procurement auction with a risk-neutral procurer and N risk-neutral suppliers. The suppliers, $i = 1, 2, ..., N$, simultaneously choose their nonnegative bids of quality q_i and price p_i to compete for a contract. Only the winner will collect his price bid p, while the quality bid q is a partially all-pay component paid by each bidder. Specifically, a type c_i of supplier i must incur a cost of $\rho C_i(q; c_i)$ to bid a quality q whether he wins the contract or not. Here, $C_i(q; c_i)$ is the total provision cost that a type c_i of supplier i must incur to provide quality q if he is selected as the winner. As in Baye, Kovenock and de Vries [5] and Siegel [14], we introduce parameter $\rho \in [0, 1]$ that represents the proportion of all pay components in the total provision cost. We assume $C_i(q; c_i) = c_i\varsigma(q)$, in which the function $\varsigma(\cdot)$ is continuous on $[0, +\infty)$, with $\varsigma(0) = \varsigma'(0) = 0, \varsigma'(q) > 0, \forall q > 0$, and $\varsigma''(q) > 0, \forall q \geq 0$.[6] For example, the class of functions $\varsigma(q) = q^\gamma, \gamma > 1$ satisfies these conditions. c_i is supplier i's private type, with high c_i leading to high marginal effort cost. We assume that the c_i are independently and identically distributed following cumulative distribution $F(\cdot)$ on $[\underline{c}, \bar{c}]$ with a positive density $f(\cdot) > 0$. We assume $0 < \underline{c} < \bar{c} < +\infty$.

The higher score wins and the score function is defined as

$$s(q, p) = q - p. \tag{1}$$

The procurer's payoff is the difference between the q and p of the winning bidder. Supplier i's payoff is $p_i - C_i(q_i)$ if he wins, otherwise it is $-\rho C_i(q_i)$. The equilibrium analysis in this paper can be easily extended to cases where the procurer's payoff takes the form of $u(q) - p$ with $u(\cdot)$ being increasing and concave, and the score rule takes a quasi-linear form of $s = v(q) - p$ with $v(\cdot)$ being increasing and concave. Please refer to Section 5 for details.

3 Equilibrium analysis and comparative statics

3.1 Equilibrium analysis

We now derive the unique equilibrium within the class of symmetric pure strategy equilibria with scores decreasing in type. There are two key observations that facilitate identifying the equilibrium. Suppose the score of type c_i is $s(c_i)$ in the equilibrium. First, if we view

[6] $\varsigma'(0)$ and $\varsigma''(0)$ refer to the right hand derivatives at zero.

the equilibrium strategy $(q(c_i), p(c_i))$ as an incentive-compatible direct mechanism, then the Myerson [12] approach, together with a zero payoff condition for type $\bar{c}$, would allow us to pin down a payment rule $p(c_i; q(\cdot))$ for any given allocation rule $q(\cdot)$ decreasing in type. Second, define the score by $s(c_i) = q(c_i) - p(c_i; q(\cdot)), \forall c_i$. Note $s(\cdot)$ is solely determined by the allocation rule $q(\cdot)$. Given supplier i with type c_i bids this score $s(c_i)$ in equilibrium, supplier i would choose a pair $(q, p) = (q(c_i), p(c_i))$ when maximizing his expected payoff. This further gives us conditions on the equilibrium bidding functions $(q(\cdot), p(\cdot))$. The restrictions on $(q(\cdot), p(\cdot))$, which are implied by the above two observations, together with the boundary condition $q(\bar{c}) = 0$, pins down a candidate equilibrium allocation rule $q(\cdot)$. If this $q(\cdot)$ is decreasing in type, and the corresponding $s(\cdot)$ is also decreasing in type, then the identified $(q(\cdot), p(\cdot))$ is the symmetric pure strategy equilibrium. Otherwise, there exists no equilibrium within the class.

We will demonstrate that the above two observations allow us to derive the unique symmetric pure strategy equilibrium with scores decreasing in type,[7] when such an equilibrium exists, and to determine when no such equilibrium exists.

3.1.1 Implications of the first observation (Myerson incentive compatibility)

We first consider the implications of the first observation. Fix any $q(c_i)$. We can view any equilibrium strategy $(q(c_i), p(c_i))$ as an incentive-compatible direct mechanism that leads to decreasing scores in type. The quality bid function $q(c_i)$ can be viewed as an allocation rule, and the price function $p(c_i)$ can be viewed as a payment rule. Then the Myerson approach allows us to pin down the payment rule $p(c_i)$ as a function of the allocation rule $q(\cdot)$. The following provides the details.

Given a $(q(c_i), p(c_i))$ pair that leads to scores that are decreasing in type, consider a type c_i supplier's problem of choosing a $\tilde{c}_i$ to maximize his expected payoff when the other suppliers are truthful:

$$\max_{\tilde{c}_i} \pi_i(\tilde{c}_i, c_i) = [1 - F(\tilde{c}_i)]^{N-1}[p(\tilde{c}_i) - (1-\rho)c_i\varsigma(q(\tilde{c}_i))] - \rho c_i \varsigma(q(\tilde{c}_i)). \tag{2}$$

If $(q(\cdot), p(\cdot))$ is a pure strategy equilibrium, then $\tilde{c}_i^*$, the solution to (2), satisfies $\tilde{c}_i^* = c_i$, $\forall c_i$. In other words, the following incentive compatibility (IC) condition holds:

$$c_i = \arg\max_{\tilde{c}_i} \pi_i(\tilde{c}_i, c_i) = [1 - F(\tilde{c}_i)]^{N-1}[p(\tilde{c}_i) - (1-\rho)c_i\varsigma(q(\tilde{c}_i))] - \rho c_i \varsigma(q(\tilde{c}_i)). \tag{3}$$

[7] The identified equilibrium score, quality and price bids are all differentiable in type.

If $q'(c_i) < 0$, the single crossing condition holds:

$$\begin{aligned}\frac{\partial^2 \pi_i(\tilde{c}_i, c_i)}{\partial \tilde{c}_i \partial c_i} &= -\frac{d\{[1-F(\tilde{c}_i)]^{N-1}(1-\rho)\varsigma(q(\tilde{c}_i)) + \rho\varsigma(q(\tilde{c}_i))\}}{d\tilde{c}_i} \\ &= (N-1)[1-F(\tilde{c}_i)]^{N-2} f(\tilde{c}_i)(1-\rho)\varsigma(q(\tilde{c}_i)) \\ &\quad -[1-F(\tilde{c}_i)]^{N-1}(1-\rho)\varsigma'(q(\tilde{c}_i))q'(\tilde{c}_i) - \rho\varsigma'(q(\tilde{c}_i))q'(\tilde{c}_i) \\ &> 0,\end{aligned}$$

which means such $q(c_i)$ can be supported by the pricing rule that will be identified below.

The IC condition (3), together with the envelope theorem, thus leads to

$$\frac{d\pi_i(c_i, c_i)}{dc_i} = -\{[1-F(c_i)]^{N-1}(1-\rho)\varsigma(q(c_i)) + \rho\varsigma(q(c_i))\}. \tag{4}$$

Note that in equilibrium with scores monotonic in type, regardless of the value of ρ, we must have $\pi_i(\bar{c}, \bar{c}) = 0$. In equilibrium, the least efficient type $\bar{c}$ never wins and never incurs a cost of quality.[8]

Using the envelope condition (4), we have

$$\pi_i(c_i, c_i) = \int_{c_i}^{\bar{c}} \{[1-F(t)]^{N-1}(1-\rho)\varsigma(q(t)) + \rho\varsigma(q(t))\}dt. \tag{5}$$

By the definition of $\pi_i(\tilde{c}_i, c_i)$ in (2), we further have

$$\begin{aligned}\pi_i(c_i, c_i) &= [1-F(c_i)]^{N-1}[p(c_i) - (1-\rho)c_i\varsigma(q(c_i))] - \rho c_i\varsigma(q(c_i)) \\ &= \int_{c_i}^{\bar{c}} \{[1-F(t)]^{N-1}(1-\rho)\varsigma(q(t)) + \rho\varsigma(q(t))\}dt,\end{aligned}$$

which leads to the price rule

$$p(c_i) = \frac{\int_{c_i}^{\bar{c}} \{[1-F(t)]^{N-1}(1-\rho)\varsigma(q(t)) + \rho\varsigma(q(t))\}dt + \rho c_i\varsigma(q(c_i))}{[1-F(c_i)]^{N-1}} + (1-\rho)c_i\varsigma(q(c_i)) \geq 0. \tag{6}$$

3.1.2 Implications of the second observation (optimal bids for a given score)

We now turn to the implication of the second observation. Suppose $(q(c_i), p(c_i))$ is a symmetric pure strategy equilibrium. By implication of the first observation, we must have that (6) holds. Define

$$s(c_i) = s(q(c_i), p(c_i)). \tag{7}$$

[8] Note $q(\bar{c}) = 0$ for $\rho = 1$.

The second observation says that any equilibrium $(q(c_i), p(c_i))$ must solve the following optimization problem:

$$\begin{aligned} (q(c_i), p(c_i)) &= \arg\max_{(q,p)} \{[1 - F(c_i)]^{N-1}[p - (1-\rho)c_i\varsigma(q)] - \rho c_i \varsigma(q)\} \\ s.t. \quad &: \quad s = q - p = s(c_i). \end{aligned}$$

The corresponding Lagrangian for the above problem is

$$\max_{(q,p,\lambda)} L(q,p,\lambda) = [1-F(c_i)]^{N-1}[p - (1-\rho)c_i\varsigma(q)] - \rho c_i\varsigma(q) + \lambda[(q-p) - s(c_i)].$$

The first-order conditions are as follows.

$$\begin{aligned} \frac{L(q,p,\lambda)}{\partial q} &= -[1-F(c_i)]^{N-1}(1-\rho)c_i\varsigma'(q) - \rho c_i\varsigma'(q) + \lambda = 0, \\ \frac{L(q,p,\lambda)}{\partial p} &= [1-F(c_i)]^{N-1} - \lambda = 0, \\ \frac{L(q,p,\lambda)}{\partial \lambda} &= (q-p) - s(c_i) = 0. \end{aligned}$$

We thus have

$$\varsigma'(q) = \frac{[1-F(c_i)]^{N-1}}{c_i\{[1-F(c_i)]^{N-1}(1-\rho) + \rho\}} = \frac{1}{c_i\left[(1-\rho) + \frac{\rho}{[1-F(c_i)]^{N-1}}\right]}. \tag{8}$$

We can thus pin down the candidate equilibrium quality

$$q(c_i) = (\varsigma')^{-1}\left(\frac{1}{c_i\{(1-\rho) + \frac{\rho}{[1-F(c_i)]^{N-1}}\}}\right). \tag{9}$$

Note that the quality bid $q(c_i)$ is identified by only using the second observation. This is the case for the difference-form score function that we consider, but it is not the case in general. Please refer to Appendix B for the procedure for identifying $q(c_i)$ without solving for the equilibrium score bidding function for a general score function. In particular, the case of a ratio score function is studied there as an example.

Clearly, for an all-pay quality bid with $\rho = 1$, we have $q(\bar{c}) = (\varsigma')^{-1}(0) = 0$.[9] For a winner-pay quality bid with $\rho = 0$, we have $q(\bar{c}) = (\varsigma')^{-1}(\frac{1}{\bar{c}}) > 0$ since $\varsigma''(\cdot) > 0$. Clearly, we have $q'(\cdot) < 0$, since

$$q'(c_i) = \frac{\left[\frac{1}{c_i\{(1-\rho) + \frac{\rho}{[1-F(c_i)]^{N-1}}\}}\right]'}{\varsigma''(q(c_i))} < 0.$$

[9] In general, $q(\bar{c}) = 0$ holds for any $\rho \in (0, 1]$.

Since $q(c_i)$ is decreasing, the above characterized $(q(\cdot), p(\cdot))$ would constitute a symmetric pure strategy equilibrium within the class we consider if and only if the corresponding score function $s(c_i) = q(c_i) - p(c_i)$ is decreasing in type. This is indeed the case as verified by the following lemma, whose proof is relegated to appendix A.

Lemma 1 *$s'(c_i) = q'(c_i) - p'(c_i) < 0$.*

We thus have the following theorem.

Theorem 1 *The symmetric bidding strategy $(q(\cdot), p(\cdot))$ identified in (9) and (6) constitutes the unique symmetric pure strategy equilibrium with scores decreasing in types. Moreover, $q(\cdot)$ is decreasing in type.*

The way we establish the equilibrium implies that the following two remarks are straightforward.

Remark 1 *Our equilibrium characterization requires $\varsigma'(0) = 0$, which says marginal cost must be zero when quality is zero. Che and Gale [8] also require the same condition in their equilibrium analysis in a complete information score auction setting. If $\varsigma'(0) > 0$, the equilibrium involves pooling. For example, consider $\rho = 1$. Define the cutoff $\hat{c}$ by $\varsigma'(0) = \frac{[1-F(\hat{c})]^{N-1}}{\hat{c}}$. In equilibrium, all types in $[\hat{c}, \bar{c}]$ place the same bid as type $\bar{c}$.*[10]

We have the following two additional remarks on applying our procedure to a more general environment.

Remark 2 *The above two-step procedure for identifying the symmetric pure strategy equilibrium if it exists can be generalized to an environment with an arbitrary number of suppliers and a general class of score functions $s(q, p)$. The details are included in Appendix B. In an application there with the score function $s = s(q, p) = \frac{q}{p}$ and $N \geq 2$ suppliers,*[11] *we show that if $\varsigma'(0) = 0$, then a symmetric pure strategy equilibrium generating scores decreasing in type does not exist for the case of all-pay quality bids. However, if $\varsigma'(0) > 0$, then the equilibrium for the case of all-pay quality bids is identified as*

$$
\begin{aligned}
q(c_i) &= (\varsigma')^{-1}\left(\frac{\bar{c}\varsigma'(0)}{c_i}\right), \\
p(c_i) &= \frac{\int_{c_i}^{\bar{c}} \varsigma(q(t))dt + c_i\varsigma(q(c_i))}{[1 - F(c_i)]^{N-1}}.
\end{aligned}
$$

[10] This is implied by (9).

[11] The equilibrium analysis in the application applies to a score function of $s = s(q, p) = \frac{q^{\mu}}{p^{\tau}}$ with $\mu, \tau > 0$.

Remark 3 *Following the insights of Hanazono, Nakabayashi and Tsuruoka [10], there is an alternative approach, which instead requires identifying an equilibrium score function as an intermediate step before decomposing it into quality and price bids. The details are as follows.*[12]

Assume a general score rule of $s(q,p)$ and $N \geq 2$ suppliers. From the second observation, if $s(c_i)$ is the equilibrium score, we can solve for

$$\begin{aligned}(q(c_i|s(c_i)), p(c_i|s(c_i))) &= \arg\max_{(q,p)}[1-F(c_i)]^{N-1}[p-(1-\rho)c_i\varsigma(q)] - \rho c_i\varsigma(q), \\ s.t. &: \quad s(q,p) = s(c_i).\end{aligned}$$

Then consider supplier i's optimization problem

$$\max_{\tilde{c}_i} \pi_i(\tilde{c}_i, c_i) = [1-F(\tilde{c}_i)]^{N-1}[p(\tilde{c}_i|s(\tilde{c}_i)) - (1-\rho)c_i\varsigma(q(\tilde{c}_i|s(\tilde{c}_i)))] - \rho c_i\varsigma(q(\tilde{c}_i|s(\tilde{c}_i))).$$

If $s(c_i)$ is an equilibrium, then we must have $\tilde{c}_i^* = c_i$. This gives the following first-order condition

$$\begin{aligned}&-(N-1)[1-F(c_i)]^{N-2}f(\tilde{c}_i)][p(\tilde{c}_i|s(\tilde{c}_i)) - (1-\rho)c_i\varsigma(q(\tilde{c}_i|s(\tilde{c}_i)))] \\ &+[1-F(c_i)]^{N-1}[p_{c_i}(\tilde{c}_i|s(\tilde{c}_i)) + p_s(\tilde{c}_i|s(\tilde{c}_i))s'(\tilde{c}_i) \\ &\qquad - (1-\rho)c_i\varsigma'(q(\tilde{c}_i|s(\tilde{c}_i)))(q_{c_i}(\tilde{c}_i|s(\tilde{c}_i)) + q_s(\tilde{c}_i|s(\tilde{c}_i))s'(\tilde{c}_i))] \\ &-\rho c_i\varsigma'(q(\tilde{c}_i|s(\tilde{c}_i)))(q_{c_i}(\tilde{c}_i|s(\tilde{c}_i)) + q_s(\tilde{c}_i|s(\tilde{c}_i))s'(\tilde{c}_i)) \\ = {}& 0, \text{ when } \tilde{c}_i = c_i.\end{aligned}$$

Under the proper conditions, this equation would pin down a solution of $s(c_i)$ with applicable boundary condition. If $s(c_i)$ is decreasing in type, then the equilibrium is identified as $(q(\cdot), p(\cdot)) = (q(c_i|s(c_i)), p(c_i|s(c_i)))$. The procedure proposed in this paper has the merit of saving the middle step of solving for the equilibrium score by first solving for the quality bid directly.

3.2 The case with minimum score

In this subsection, we demonstrate how our approach accommodates the case with minimum score requirement. The new complication of a minimum score $\hat{s}(>0)$ lies in that it induces endogenous participation of potential suppliers. We focus on the entry equilibrium where

[12] Our approach avoids this additional intermediate step of solving the score bidding function, which could involve substantial calculations.

all suppliers adopt a symmetric entry threshold $\hat{c}$, and we look for a symmetric bidding equilibrium across participants. In our analysis, we assume that the number of participants is not disclosed when they make their bidding decisions.

Our strategy for identifying the entry and bidding equilibrium is as follows. We start with an entry threshold $\hat{c} \in [\underline{c}, \bar{c}]$, and pin down the corresponding minimum score $\hat{s}$ that induces $\hat{c}$, and the corresponding bidding equilibrium.

Note that it must be true that the entry threshold type $\hat{c}$'s expected payoff is zero at equilibrium, i.e. $\pi_i(\hat{c}, \hat{c}) = 0$. First, the second observation in Section 3.1 remains valid. Therefore, the equilibrium quality bid $q(c_i)$ of any participating type remains the same as identified in (9). We next pin down their equilibrium price bids.

Using $\pi_i(\hat{c}, \hat{c}) = 0$ and the envelope condition (4) that is derived based on the first observation in Section 3.1, we have

$$\pi_i(c_i, c_i; \hat{c}) = \int_{c_i}^{\hat{c}} \{[1 - F(t)]^{N-1}(1-\rho)\varsigma(q(t)) + \rho\varsigma(q(t))\}dt. \tag{10}$$

By the definition of $\pi_i(\tilde{c}_i, c_i)$ in (2) and (10), we further have the following equilibrium price rule

$$p(c_i; \hat{c}) = \frac{\int_{c_i}^{\hat{c}} \{[1 - F(t)]^{N-1}(1-\rho)\varsigma(q(t)) + \rho\varsigma(q(t))\}dt + \rho c_i \varsigma(q(c_i))}{[1 - F(c_i)]^{N-1}} + (1-\rho)c_i\varsigma(q(c_i)) \geq 0. \tag{11}$$

Clearly, we have

$$p(c_i; \hat{c}) = p(c_i; \bar{c}) - \frac{\int_{\hat{c}}^{\bar{c}} \{[1 - F(t)]^{N-1}(1-\rho)\varsigma(q(t)) + \rho\varsigma(q(t))\}dt}{[1 - F(c_i)]^{N-1}}.$$

Let $s(c_i; \hat{c}) = p(c_i; \hat{c}) - q(c_i)$ for $c_i \leq \hat{c}$. We have that

$$s(c_i; \hat{c}) = s(c_i; \bar{c}) + \frac{\int_{\hat{c}}^{\bar{c}} \{[1 - F(t)]^{N-1}(1-\rho)\varsigma(q(t)) + \rho\varsigma(q(t))\}dt}{[1 - F(c_i)]^{N-1}}, \forall c_i \leq \hat{c}.$$

Clearly, $s(c_i; \hat{c})$ decreases with $\hat{c}$. We next show that $s(c_i; \hat{c})$ decreases with $c_i \leq \hat{c}$. From (11), we have that $\forall c_i \leq \hat{c}$,

$$\begin{aligned} & p(c_i; \hat{c})[1 - F(c_i)]^{N-1} \\ = & \int_{c_i}^{\hat{c}} \{[1 - F(t)]^{N-1}(1-\rho)\varsigma(q(t)) + \rho\varsigma(q(t))\}dt + \rho c_i\varsigma(q(c_i)) + [1 - F(c_i)]^{N-1}(1-\rho)c_i\varsigma(q(c_i)). \end{aligned}$$

Thus, we have

$$\frac{d\left\{p(c_i; \hat{c})[1 - F(c_i)]^{N-1}\right\}}{dc_i} = \rho c \varsigma'(q(c_i))q'(c_i) + c_i \frac{d\{(1 - F(c_i))^{N-1}(1-\rho)\varsigma(q(c_i))\}}{dc_i}.$$

It follows that

$$
\begin{aligned}
& (1-F(c_i))^{N-1}\frac{dp(c_i;\hat{c})}{dc_i} - (N-1)(1-F(c_i))^{N-2}f(c)p(c_i;\hat{c}) \\
= \; & \rho c \varsigma'(q(c_i))q'(c_i) + c_i \frac{d\{(1-F(c_i))^{N-1}(1-\rho)\varsigma(q(c_i))\}}{dc_i}.
\end{aligned}
$$

Therefore, we have

$$
\begin{aligned}
& \frac{dp(c_i;\hat{c})}{dc_i} - \frac{(N-1)f(c_i)}{1-F(c_i)}p(c_i;\hat{c}) \\
= \; & q'(c_i)\frac{c_i[\rho+(1-\rho)(1-F(c_i))^{N-1}]}{(1-F(c_i))^{N-1}}\varsigma'(q(c_i)) - \frac{(N-1)f(c_i)}{1-F(c_i)}c_i(1-\rho)\varsigma(q(c_i)).
\end{aligned}
$$

Note that from (8), we have $\frac{c_i[\rho+(1-\rho)(1-F(c_i))^{N-1}]}{(1-F(c_i))^{N-1}}\varsigma'(q(c_i)) = 1$. Therefore, we have

$$
\frac{dp(c_i;\hat{c})}{dc_i} = q'(c_i) - \frac{(N-1)f(c_i)}{1-F(c_i)}c_i(1-\rho)\varsigma(q(c_i)) + \frac{(N-1)f(c_i)}{1-F(c_i)}p(c_i),
$$

which gives

$$
\frac{ds(c_i;\hat{c})}{dc_i} = \frac{dp(c_i;\hat{c})}{dc_i} - q'(c_i) = \frac{(N-1)f(c_i)}{1-F(c_i)}\{c_i(1-\rho)\varsigma(q(c_i)) - p(c_i)\}.
$$

Note that from (11), we have $c_i(1-\rho)\varsigma(q(c_i)) - p(c_i) < 0$, $\forall c_i \leq \hat{c}$. We thus have $\frac{ds(c_i;\hat{c})}{dc_i} < 0$, $\forall c_i \leq \hat{c}$.

Now that $s(c_i;\hat{c})$ decreases in both c_i and $\hat{c}$, we must have $s(\hat{c};\hat{c})$ decreases in $\hat{c} \in [\underline{c}, \bar{c}]$. Note that $s(\bar{c};\bar{c}) = 0$ as $q(\bar{c}) = 0$ and $p(\bar{c};\bar{c}) = 0$. Let $\bar{s} = s(\underline{c};\underline{c}) > 0$. Then for any $\hat{s} \in [0, \bar{s}]$, there exists $\hat{c} \in [\underline{c}, \bar{c}]$, such that $s(\hat{c};\hat{c}) = \hat{s}$. We thus have the following results.

Proposition 1 *For any given $\hat{s} \in [0, \bar{s}]$ such that $s(\hat{c};\hat{c}) = \hat{s}$, the symmetric entry threshold is $\hat{c}$, and the symmetric equilibrium bidding strategy is $\{q(c_i), p(c_i;\hat{c})\}, \forall c_i \leq \hat{c}, \forall i$. For any $\hat{s} \geq \bar{s}$, all types of bidders are shut down.*

3.2.1 Optimal minimum score

We next investigate whether a minimum score requirement would improve the procurer's payoff under the difference score rule. We first write the procurer's expected payoff as a function of $q(\cdot)$, which is the equilibrium quality bidding function $q(\cdot)$ defined as in (9) under the concerned score rule.

Suppose the entry threshold is $\hat{c}$. Let $c_N = \min\{c_i\}$ and $F_N(c_N) = 1 - [1 - F(c_N)]^N$ be the cumulative distribution function of c_N. Then, the total expected surplus of the procurer and suppliers is as follows:

$$\begin{aligned} TS &= \int_{\underline{c}}^{\hat{c}} [q(c_N) - (1-\rho)c_N\varsigma(q(c_N))]\, dF_N(c_N) - N\rho \int_{\underline{c}}^{\hat{c}} c\varsigma(q(c))dF(c) \\ &= N\int_{\underline{c}}^{\hat{c}} \left\{[q(c) - (1-\rho)c\varsigma(q(c))]\,[1-F(c)]^{N-1} - \rho c\varsigma(q(c))\right\} f(c)dc. \end{aligned}$$

From (4), we have that a representative supplier's conditional expected payoff has a slope of

$$\frac{d\pi_i(c,c)}{dc} = -\{[1-F(c)]^{N-1}(1-\rho) + \rho\}\varsigma(q(c)).$$

Recall $\pi_i(\hat{c},\hat{c}) = 0$. Therefore, we have that a representative supplier's conditional expected payoff is

$$\pi_i(c,c) = \int_c^{\hat{c}} \{[1-F(c')]^{N-1}(1-\rho) + \rho\}\varsigma(q(c'))dc', \forall c \le \hat{c}.$$

A representative supplier's expected payoff is thus as follows.

$$\begin{aligned} E\pi_i &= \int_{\underline{c}}^{\hat{c}} \left[\int_c^{\hat{c}} \{[1-F(c')]^{N-1}(1-\rho) + \rho\}\varsigma(q(c'))dc'\right] f(c)dc \\ &= \int_{\underline{c}}^{\hat{c}} \left[\int_{\underline{c}}^{c'} f(c)dc\right] \{[1-F(c')]^{N-1}(1-\rho) + \rho\}\varsigma(q(c'))dc' \\ &= \int_{\underline{c}}^{\hat{c}} F(c')\{[1-F(c')]^{N-1}(1-\rho) + \rho\}\varsigma(q(c'))dc' \\ &= \int_{\underline{c}}^{\hat{c}} \frac{F(c)}{f(c)}\{[1-F(c)]^{N-1}(1-\rho) + \rho\}\varsigma(q(c))f(c)dc. \end{aligned}$$

Therefore, the procurer's expected surplus is

$$\begin{aligned} & PS(\hat{c}) = TS - N\cdot E\pi_i \\ = \;& N\int_{\underline{c}}^{\hat{c}} \left\{[q(c) - (1-\rho)c\varsigma(q(c))]\,[1-F(c)]^{N-1} - \rho c\varsigma(q(c))\right\} f(c)dc \\ & -N\int_{\underline{c}}^{\hat{c}} \frac{F(c)}{f(c)}\{[1-F(c)]^{N-1}(1-\rho) + \rho\}\varsigma(q(c))f(c)dc \\ = \;& N\int_{\underline{c}}^{\hat{c}} \left\{q(c)\,[1-F(c)]^{N-1} - \left[(1-\rho)\,[1-F(c)]^{N-1} + \rho\right]\left(c + \frac{F(c)}{f(c)}\right)\varsigma(q(c))\right\} f(c)dc. \quad (12) \end{aligned}$$

We thus have

$$\frac{dPS(\hat{c})}{d\hat{c}} = N\left\{q(\hat{c})\left[1-F(\hat{c})\right]^{N-1} - \left[(1-\rho)\left[1-F(\hat{c})\right]^{N-1}+\rho\right]\left(\hat{c}+\frac{F(\hat{c})}{f(\hat{c})}\right)\varsigma(q(\hat{c}))\right\}f(\hat{c}). \tag{13}$$

Consider $\varsigma(q) = q^{\gamma}$. In this case, $q(c) = \left(\frac{1}{c\gamma}\right)^{\frac{1}{\gamma-1}}(1-F(c))^{\frac{N-1}{\gamma-1}}$ from (9). Therefore, we have

$$\begin{aligned}\frac{dPS(\hat{c})}{d\hat{c}} &\leq N\left\{q(\hat{c})\left[1-F(\hat{c})\right]^{N-1} - \rho\left(\hat{c}+\frac{F(\hat{c})}{f(\hat{c})}\right)\varsigma(q(\hat{c}))\right\}f(\hat{c}) \\ &= Nf(\hat{c})\left(\frac{1}{\hat{c}\gamma}\right)^{\frac{\gamma}{\gamma-1}}(1-F(\hat{c}))^{\frac{(N-1)\gamma}{\gamma-1}}\hat{c}\left\{\gamma-\left(\rho+\rho\frac{F(\hat{c})}{\hat{c}f(\hat{c})}\right)\right\}.\end{aligned}$$

Thus, we have $\frac{dPS(\hat{c})}{d\hat{c}} < 0$ if $\hat{c}$ is close to $\bar{c}$ when $\gamma < \rho + \rho\frac{1}{\bar{c}f(\bar{c}))}$. This means that it must be the case that a zero minimum score is sub-optimal when $1 < \gamma < \rho + \frac{\rho}{\bar{c}f(\bar{c}))}$.[13] We summarize this result in the following proposition.

Proposition 2 *Assume $\varsigma(q) = q^{\gamma}$. When $1 < \gamma < \rho + \frac{\rho}{\bar{c}f(\bar{c}))}$, we must have that the optimal minimum score is strictly above zero.*

The above proposition illustrates that in general under a given score rule setting a minimum score could improve the procurer's expected payoff. The optimal interior $\hat{c}^*$ in general can be identified by the solutions of $\frac{dPS(\hat{c})}{d\hat{c}} = 0$ with $\frac{dPS(\hat{c})}{d\hat{c}}$ being described in (13) in which $q(\cdot)$ is defined as in (9). It follows that minimum score $\hat{s}^* = s(\hat{c}^*; \hat{c}^*)$ would induce entry threshold $\hat{c}^*$ and thus is optimal. For example, in the setting of Proposition 2, such an optimal $\hat{c}^* \in (\underline{c}, \bar{c})$ must exist as $\frac{dPS(\hat{c})}{d\hat{c}} > 0$ at $\hat{c} = \underline{c}$.

3.3 The case with minimum quality requirement

We now turn to the equilibrium analysis with minimum quality requirements. In this case, we assume that the minimum quality bid $\hat{q}$ must fall in $(0, q(\underline{c}))$, where $q(\cdot)$ is the unrestricted equilibrium quality bidding strategy defined in (9).

Like the case with minimum score requirement, minimum quality requirement induces an endogenous entry threshold $\hat{c}$. It remains true that the entry threshold type $\hat{c}$'s expected payoff is zero at equilibrium, i.e. $\pi_i(\hat{c}, \hat{c}) = 0$.

The second observation in Section 3.1 also remains valid. Therefore, the equilibrium quality bid $q(c_i)$ of any participating type remains the same as identified in (9). In particular,

[13] Note that $\rho + \frac{\rho}{\bar{c}f(\bar{c}))} > 1$ if ρ is close to 1 or $f(\bar{c})$ is small.

the entry threshold type $\hat{c}$ must bid the minimum quality $\hat{q}$. This means that we can immediately pin down $\hat{c}$:

$$\hat{c} = q^{-1}(\hat{q}).$$

We can then pin down the equilibrium price bidding strategy following the same procedure as in Section 3.2. The equilibrium price bids are given by (11). These results are summarized in the following proposition.

Proposition 3 *For any given $\hat{q} \in (0, q(\underline{c}))$, the symmetric entry threshold is $\hat{c} = q^{-1}(\hat{q})$, and the symmetric equilibrium bidding strategy is $\{q(c_i), p(c_i; \hat{c})\}, \forall c_i \leq \hat{c}$, $\forall i$.*

By Propositions 1 and 3, the same entry threshold and bidding strategy can be induced by either a properly set minimum score or minimum quality requirement.

3.4 The case with multi-dimensional quality

In this subsection, we investigate how to extend our analysis to certain cases in which the quality is multi-dimensional. For this purpose, let us assume that quality is $K(\geq 2)$ dimensional, i.e. $\mathbf{q} = (q_1, q_2, \cdots, q_K)$. We consider a given score function $S(\mathbf{q}, p) = v(\mathbf{q}) - p$. A representative supplier i's cost function is $C_i(\mathbf{q}; c_i) = c_i \varsigma(\mathbf{q})$. We next demonstrate that the above problem can be reduced to a problem with single dimensional quality bid.

Consider the following problem:

$$\min_{\mathbf{q}} \varsigma(\mathbf{q})$$
$$s.t. \quad : \quad v(\mathbf{q}) = v.$$

Let $\mathbf{q}^*(v)$ denote the optimal solution of the above problem. Define $\tilde{C}_i(v; c_i) = c_i \varsigma(\mathbf{q}^*(v))$, and $\tilde{S}(v, p) = v - p$. The equilibrium analysis under the score rule $\tilde{S}(v, p) = v - p$ and cost function $\tilde{C}_i(v; c_i)$ while treating v as a single-dimensional quality choice can be carried out following our procedure. The resultant equilibrium if it exists can be denoted by $\{v(c), p(c)\}$. It follows that the equilibrium bidding strategy in the original multi-dimensional quality setting would be $\{\mathbf{q}^*(v(c)), p(c)\}$.

3.5 Comparative statics

We now examine the comparative statics concerning ρ, i.e. the proportion of the all-pay component. We will establish that a higher ρ leads to lower quality provision in general. For the class of cost functions $\varsigma(q) = q^\gamma, \gamma > 1$, it leads to lower total surplus and suppliers' payoffs. We find that procurer payoff is in general neither monotonically increasing nor decreasing in ρ. Nevertheless, a higher ρ always leads to lower procurer payoff when $(1-\frac{1}{\gamma})-\frac{1}{\gamma c}\frac{F(c)}{f(c)}\geq 0, \forall c\in[\underline{c},\bar{c}]$. On the other hand, when γ converges to 1 from above, i.e. when the cost function $\varsigma(q)$ gets close to being linear, procurer payoff must increase with ρ when ρ is close to zero.

First, by (9), it is clear that the equilibrium quality bid decreases in ρ.

Proposition 4 *A higher proportion of all pay components leads to lower quality provision.*

From (4), we have

$$\frac{d\pi_i(c_i,c_i)}{dc_i}=-\{[1-F(c_i)]^{N-1}(1-\rho)+\rho\}\varsigma(q(c_i,\rho)), \tag{14}$$

where from (9)

$$q(c_i;\rho)=(\varsigma')^{-1}\left(\frac{[1-F(c_i)]^{N-1}}{c_i\{(1-\rho)[1-F(c_i)]^{N-1}+\rho\}}\right). \tag{15}$$

Therefore, we have

$$\begin{aligned}&\frac{d\pi_i(c_i,c_i)}{dc_i}\\ =\;& -[1-F(c_i)]^{N-1}\frac{[1-F(c_i)]^{N-1}(1-\rho)+\rho}{[1-F(c_i)]^{N-1}}\cdot(\varsigma\circ(\varsigma')^{-1})\left(\frac{[1-F(c_i)]^{N-1}}{c_i\{(1-\rho)[1-F(c_i)]^{N-1}+\rho\}}\right).\end{aligned}$$

Note $\pi_i(\bar{c},\bar{c})=0, \forall\rho$. If $|\frac{d\pi_i(c_i,c_i)}{dc_i}|$ decreases with ρ, then a higher ρ leads to lower supplier payoffs. This is the case when $\varphi=\varsigma\circ[(\varsigma')^{-1}]$ is convex as will be verified in the proof of the following proposition. Clearly, $\varsigma(q)=q^\gamma,\gamma>1$ satisfies this condition.

Proposition 5 *If $\varphi=\varsigma\circ(\varsigma')^{-1}$ is convex, then a higher ρ leads to lower supplier payoffs.*

We next turn to the total surplus and the procurer's payoff. Total surplus is

$$\begin{aligned}TS(\rho) &= \int_{\underline{c}}^{\bar{c}} q(c;\rho)d[1-(1-F(c))^N]\\ &\quad -N\cdot E_{c_i}\{(1-\rho)c_i\varsigma(q(c_i;\rho))[1-F(c_i)]^{N-1}\}-N\cdot E_{c_i}\rho c_i\varsigma(q(c_i;\rho)).\end{aligned}$$

The procurer's payoff equals the difference between the total surplus and the suppliers' payoffs. A supplier's expected payoff is

$$\begin{aligned}\pi_S(\rho) &= \int_{\underline{c}}^{\bar{c}}\pi_i(c_i,c_i)dF(c_i)\\ &= \int_{\underline{c}}^{\bar{c}}\int_{c_i}^{\bar{c}}\{[1-F(t)]^{N-1}(1-\rho)+\rho\}\varsigma(q(t;\rho))dtdF(c_i).\end{aligned}$$

Therefore, the procurer's payoff equals

$$\pi_P(\rho) = TS(\rho) - N\pi_S(\rho).$$

In the following analysis, we consider $\varsigma(q) = q^\gamma, \gamma > 1$. With this specification, we have the following results.

Lemma 2 *Assume $\varsigma(q) = q^\gamma, \gamma > 1$. We have*

$$\begin{aligned}
TS(\rho) &= N(1-\frac{1}{\gamma})E_c\left\{\left[\frac{1}{\gamma c[(1-\rho)+\frac{\rho}{(1-F(c))^{N-1}}]}\right]^{\frac{1}{\gamma-1}}(1-F(c))^{N-1}\right\};\\
\pi_S(\rho) &= E_c\left\{\frac{1}{\gamma c}\cdot\left[\frac{1}{\gamma c[(1-\rho)+\frac{\rho}{(1-F(c))^{N-1}}]}\right]^{\frac{1}{\gamma-1}}\frac{F(c)}{f(c)}[1-F(c)]^{N-1}\right\};\\
\pi_P(\rho) &= NE_c\left\{\left[\frac{1}{\gamma c[(1-\rho)+\frac{\rho}{(1-F(c))^{N-1}}]}\right]^{\frac{1}{\gamma-1}}(1-F(c))^{N-1}\left[(1-\frac{1}{\gamma})-\frac{1}{\gamma c}\frac{F(c)}{f(c)}\right]\right\}.
\end{aligned}$$

Lemma 2 immediately leads to the following results.[14]

Proposition 6 *If $\varsigma = q^\gamma, \gamma > 1$, a higher ρ leads to lower total surplus and suppliers' payoffs, and it leads to lower procurer payoff when $(1-\frac{1}{\gamma}) - \frac{1}{\gamma c}\frac{F(c)}{f(c)} \geq 0, \forall c \in [\underline{c}, \bar{c}]$.*

For example, we consider the class $F(c) = (c-1)^\eta$ on $[1, 2]$, $\eta > 0$. For this class of type distributions, we have

$$(1-\frac{1}{\gamma}) - \frac{1}{\gamma c}\frac{F(c)}{f(c)} = (1-\frac{1}{\gamma}) - \frac{1}{\eta\gamma}\frac{c-1}{c}.$$

Since $\frac{c-1}{c} < 1$, the right hand side of the above equation is always positive when

$$(1-\frac{1}{\gamma}) - \frac{1}{\eta\gamma} > 0, \text{ i.e. } 1 + \frac{1}{\eta} < \gamma.$$

We thus have the following corollary.[15]

Corollary 1 *If $\varsigma(q) = q^\gamma, \gamma > 1$, and $F(c) = (c-1)^\eta$ on $[1, 2]$ with $1 + \frac{1}{\eta} < \gamma$, then a higher ρ leads to lower procurer payoff.*

[14] The result on suppliers' payoffs in Proposition 6 is consistent with that of Proposition 5.

[15] Note that $\gamma c[(1-\rho) + \frac{\rho}{1-F(c)}]$ increases with ρ for any c.

While Proposition 6 says that a higher ρ may hurt the procurer, it remains an issue whether a higher proportion ρ can benefit the procurer in some scenarios. The following example confirms this possibility.

Let $N = 2, \varsigma(q) = q^\gamma, \gamma > 1,\ f(c) = 4 - 2c$ and $F(c) = 2(2c - \frac{c^2}{2} - \frac{3}{2})$ on $[1, 2]$. We have $\frac{F(c)}{cf(c)} = \frac{2c - \frac{c^2}{2} - \frac{3}{2}}{c(2-c)}$ and $1 - F(c) = (2 - c)^2$. Note that[16]

$$\pi_P(\rho = 0) = 2E_c\{[\frac{1}{\gamma c}]^{\frac{1}{\gamma-1}}(1 - F(c))[(1 - \frac{1}{\gamma}) - \frac{1}{\gamma}\frac{F(c)}{cf(c)}]\},\ \text{and}$$

$$\pi_P(\rho = 1) = 2E_c\{[\frac{1 - F(c)}{\gamma c}]^{\frac{1}{\gamma-1}}(1 - F(c))[(1 - \frac{1}{\gamma}) - \frac{1}{\gamma c}\frac{F(c)}{f(c)}]\}.$$

Numerical simulation reveals that all-pay quality bids ($\rho = 1$) lead to higher procurer payoff than winner-pay quality bids ($\rho = 0$), i.e. $\pi_p(\rho = 1) > \pi_p(\rho = 0)$, if and only if $\gamma \in (1, 1.18613)$, i.e. when the effort cost function is not far from a linear function. In particular, when $\gamma = 1.1$, we have $\pi_P(\rho = 0) = 8.28 \times 10^{-4}$ and $\pi_P(\rho = 1) = 1.52 \times 10^{-3}$.

A higher ρ leads to lower quality bids, which induces the suppliers to lower their price bids to make their offers attractive to the procurer. It is thus possible that the drop in price bids dominates the drop in quality bids, such that a higher ρ leads to a higher procurer payoff. When γ is close to 1, the marginal cost curve is less steep, which means that the suppliers would lower less their quality bids when ρ gets higher. As a result, it is more likely that a higher ρ may benefit the procurer.

[16] Let $a = \frac{1}{1-\gamma}$. One can verify that

$$\pi_p(\rho = 0) = 2\gamma^a \int_1^2 \left[(1 - \frac{1}{\gamma})c^a(2 - c)^2 - \frac{1}{\gamma}c^a(2 - c)(2 - \frac{c}{2} - \frac{3}{2c})\right] dc$$

$$= -(1 - \frac{1}{a})^{a-1} \left[\begin{array}{c} \frac{2^{a+5}-4}{a(a+1)(a+2)(a+3)} - \frac{4}{a(a+1)(a+2)} - \frac{2^{a+1}3-1}{a(a+1)} \\ + \frac{2^{a+4}-4}{(a+1)(a+2)} - \frac{2^{a+3}-1}{(a+2)(a+3)} + \frac{3}{a} - \frac{4}{a+1} + \frac{1}{a+2} \end{array} \right];\ \text{and}$$

$$\pi_p(\rho = 1) = 2\gamma^a \int_1^2 \left[(1 - \frac{1}{\gamma})c^a(2 - c)^{2-2a} - \frac{1}{\gamma}c^a(2 - c)^{1-2a}(2 - \frac{c}{2} - \frac{3}{2c})\right] dc$$

$$= -2(1 - \frac{1}{a})^{a-1} \left[\begin{array}{c} \frac{2^{3-a}}{a}\beta_{\frac{1}{2}}(3 - 2a, 1 + a) + 2^{3-a}\beta_{\frac{1}{2}}(2 - 2a, 1 + a) \\ -2^{2-a}\beta_{\frac{1}{2}}(2 - 2a, 2 + a) - \frac{3}{2^a}\beta_{\frac{1}{2}}(2 - 2a, a) \end{array} \right],$$

where $\beta_z(d, e) = \int_0^z x^{d-1}(1 - x)^{e-1}\, dx$ is the beta function.

4 On reimbursement of suppliers' all-pay bidding costs

4.1 Reimbursement of the losers' all pay component

Proposition 6 provides sufficient conditions under which an all-pay component would hurt the procurer. The all-pay component lowers the quality provision more significantly than it reduces the price. An interesting issue naturally arises: should the procurer reimburse the suppliers' all-pay bidding costs? In this subsection, we first study a scenario, in which only the losers' all-pay bidding costs are reimbursed. In the next subsection, we further examine a scenario, in which all suppliers' all pay bidding costs are reimbursed.

With such reimbursement, suppliers' bidding behavior would be identical to that in our model with $\rho = 0$. Therefore, the results of Section 3.5 must lead to the following results.

Proposition 7 *When the procurer reimburses the all pay component of the losers, then this leads to (i) higher quality provision in general, and (ii) higher supplier payoff for every type when $\varphi = \varsigma \circ (\varsigma')^{-1}$ is convex (e.g. when $\varsigma(q) = q^\gamma, \gamma > 1$).*

It is pretty intuitive that the reimbursement would increase quality provision and supplier payoff as it reduces the suppliers' marginal provision cost. It remains to further investigate whether reimbursing the all-pay component of the losers can generate a higher procurer payoff, and how the total surplus changes. For the sake of tractability, we investigate these issues under the assumption that $\varsigma(q) = q^\gamma, \gamma > 1$. Note that under this specification, we have that $q(c; \rho = 0) = (\gamma c)^{-\frac{1}{\gamma-1}}$ by (9) and thus $\varsigma(q(c; \rho = 0)) = (\gamma c)^{-\frac{\gamma}{\gamma-1}}$.

We first look at total surplus. In a procurement auction with all pay component of quality bids, i.e. $\rho > 0$, by Lemma 2, the total surplus is

$$\begin{aligned} TS(\rho) &= N(1-\frac{1}{\gamma})E_c\left\{\left[\frac{1}{\gamma c[(1-\rho)+\frac{\rho}{(1-F(c))^{N-1}}]}\right]^{\frac{1}{\gamma-1}}(1-F(c))^{N-1}\right\} \\ &= N(\gamma-1)\gamma^{-\frac{\gamma}{\gamma-1}}E_c\left\{c^{-\frac{1}{\gamma-1}}[(1-\rho)(1-F(c))^{N-1}+\rho]^{-\frac{1}{\gamma-1}}(1-F(c))^{\frac{(N-1)\gamma}{\gamma-1}}\right\}. \end{aligned} \quad (16)$$

Let $c^{(1)}$ denote the lowest cost parameter of the suppliers. In the second benchmark of winner-pay quality bids, in which both suppliers incur their costs of placing their quality bids but the losing supplier's cost of placing his quality bid is reimbursed by the procurer, the total surplus is

$$
\begin{aligned}
& TS^R(\rho) = TS(0) - N\rho E_c[c\varsigma(q(c;\rho=0))] + \rho E_{c^{(1)}}[c^{(1)}\varsigma(q(c^{(1)};\rho=0))] \\
= \;& N(1-\frac{1}{\gamma})E_c\left\{\left[\frac{1}{\gamma c}\right]^{\frac{1}{\gamma-1}}(1-F(c))^{N-1}\right\} - N\rho E_c[c(\gamma c)^{-\frac{\gamma}{\gamma-1}}] + \rho E_{c^{(1)}}[c^{(1)}(\gamma c^{(1)})^{-\frac{\gamma}{\gamma-1}}] \\
= \;& N(1-\frac{1}{\gamma})E_c\left\{\left[\frac{1}{\gamma c}\right]^{\frac{1}{\gamma-1}}(1-F(c))^{N-1}\right\} - N\rho E_c[\gamma^{-\frac{\gamma}{\gamma-1}}c^{-\frac{1}{\gamma-1}}] + \rho E_{c^{(1)}}[\gamma^{-\frac{\gamma}{\gamma-1}}(c^{(1)})^{-\frac{1}{\gamma-1}}] \\
= \;& N(1-\frac{1}{\gamma})E_c\left\{\left[\frac{1}{\gamma c}\right]^{\frac{1}{\gamma-1}}(1-F(c))^{N-1}\right\} - N\rho E_c[\gamma^{-\frac{\gamma}{\gamma-1}}c^{-\frac{1}{\gamma-1}}] \\
& + N\rho E_c[\gamma^{-\frac{\gamma}{\gamma-1}}c^{-\frac{1}{\gamma-1}}(1-F(c))^{N-1}] \\
= \;& N(\gamma-1)\gamma^{-\frac{\gamma}{\gamma-1}}E_c[c^{-\frac{1}{\gamma-1}}(1-F(c))^{N-1}] - N\rho E_c\left\{\gamma^{-\frac{\gamma}{\gamma-1}}c^{-\frac{1}{\gamma-1}}[1-(1-F(c))^{N-1}]\right\}.
\end{aligned}
$$

Proposition 8 *Suppose $\varsigma = q^\gamma, \gamma > 1$. When the procurer reimburses the all-pay component of the losers, it generates lower total surplus and lower procurer payoffs.*

Recall that from Proposition 7, the reimbursement would increase quality provision and supplier payoff. Since social marginal quality provision cost remains the same, the enhanced quality provision can be overly costly to society. As a result, suppliers' optimal decisions on quality provision under such subsidy of reimbursement tend to be less efficient. This further means that a lower procurer payoff must be in place under such reimbursement as the supplier payoff gets higher.

4.2 Reimbursement of all suppliers' all pay component

To check the robustness of the results in Section 4.1, we now turn to an alternative scenario of reimbursing the suppliers' all-pay bidding costs, i.e. the procurer reimburses all suppliers' all-pay bidding costs.

We first derive the bidding equilibrium $(\tilde{q}(c_i,\rho), \tilde{p}(c_i,\rho))$ with such reimbursement. Suppose all other suppliers are adopting the hypothetical equilibrium. A representative supplier's payoff function

$$\pi_i(\tilde{c}_i, c_i) = [1-F(\tilde{c}_i)]^{N-1}[\tilde{p}(\tilde{c}_i,\rho) - (1-\rho)c_i\varsigma(\tilde{q}(\tilde{c}_i,\rho))],$$

if his type is c_i and he bids $(\tilde{q}(\tilde{c}_i,\rho), \tilde{p}(\tilde{c}_i,\rho))$.

Following the same procedure of Section 3.1, we have

$$\frac{d\pi_i(c_i,c_i)}{dc_i}=-[1-F(c_i)]^{N-1}(1-\rho)\varsigma(\tilde{q}(c_i,\rho)).$$

The equilibrium price bid is given by

$$\tilde{p}(c_i,\rho)=\frac{\int_{c_i}^{\bar{c}}\{[1-F(t)]^{N-1}(1-\rho)\varsigma(\tilde{q}(t,\rho))\}dt}{[1-F(c_i)]^{N-1}}+(1-\rho)c_i\varsigma(\tilde{q}(c_i,\rho))\geq 0.$$

The equilibrium quality bid is given by

$$\tilde{q}(c_i,\rho)=(\varsigma')^{-1}\left(\frac{1}{c_i(1-\rho)}\right),\tag{17}$$

which says that reimbursing the all-pay component of all suppliers leads to higher quality provision.

The case where the procurer reimburses everyone their all pay component means that the actual cost paid by losing suppliers is 0 and the winner pays $(1-\rho)c_i\varsigma(q_i)$. Proposition 7 reveals that when $\varphi=\varsigma\circ(\varsigma')^{-1}$ is convex, the winner pay case in which the actual cost paid by losing suppliers is 0 and the winner pays $c_i\varsigma(q_i)$ entails a higher supplier payoff for every type. We next show that compared to the above winner pay case, when $\varphi=\varsigma\circ(\varsigma')^{-1}$ is convex, reimbursing the all pay component of all suppliers leads to a higher supplier payoff.

Note that the case in which all pay components of all suppliers are reimbursed is equivalent to a winner-pay auction in which the modified cost type is $\breve{c}_i=(1-\rho)c_i,\forall i$. Applying (14) and (15), we have that in the winner pay auction with original cost type c_i, the slope of the supplier payoff is given by

$$\frac{d\pi_i(c_i,c_i)}{dc_i}=-[1-F(c_i)]^{N-1}\varphi(\frac{1}{c_i}).\tag{18}$$

In the winner pay auction with modified cost type $\tilde{c}_i$, the slope of the supplier's payoff is given by

$$\frac{d\tilde{\pi}_i(c_i,c_i)}{dc_i}=-[1-F(c_i)]^{N-1}\varphi\left(\frac{1}{(1-\rho)c_i}\right)(1-\rho).\tag{19}$$

By the same procedure in the proof of Proposition 7, we have that $\varphi\left(\frac{1}{(1-\rho)c_i}\right)(1-\rho)$ increases with $\rho\in[0,1)$ when φ is convex. Note that $\pi_i(\bar{c},\bar{c})=\tilde{\pi}_i(\bar{c},\bar{c})=0$ and $\left|\frac{d\pi_i(c_i,c_i)}{dc_i}\right|<\left|\frac{d\tilde{\pi}_i(c_i,c_i)}{dc_i}\right|$. We thus have the winner pay auction with $\tilde{c}_i$ entails a higher supplier payoff.

We summarize the above results in the following proposition.

Proposition 9 *When the procurer reimburses the all-pay component of all suppliers, then this leads to (i) higher quality provision in general, and (ii) higher supplier payoffs when* $\varphi=\varsigma\circ(\varsigma')^{-1}$ *is convex (e.g. when* $\varsigma(q)=q^\gamma,\gamma>1$*).*

Relative to the setting of Proposition 7, the suppliers' marginal quality provision costs are even lower. It is thus intuitive that the quality provision and the suppliers' payoff can get even higher.

It remains to further investigate whether reimbursing the all-pay component of all suppliers can generate a higher procurer payoff, and how the total surplus changes. For the sake of tractability, we investigate these issues under the assumption that $\varsigma(q) = q^\gamma, \gamma > 1$. Note that under this specification, we have that $\tilde{q}(c,\rho) = (\gamma(1-\rho)c)^{-\frac{1}{\gamma-1}}$ by (17) and thus $\varsigma(\tilde{q}(c,\rho)) = (\gamma(1-\rho)c)^{-\frac{\gamma}{\gamma-1}}$.

We first look at total surplus. Recall that in a procurement auction with all pay component of quality bids, i.e. $\rho > 0$, the total surplus $TS(\rho)$ is given by (16). When all suppliers' all pay component quality provision costs are reimbursed by the procurer, the total surplus is

$$
\begin{aligned}
& TS^{R,A}(\rho) \\
= {} & \int_{\underline{c}}^{\bar{c}} \tilde{q}(c,\rho) d[1-(1-F(c))^N] - N \cdot E_{c_i}\{(1-\rho)c_i\varsigma(\tilde{q}(c_i,\rho))[1-F(c_i)]^{N-1}\} \\
& - N \cdot E_{c_i}\rho c_i \varsigma(\tilde{q}(c_i,\rho)) \\
= {} & NE_\tau \left\{ \left[\frac{1}{\gamma\tau(1-\rho)}\right]^{\frac{1}{\gamma-1}} (1-F(\tau))^{N-1} \right\} \\
& - N \cdot E_\tau \left\{ (1-\rho)\tau \left[\frac{1}{\gamma\tau(1-\rho)}\right]^{\frac{\gamma}{\gamma-1}} (1-F(\tau))^{N-1} \right\} - N \cdot E_\tau \left\{ \rho\tau \left[\frac{1}{\gamma\tau(1-\rho)}\right]^{\frac{\gamma}{\gamma-1}} \right\} \\
= {} & NE_c \left\{ \left[\frac{1}{\gamma c(1-\rho)}\right]^{\frac{1}{\gamma-1}} (1-F(c))^{N-1} \right\} \\
& - N \cdot E_c \left\{ \left[\frac{1}{\gamma c(1-\rho)}\right]^{\frac{\gamma}{\gamma-1}} c\left[(1-\rho)(1-F(c))^{N-1} + \rho\right] \right\} \\
= {} & N\gamma^{-\frac{\gamma}{\gamma-1}}\gamma(1-\rho)^{-\frac{1}{\gamma-1}} E_c \left\{ c^{-\frac{1}{\gamma-1}}(1-F(c))^{N-1} \right\} \\
& - N\gamma^{-\frac{\gamma}{\gamma-1}}(1-\rho)^{-\frac{\gamma}{\gamma-1}} \cdot E_c \left\{ c^{-\frac{1}{\gamma-1}} \left[(1-\rho)(1-F(c))^{N-1} + \rho\right] \right\}.
\end{aligned}
$$

In the following proposition, we will establish that $TS^{R,A}(\rho) \le TS(\rho)$. The procurer's payoff is simply the difference between the total surplus and the supplies' surplus. As a result, we must have that the procurer gets a higher payoff without reimbursing the suppliers' all-pay bidding costs.

Proposition 10 *Suppose $\varsigma = q^{\gamma}, \gamma > 1$. When the procurer reimburses the all-pay component of all suppliers, it generates a lower total surplus and lower procurer payoffs.*

On one hand, relative to the setting of Proposition 8, the suppliers' marginal quality provision costs are even lower, which means that the quality provision and the suppliers' payoff can get even higher in the setting of Proposition 10. On the other hand, as the enhanced quality provision gets further away from the efficient level, the total surplus must further decrease. As a result, a lower procurer payoff must entail.

5 Optimal score rule

We now turn to deriving the procurer-optimal score rule. In this section, the procurer's payoff is generalized to be $u(q) - p$, where $\{q, p\}$ is the bid of the winning bidder. Here $u(\cdot)$ is an increasing concave function with $u(0) = 0$. Supplier i's payoff is the same as specified before. We search for the optimum among all score rules that induce a monotone equilibrium score. In this section, we assume that the virtual cost function $c + \frac{F(c)}{f(c)}$ increases in c.

We first write the procurer's expected payoff as a function of equilibrium $q(\cdot)$. Recall have let $c_N = \min\{c_i\}$ and $F_N(c_N) = 1 - [1 - F(c_N)]^N$ be the cumulative distribution function of c_N. The total expected surplus of the procurer and suppliers is as follows:

$$\begin{aligned} TS &= \int_{\underline{c}}^{\bar{c}} [u(q(c_N)) - (1-\rho)c_N\varsigma(q(c_N))]\, dF_N(c_N) - N\rho \int_{\underline{c}}^{\bar{c}} c\varsigma(q(c))dF(c) \\ &= N \int_{\underline{c}}^{\bar{c}} \left\{ [u(q(c)) - (1-\rho)c\varsigma(q(c))]\, [1 - F(c)]^{N-1} - \rho c\varsigma(q(c)) \right\} f(c)dc. \qquad (20) \end{aligned}$$

From (14), we have that a representative supplier's conditional expected payoff has a slope of

$$\frac{d\pi_i(c,c)}{dc} = -\{[1 - F(c)]^{N-1}(1-\rho) + \rho\}\varsigma(q(c)).$$

Recall $\pi_i(\bar{c}, \bar{c}) = 0$. Therefore, we have that a representative supplier's conditional expected payoff is

$$\pi_i(c,c) = \int_{c}^{\bar{c}} \{[1 - F(c')]^{N-1}(1-\rho) + \rho\}\varsigma(q(c'))dc'.$$

It follows that a representative supplier's expected payoff is

$$
\begin{aligned}
E\pi_i &= \int_{\underline{c}}^{\bar{c}} \left[\int_{c}^{\bar{c}} \{[1-F(c')]^{N-1}(1-\rho)+\rho\}\varsigma(q(c'))dc'\right] f(c)dc \\
&= \int_{\underline{c}}^{\bar{c}} \left[\int_{\underline{c}}^{c'} f(c)dc\right] \{[1-F(c')]^{N-1}(1-\rho)+\rho\}\varsigma(q(c'))dc' \\
&= \int_{\underline{c}}^{\bar{c}} F(c')\{[1-F(c')]^{N-1}(1-\rho)+\rho\}\varsigma(q(c'))dc' \\
&= \int_{\underline{c}}^{\bar{c}} \frac{F(c)}{f(c)}\{[1-F(c)]^{N-1}(1-\rho)+\rho\}\varsigma(q(c))f(c)dc.
\end{aligned}
$$

Therefore, the procurer's expected surplus is

$$
\begin{aligned}
& PS(q(\cdot)) = TS - N\cdot E\pi_i \\
= {} & N\int_{\underline{c}}^{\bar{c}} \left\{[u(q(c))-(1-\rho)c\varsigma(q(c))]\,[1-F(c)]^{N-1} - \rho c\varsigma(q(c))\right\} f(c)dc \\
& -N\int_{\underline{c}}^{\bar{c}} \frac{F(c)}{f(c)}\{[1-F(c)]^{N-1}(1-\rho)+\rho\}\varsigma(q(c))f(c)dc \\
= {} & N\int_{\underline{c}}^{\bar{c}} \left\{u(q(c))\,[1-F(c)]^{N-1} - \left[(1-\rho)\,[1-F(c)]^{N-1}+\rho\right](c+\frac{F(c)}{f(c)})\varsigma(q(c))\right\} f(c)dc. \quad (21)
\end{aligned}
$$

Pointwise maximization leads to the following $q^*(c)$:

$$
\frac{\varsigma'(q^*(c))}{u'(q^*(c))} = \frac{[1-F(c)]^{N-1}}{\left[(1-\rho)\,[1-F(c)]^{N-1}+\rho\right]}\cdot\frac{1}{c+\frac{F(c)}{f(c)}}. \tag{22}
$$

Note that both $\frac{[1-F(c)]^{N-1}}{\left[(1-\rho)[1-F(c)]^{N-1}+\rho\right]}$ and $\frac{1}{c+\frac{F(c)}{f(c)}}$ (under the assumption of increasing virtual cost $c+\frac{F(c)}{f(c)}$) decrease with c, and $\frac{\varsigma'(q)}{u'(q)}$ increases with q. We thus have

$$
q^*(c) = \left(\frac{\varsigma'}{u'}\right)^{-1}\left(\frac{[1-F(c)]^{N-1}}{\left[(1-\rho)\,[1-F(c)]^{N-1}+\rho\right]}\cdot\frac{1}{c+\frac{F(c)}{f(c)}}\right),
$$

which is decreasing in c. Clearly, we have $q^*(\bar{c})=0$ as $\frac{\varsigma'(0)}{u'(0)}=0$ under assumptions. It is clear that $q^*(c)$ decreases with ρ.

We now look at how to induce $q^*(c)$ using a properly set score rule. Specifically, we choose $v(q)$ in the following quasi-linear specification $s=v(q)-p$ to induce $q^*(c)$, which generates an upper bound $TS(q^*(\cdot))$ for procurer expected payoff.

With score function $s = v(q) - p$, any equilibrium $(q(c_i), p(c_i))$ must solve the following optimization problem:

$$\begin{aligned}
(q(c_i), p(c_i)) &= \arg\max_{(q,p)} \{[1 - F(c_i)]^{N-1}[p - (1-\rho)c_i\varsigma(q)] - \rho c_i \varsigma(q)\} \\
s.t. &: \quad s = v(q) - p = s(c_i),
\end{aligned}$$

where $s(c_i)$ denotes the equilibrium score for type c_i.

The corresponding Lagrangian for the above problem is

$$\max_{(q,p,\lambda)} L(q,p,\lambda) = [1 - F(c_i)]^{N-1}[p - (1-\rho)c_i\varsigma(q)] - \rho c_i\varsigma(q) + \lambda[(v(q) - p) - s(c_i)].$$

The first order conditions are as follows:

$$\begin{aligned}
\frac{L(q,p,\lambda)}{\partial q} &= -[1 - F(c_i)]^{N-1}(1-\rho)c_i\varsigma'(q) - \rho c_i \varsigma'(q) + \lambda v'(q) = 0, \\
\frac{L(q,p,\lambda)}{\partial p} &= [1 - F(c_i)]^{N-1} - \lambda = 0, \\
\frac{L(q,p,\lambda)}{\partial \lambda} &= (v(q) - p) - s(c_i) = 0.
\end{aligned}$$

We thus have

$$\frac{\varsigma'(q)}{v'(q)} = \frac{[1 - F(c_i)]^{N-1}}{c_i\{[1 - F(c_i)]^{N-1}(1-\rho) + \rho\}}. \tag{23}$$

Let $c(q;\rho)$ denote the inverse function of $q^*(c)$. From (22) and (23), to induce $q^*(c)$, we should set $v^*(\cdot)$ such that

$$v^{*\prime}(q) = \frac{c(q;\rho)}{(c(q;\rho) + \frac{F(c(q;\rho))}{f(c(q;\rho))})} u'(q) > 0. \tag{24}$$

We further set $v^*(0) = 0$ to fully pin down the $v^*(\cdot)$ function.

We verify that $\frac{\varsigma'(q)}{v^{*\prime}(q)}$ increases in q for $q \le q^*(\underline{c})$. Note $q^*(c)$ decreases with c. To show $\frac{\varsigma'(q)}{v^{*\prime}(q)}$ increases in q for $q \le q^*(\underline{c})$, it suffices to show $\frac{\varsigma'(q^*(c))}{v^{*\prime}(q^*(c))}$ decreases in c. Note that

$$\begin{aligned}
\frac{\varsigma'(q^*(c))}{v^{*\prime}(q^*(c))} &= \frac{\varsigma'(q^*(c))}{u'(q^*(c))} \left(\frac{c(q^*(c);\rho) + \frac{F(c(q^*(c);\rho))}{f(c(q^*(c);\rho))}}{c(q^*(c);\rho)} \right) \\
&= \frac{[1 - F(c)]^{N-1}}{\left[(1-\rho)[1 - F(c)]^{N-1} + \rho\right]} \cdot \frac{1}{c + \frac{F(c)}{f(c)}} \left(\frac{c + \frac{F(c)}{f(c)}}{c} \right) \\
&= \frac{[1 - F(c)]^{N-1}}{\left[(1-\rho)[1 - F(c)]^{N-1} + \rho\right]} \cdot \frac{1}{c}.
\end{aligned}$$

Therefore, $\frac{\varsigma'(q^*(c))}{v^{*\prime}(q^*(c))}$ decreases in c, since both terms $\frac{[1-F(c)]^{N-1}}{\left[(1-\rho)[1-F(c)]^{N-1}+\rho\right]}$ and $\frac{1}{c}$ decrease in c. We thus have that $\frac{\varsigma'(q)}{v^{*\prime}(q)}$ increases in q for $q \leq q^*(\underline{c})$, which means that (23) uniquely pins down $q^*(c)$ under restriction $q \leq q^*(\underline{c})$.

Consider a procurement with the score rule $s^*(q,p) = v^*(q) - p$, for $q \leq q^*(\underline{c})$; $s^*(q,p) = v^*(q^*(\underline{c})) - p$, for $q > q^*(\underline{c})$. Our procedure of Section 3.1 leads to the following result.

Proposition 11 *A first-price procurement with quasi-linear score rule $s^*(q,p)$ maximizes the procurer's expected payoff when virtual cost function $c + \frac{F(c)}{f(c)}$ is increasing. It induces equilibrium quality bid $q^*(c)$ and the following price bid:*

$$p^*(c) = \frac{\int_c^{\bar{c}}\{[1-F(t)]^{N-1}(1-\rho)\varsigma(q^*(t)) + \rho\varsigma(q^*(t))\}dt + \rho c\varsigma(q^*(c))}{[1-F(c)]^{N-1}} + (1-\rho)c\varsigma(q^*(c)) \geq 0. \tag{25}$$

The equilibrium score decreases in c. All types participate at the optimum.

Following our procedure, we can establish the candidate equilibrium $(q^*(c), p^*(c))$ under score rule $s^*(q,p)$. To save space, we omit the details. We next verify that the equilibrium score function $s^*(q^*(c), p^*(c)) = v^*(q^*(c)) - p^*(c)$ is decreasing in c, i.e. $\frac{ds^*(q^*(c),p^*(c))}{dc} = v^{*\prime}(q^*(c))q^{*\prime}(c) - p^{*\prime}(c) < 0$.

Note that from (25), we have

$$\begin{aligned}
& (1-F(c))^{N-1}p^*(c) \\
= & \int_c^{\bar{c}}\{[1-F(t)]^{N-1}(1-\rho)\varsigma(q^*(t)) \\
& +\rho\varsigma(q^*(t))\}dt + \rho c\varsigma(q^*(c)) + (1-F(c))^{N-1}(1-\rho)c\varsigma(q^*(c)).
\end{aligned}$$

Therefore, we have

$$[(1-F(c))^{N-1}p^*(c)]' = \rho c\varsigma'(q^*(c))q^{*\prime}(c) + c\frac{d\{(1-F(c))^{N-1}(1-\rho)\varsigma(q^*(c))\}}{dc},$$

which gives

$$\begin{aligned}
& (1-F(c))^{N-1}p^{*\prime}(c) - (N-1)(1-F(c))^{N-2}f(c)p^*(c) \\
= & \rho c\varsigma'(q^*(c))q^{*\prime}(c) + c\frac{d\{(1-F(c))^{N-1}(1-\rho)\varsigma(q^*(c))\}}{dc},
\end{aligned}$$

which leads to

$$\begin{aligned}
& p^{*\prime}(c) - \frac{(N-1)f(c)}{1-F(c)}p^*(c) \\
= & q^{*\prime}(c)\frac{c[\rho + (1-\rho)(1-F(c))^{N-1}]}{(1-F(c))^{N-1}}\varsigma'(q^*(c)) - \frac{(N-1)f(c)}{1-F(c)}c(1-\rho)\varsigma(q^*(c)).
\end{aligned}$$

Note that from (23), we have $\frac{c[\rho+(1-\rho)(1-F(c))^{N-1}]}{(1-F(c))^{N-1}}\varsigma'(q^*(c)) = v^{*\prime}(q^*(c))$. We thus have

$$p^{*\prime}(c) = q^{*\prime}(c)v^{*\prime}(q^*(c)) - \frac{(N-1)f(c)}{1-F(c)}c(1-\rho)\varsigma(q^*(c)) + \frac{(N-1)f(c)}{1-F(c)}p^*(c),$$

which means that

$$\frac{ds^*(q^*(c), p^*(c))}{dc} = q^{*\prime}(c)v'(q^*(c)) - p^{*\prime}(c) = \frac{(N-1)f(c)}{1-F(c)}\{c(1-\rho)\varsigma(q^*(c)) - p^*(c)\}.$$

Note that from (25), we have $c(1-\rho)\varsigma(q^*(c)) - p^*(c) < 0$. We thus have $\frac{ds^*(q^*(c),p^*(c))}{dc} < 0$. Therefore, $\{q^*(c), p^*(c)\}$ is the symmetric equilibrium bidding strategy in the Proposition 11 procurement. The procurement induces $q^*(c)$ and $\pi_i(\bar{c}, \bar{c}) = 0$. It thus achieves the established bound $PS(q^*(\cdot))$, which means that the Proposition 11 procurement maximizes the procurer's expected payoff.

It is useful to explicitly point out that as in Myerson [12], we have a version of **payoff equivalence theorem** in our setting. The procurer and suppliers' expected payoffs are solely determined by the allocation rule $q(\cdot)$ and the least efficient types' payoff $\pi_i(\bar{c}, \bar{c})$. The identified $q^*(\cdot)$ actually renders an upper bound for the procurer's expected payoff among all possible score rules that induce decreasing $q(\cdot)$. This means that even if we allow endogenous entry when deriving the optimal score rule, we would still have the same $q^*(\cdot)$. Therefore, the optimal procurement should not induce partial entry as $q^*(c) > 0$, $\forall c \leq \bar{c}$.

Corollary 2 *Relative to the procurer's payoff function, the optimal score rule values quality q less, and the score rule function increases in q with a lower rate than the procure payoff function.*

The above results are clear since $u(0) = v^*(0) = 0$ and $v^{*\prime}(q) < u'(q)$ by (24). The intuition behind this is as follows. When the supplies have a convex quality provision cost function, providing them higher incentive to provide higher quality means that the prices they charge would increase faster than the qualities do. Recall that the procurer measures quality q by a concave function u, and thus the marginal utility of q decreases. On the other hand, her disutility of paying price p is constant. As a result, the procurer would benefit by providing the suppliers with less incentive in quality provision.

We further have the following result.

Corollary 3 *Suppose the reverse hazard rate $\frac{F(c)}{f(c)}$ is weakly convex, then $v^*(q)$ increases with higher rate in q when ρ is higher. Therefore, the optimal score rule should value quality more when ρ is higher.*

Note that $\frac{c(q;\rho)}{(c(q;\rho)+\frac{F(c(q;\rho))}{f(c(q;\rho))})} = \frac{1}{1+\frac{F(c(q;\rho))}{f(c(q;\rho))}/c(q;\rho)}$. With a higher ρ, we have $c(q;\rho)$ is lower. Since $\underline{c} > 0$, we thus have $\frac{F(c(q;\rho))}{f(c(q;\rho))}/c(q;\rho)$ strictly decreases with ρ when $\frac{F(c)}{f(c)}$ is weakly convex. It follows from (24) that $v^{*\prime}(q)$ gets higher. The intuition is as follows. When the virtual cost function is convex, it gets more difficult to incentivize the less efficient types. The procurer should thus provide more incentive to the suppliers at optimum by setting a score rule that is more responsive to quality.[17]

Similar result can be established when number of bidders N gets higher, when $\rho > 0$. For fixed $\rho > 0$, $q^*(c)$ drops when N gets higher. We denote the inverse function of $q^*(c)$ by $c(q;N)$ to explicitly capture the impact of N. Clearly, $c(q;N)$ decreases with N. We thus have $\frac{F(c(q;N))}{f(c(q;N))}/c(q;N)$ strictly decreases with N when $\frac{F(c)}{f(c)}$ is weakly convex.

Corollary 4 *Suppose the reverse hazard rate $\frac{F(c)}{f(c)}$ is weakly convex and $\rho > 0$, then $v^*(q)$ increases with higher rate in q when N is higher. Therefore, the optimal score rule should value quality more when N is higher.*

5.1 Restricted optimum

In situations where the procurer has no full flexibility in choosing the optimal score rule, our procedure can be adapted to identify the restricted optimum among a family of score rules.

For example, if the procurer is restricted to a family of linear score rules in the following form: $s = \mu q - p$ where $\mu > 0$. Assume $u(q) = q$ and $\varsigma(q) = q^2$. From (23), we have

$$q(c;\mu) = \frac{\mu}{2}\frac{[1-F(c)]^{N-1}}{c\{[1-F(c)]^{N-1}(1-\rho)+\rho\}}.$$

Plugging $q(c;\mu)$ into $PS(q(\cdot))$, we have

$$\begin{aligned} PS(\mu) &= N\int_{\underline{c}}^{\bar{c}} \left\{ \begin{array}{c} \frac{\mu}{2}\frac{[1-F(c)]^{N-1}}{c\{[1-F(c)]^{N-1}(1-\rho)+\rho\}}\left[1-F(c)\right]^{N-1} \\ -\left[(1-\rho)\left[1-F(c)\right]^{N-1}+\rho\right](c+\frac{F(c)}{f(c)})\left(\frac{\mu}{2}\frac{[1-F(c)]^{N-1}}{c\{[1-F(c)]^{N-1}(1-\rho)+\rho\}}\right)^2 \end{array} \right\} f(c)dc \\ &= N\int_{\underline{c}}^{\bar{c}} \left\{ \frac{[1-F(c)]^{2(N-1)}}{c\{[1-F(c)]^{N-1}(1-\rho)+\rho\}}\left(\frac{\mu}{2}-(c+\frac{F(c)}{f(c)})\left(\frac{\mu}{2}\right)^2\frac{1}{c}\right)\right\} f(c)dc. \end{aligned}$$

Therefore, we identify the following optimum:

$$\mu^* = \frac{\int_{\underline{c}}^{\bar{c}} \frac{[1-F(c)]^{2(N-1)}}{c\{[1-F(c)]^{N-1}(1-\rho)+\rho\}} f(c)dc}{\int_{\underline{c}}^{\bar{c}} \frac{[1-F(c)]^{2(N-1)}}{c\{[1-F(c)]^{N-1}(1-\rho)+\rho\}}(1+\frac{F(c)}{cf(c)})f(c)dc} < 1.$$

[17] The case of concave $\frac{F(c)}{f(c)}$ is more complicated.

6 Concluding remarks

In this paper, we study score procurements with an all-pay component of quality bid. Our methodological innovation in this paper lies in that we come up with a two-step procedure for deriving the bidding equilibrium in a score auction with a general score function, which allows us to identify the quality and price bids directly if there exists a symmetric pure strategy equilibrium that renders scores decreasing in types. This procedure does not rely on solving for the equilibrium score function as a middle step. Our procedure also reveals if such an equilibrium does not exist. Our approach is based on our new perspective of viewing the equilibrium quality bidding strategy as the allocation rule and the price bidding strategy as the transfer rule in a truthful direct mechanism.

We have demonstrated that our perspective and procedure are particularly convenient in deriving the symmetric entry and bidding equilibrium when a minimum score or quality is imposed by the procurer. We illustrate that for a given score rule, typically imposing a minimum score or quality requirement can improve the procurer payoff.

Relying on our perspective, we further establish a version of the payoff equivalence theorem in our setting with all pay quality features. This result allows us to establish a tight upper bound for the procurer payoff and the optimal allocation rule (i.e. quality bidding strategy) that achieves it. We identify an optimal score rule that is quasi-linear, which induces the optimal allocation as an equilibrium bidding strategy. We find that the optimal score rule must be less responsive to quality or values less the quality than the procurer's payoff function does. Moreover, when suppliers' type distribution has a convex reverse hazard rate, the optimal score rule is more responsive to quality or values more the quality when placing the quality bid gets more costly or there are more suppliers.

We also study the impact of an all-pay component of quality bids in score procurements. Our focus is on the equilibrium quality, supplier payoff, procurer payoff, and total surplus. Under plausible conditions on the effort cost function, we find that a lower all-pay quality component increases the quality provision, the supplier payoff, and the total surplus. However, it could either increase or reduce the procurer's payoff. If the procurer reimburses the suppliers' all-pay quality provision costs, this would increase the quality provision and the supplier payoff, but it reduces the total surplus and procurer's payoff.

These findings thus can be viewed as additional justifications for a procurer's requirement that suppliers actually bear the costs of developing the product as part of their bids. Doing so not only guarantees that the procurer acquires exactly the quality that the winning supplier bids, but also could increase the procurer's payoff.

Appendix A

Proof of Lemma 1

Note that from (6), we have

$$
\begin{aligned}
& (1-F(c_i))^{N-1}p(c_i) \\
= & \int_{c_i}^{\bar{c}} \{[1-F(t)]^{N-1}(1-\rho)\varsigma(q(t)) + \rho\varsigma(q(t))\}dt \\
& +\rho c_i\varsigma(q(c_i)) + (1-F(c_i))^{N-1}(1-\rho)c_i\varsigma(q(c_i)).
\end{aligned}
$$

Note that from (8), we have $\frac{c_i[\rho+(1-\rho)(1-F(c_i))^{N-1}]}{(1-F(c_i))^{N-1}}\varsigma'(q(c_i)) = 1$. We thus have

$$
\begin{aligned}
& [(1-F(c_i))^{N-1}p(c_i)]' = \rho c_i\varsigma'(q(c_i))q'(c_i) + c_i\frac{d\{(1-F(c_i))^{N-1}(1-\rho)\varsigma(q(c_i))\}}{dc_i} \\
\Leftrightarrow \quad & (1-F(c_i))^{N-1}p'(c_i) - (N-1)(1-F(c_i))^{N-2}f(c_i)p(c_i) \\
= \quad & \rho c_i\varsigma'(q(c_i))q'(c_i) + c_i\frac{d\{(1-F(c_i))^{N-1}(1-\rho)\varsigma(q(c_i))\}}{dc_i} \\
\Leftrightarrow \quad & p'(c_i) - \frac{(N-1)f(c_i)}{1-F(c_i)}p(c_i) \\
= \quad & q'(c_i)\frac{c_i[\rho+(1-\rho)(1-F(c_i))^{N-1}]}{(1-F(c_i))^{N-1}}\varsigma'(q(c_i)) - \frac{(N-1)f(c_i)}{1-F(c_i)}c_i(1-\rho)\varsigma(q(c_i)) \\
\Leftrightarrow \quad & p'(c_i) = q'(c_i) - \frac{(N-1)f(c_i)}{1-F(c_i)}c_i(1-\rho)\varsigma(q(c_i)) + \frac{(N-1)f(c_i)}{1-F(c_i)}p(c_i) \\
\Leftrightarrow \quad & s'(c_i) = q'(c_i) - p'(c_i) = \frac{(N-1)f(c_i)}{1-F(c_i)}\{c_i(1-\rho)\varsigma(q(c_i)) - p(c_i)\}.
\end{aligned}
$$

Note that from (6), we have $c_i(1-\rho)\varsigma(q(c_i)) - p(c_i) < 0$. We thus have $s'(c_i) < 0$.

Proof of Lemma 2

By (9), we have $q(c_i;\rho) = \left[\frac{1}{\gamma c_i[(1-\rho)+\frac{\rho}{[1-F(c_i)]^{N-1}}]}\right]^{\frac{1}{\gamma-1}}$. The expected total surplus is thus

$$
\begin{aligned}
& TS(\rho) \\
= & \int_{\underline{c}}^{\bar{c}} q(\tau;\rho) d[1-(1-F(\tau))^N] - N\cdot E_{c_i}\{(1-\rho)c_i\varsigma(q(c_i;\rho))[1-F(c_i)]^{N-1}\} \\
& -N\cdot E_{c_i}[\rho c_i \varsigma(q(c_i;\rho))] \\
= & NE_\tau\left\{\left[\frac{1}{\gamma\tau[(1-\rho)+\frac{\rho}{(1-F(\tau))^{N-1}}]}\right]^{\frac{1}{\gamma-1}}(1-F(\tau))^{N-1}\right\} \\
& -N\cdot E_\tau\left\{(1-\rho)\tau\left[\frac{1}{\gamma\tau[(1-\rho)+\frac{\rho}{(1-F(\tau))^{N-1}}]}\right]^{\frac{\gamma}{\gamma-1}}(1-F(\tau))^{N-1}\right\} \\
& -N\cdot E_\tau\left\{\rho\tau\left[\frac{1}{\gamma\tau[(1-\rho)+\frac{\rho}{(1-F(\tau))^{N-1}}]}\right]^{\frac{\gamma}{\gamma-1}}\right\} \\
= & NE_\tau\left\{\left[\frac{1}{\gamma\tau[(1-\rho)+\frac{\rho}{(1-F(\tau))^{N-1}}]}\right]^{\frac{1}{\gamma-1}}(1-F(\tau))^{N-1}\right\} \\
& -N\cdot E_\tau\left\{\left[\frac{1}{\gamma\tau[(1-\rho)+\frac{\rho}{(1-F(\tau))^{N-1}}]}\right]^{\frac{\gamma}{\gamma-1}}\tau\left[(1-\rho)-\frac{\rho}{(1-F(\tau))^{N-1}}\right](1-F(\tau))^{N-1}\right\} \\
= & NE_\tau\left\{\left[\frac{1}{\gamma\tau[(1-\rho)+\frac{\rho}{(1-F(\tau))^{N-1}}]}\right]^{\frac{1}{\gamma-1}}(1-F(\tau))^{N-1}\right\} \\
& -N\cdot E_\tau\left\{\left[\frac{1}{\gamma\tau[(1-\rho)+\frac{\rho}{(1-F(\tau))^{N-1}}]}\right]^{\frac{1}{\gamma-1}}(1-F(\tau))^{N-1}\frac{1}{\gamma}\right\} \\
= & N(1-\frac{1}{\gamma})E_\tau\left\{\left[\frac{1}{\gamma\tau[(1-\rho)+\frac{\rho}{(1-F(\tau))^{N-1}}]}\right]^{\frac{1}{\gamma-1}}(1-F(\tau))^{N-1}\right\}.
\end{aligned}
$$

Suppliers' expected surplus is

$$
\begin{aligned}
& N\pi_S(\rho) \\
= \;& N\int_{\underline{c}}^{\bar{c}}\int_{c}^{\bar{c}}\{[1-F(\tau)]^{N-1}(1-\rho)+\rho\}\varsigma(q(\tau;\rho))d\tau dF(c) \\
= \;& N\int_{\underline{c}}^{\bar{c}}\int_{0}^{\tau}\{[1-F(\tau)]^{N-1}(1-\rho)+\rho\}\varsigma(q(\tau))dF(c)d\tau \\
= \;& N\int_{\underline{c}}^{\bar{c}}\{[1-F(\tau)]^{N-1}(1-\rho)+\rho\}\varsigma(q(\tau))F(\tau)d\tau \\
= \;& NE_\tau\left\{[(1-\rho)+\frac{\rho}{(1-F(\tau))^{N-1}}][\frac{1}{\gamma\tau[(1-\rho)+\frac{\rho}{(1-F(\tau))^{N-1}}]}]^{\frac{\gamma}{\gamma-1}}\frac{F(\tau)}{f(\tau)}[1-F(\tau)]^{N-1}\right\} \\
= \;& NE_\tau\left\{\frac{1}{\gamma\tau}\cdot\left[\frac{1}{\gamma\tau[(1-\rho)+\frac{\rho}{(1-F(\tau))^{N-1}}]}\right]^{\frac{1}{\gamma-1}}\frac{F(\tau)}{f(\tau)}[1-F(\tau)]^{N-1}\right\}.
\end{aligned}
$$

Therefore, we have the following procurer's expected payoff

$$
\begin{aligned}
\pi_P(\rho) \;=\;& TS(\rho)-N\pi_S(\rho) \\
=\;& N(1-\frac{1}{\gamma})E_\tau\left\{\left[\frac{1}{\gamma\tau[(1-\rho)+\frac{\rho}{(1-F(\tau))^{N-1}}]}\right]^{\frac{1}{\gamma-1}}(1-F(\tau))^{N-1}\right\} \\
& -NE_\tau\left\{\frac{1}{\gamma\tau}\cdot\left[\frac{1}{\gamma\tau[(1-\rho)+\frac{\rho}{(1-F(\tau))^{N-1}}]}\right]^{\frac{1}{\gamma-1}}\frac{F(\tau)}{f(\tau)}[1-F(\tau)]^{N-1}\right\} \\
=\;& NE_\tau\left\{\left[\frac{1}{\gamma\tau[(1-\rho)+\frac{\rho}{(1-F(\tau))^{N-1}}]}\right]^{\frac{1}{\gamma-1}}(1-F(\tau))^{N-1}\left[(1-\frac{1}{\gamma})-\frac{1}{\gamma\tau}\frac{F(\tau)}{f(\tau)}\right]\right\}.
\end{aligned}
$$

Proof of Proposition 5

Recall

$$
\begin{aligned}
& \frac{d\pi_i(c_i,c_i)}{dc_i} \\
= \;& -[1-F(c_i)]^{N-1}\frac{[1-F(c_i)]^{N-1}(1-\rho)+\rho}{[1-F(c_i)]^{N-1}}\cdot(\varsigma\circ(\varsigma')^{-1})\left(\frac{[1-F(c_i)]^{N-1}}{c_i\{(1-\rho)[1-F(c_i)]^{N-1}+\rho\}}\right).
\end{aligned}
$$

Let $x(\rho) = \frac{[1-F(c_i)]^{N-1}}{c_i\{(1-\rho)[1-F(c_i)]^{N-1}+\rho\}}$. We have $x(\rho) > 0$, $x'(\rho) < 0$. Thus the monotonicity of $|\frac{d\pi_i(c_i,c_i)}{dc_i}|$ with respect to ρ is equivalent to the monotonicity of $\frac{1}{x}\varphi(x)$, $x \geq 0$. Note $\varphi(0) = 0$ and $\lim_{x\to 0} \frac{1}{x}\varphi(x) = \varphi'(0) \geq 0$, since φ increases. In addition, $[\frac{1}{x}\varphi(x)]' = \frac{1}{x^2}[x\varphi'(x) - \varphi(x)]$. $x\varphi'(x) - \varphi(x)$ increases with x since its derivative is $x\varphi''(x) \geq 0$ and reaches zero when $x = 0$. We thus have $[\frac{1}{x}\varphi(x)]' \geq 0$, i.e. $|\frac{d\pi_i(c_i,c_i)}{dc_i}|$ decreases with ρ.

Proof of Proposition 8

Let $x = F(c) \in [0,1]$, and

$$\varsigma(x) = (1-x)^{N-1} - \frac{\rho}{\gamma-1}[1-(1-x)^{N-1}] - [(1-\rho)(1-x)^{N-1}+\rho]^{-\frac{1}{\gamma-1}}(1-x)^{\frac{(N-1)\gamma}{\gamma-1}}.$$

Note that

$$\frac{TS^R(\rho) - TS(\rho)}{N(\gamma-1)\gamma^{-\frac{\gamma}{\gamma-1}}} = E_c\left\{c^{\frac{-1}{\gamma-1}}\varsigma(F(c))\right]\}.$$

Note $\varsigma(0) = 0$ and

$$\begin{aligned} & \frac{\varsigma'(x)}{N-1}(1-x)^{-(N-2)}(\gamma-1) \\ = & -[\gamma-(1-\rho)] + [\gamma\alpha(x) - (1-\rho)\alpha^\gamma(x)], \end{aligned}$$

where

$$\alpha(x) = [(1-\rho)(1-x)^{N-1}+\rho]^{-\frac{1}{\gamma-1}}(1-x)^{\frac{N-1}{\gamma-1}} \in [0,1].$$

It is easily to verify $\alpha'(x) \leq 0$, as $\alpha(x) = \{[(1-\rho)z(x)+\rho]^{-1}z(x)\}^{\frac{1}{\gamma-1}}$, where $z(x) = (1-x)^{N-1}$.

Let $\omega(\alpha) = \gamma\alpha - (1-\rho)\alpha^\gamma$. Note that

$$\omega'(\alpha) = \gamma[1-(1-\rho)\alpha^{\gamma-1}] > 0.$$

We thus have $\varsigma'(x) \leq 0$, which means that $\varsigma(x) \leq 0$ as $\varsigma(0) = 0$. As a result,

$$TS^R(\rho) - TS(\rho) \leq 0,$$

which says reimbursing the all pay component of the losers leads to lower total surplus.

Recall that Proposition 7 says that reimbursing the all pay component of the losers leads to higher suppliers' payoffs if $\varsigma(q) = q^\gamma, \gamma > 1$. This implies that the procurer must enjoy a higher payoff when she does not reimburse the all pay component of the losers.

Proof of Proposition 10

Note that

$$\begin{aligned}
&\frac{TS^{R,A}(\rho)-TS(\rho)}{N\gamma^{-\frac{\gamma}{\gamma-1}}} \\
= \quad & E_c\left\{c^{\frac{-1}{\gamma-1}}\left[\begin{array}{c}\gamma(1-\rho)^{-\frac{1}{\gamma-1}}(1-F(c))^{N-1}-(1-\rho)^{-\frac{\gamma}{\gamma-1}}\left[(1-\rho)(1-F(c))^{N-1}+\rho\right] \\ -(\gamma-1)[(1-\rho)(1-F(c))^{N-1}+\rho]^{-\frac{1}{\gamma-1}}(1-F(c))^{\frac{(N-1)\gamma}{\gamma-1}}\end{array}\right]\right\}.
\end{aligned}$$

Let $x=F(c)\in[0,1]$, and

$$\begin{aligned}
\xi(x) \quad = \quad & \gamma(1-\rho)^{-\frac{1}{\gamma-1}}(1-x)^{N-1}-(1-\rho)^{-\frac{\gamma}{\gamma-1}}\left[(1-\rho)(1-x)^{N-1}+\rho\right] \\
& -(\gamma-1)[(1-\rho)(1-x)^{N-1}+\rho]^{-\frac{1}{\gamma-1}}(1-x)^{\frac{(N-1)\gamma}{\gamma-1}}.
\end{aligned}$$

Note that

$$\begin{aligned}
\xi(0) \quad &= \quad \gamma(1-\rho)^{-\frac{1}{\gamma-1}}-(1-\rho)^{-\frac{\gamma}{\gamma-1}}-(\gamma-1) \\
&= \quad \gamma y-y^{\gamma}-(\gamma-1),
\end{aligned}$$

where $y=(1-\rho)^{-\frac{1}{\gamma-1}}\geq 1$. Note $\frac{d\xi(0)}{dy}=\gamma(1-y^{\gamma-1})\leq 0$. Thus, $\xi(0)\leq(\gamma y-y^{\gamma}-(\gamma-1))|_{y=1}=0$.

$$\begin{aligned}
&\frac{\xi'(x)}{(N-1)}(1-x)^{-(N-2)} \\
= \quad & -\gamma(1-\rho)^{-\frac{1}{\gamma-1}}+(1-\rho)^{-\frac{1}{\gamma-1}}-[(1-\rho)(1-x)^{N-1}+\rho]^{-\frac{\gamma}{\gamma-1}}(1-\rho)(1-x)^{\frac{(N-1)\gamma}{\gamma-1}} \\
& +\gamma[(1-\rho)(1-x)^{N-1}+\rho]^{-\frac{1}{\gamma-1}}(1-x)^{\frac{(N-1)}{\gamma-1}} \\
= \quad & (1-\rho)^{-\frac{1}{\gamma-1}}(1-\gamma)+\left[\gamma\alpha(x)-(1-\rho)\alpha^{\gamma}(x)\right],
\end{aligned}$$

where $\alpha(x)=[(1-\rho)(1-x)^{N-1}+\rho]^{-\frac{1}{\gamma-1}}(1-x)^{\frac{N-1}{\gamma-1}}\in[0,1]$, as in the proof of Proposition 8.

Let $\omega(\alpha)=\gamma\alpha-(1-\rho)\alpha^{\gamma}$. Note that $\omega'(\alpha)=\gamma[1-(1-\rho)\alpha^{\gamma-1}]>0$, as in the proof of Proposition 8. Therefore, we have

$$\begin{aligned}
&\frac{\xi'(x)}{(N-1)}(1-x)^{-(N-2)} \\
\leq \quad & (1-\rho)^{-\frac{1}{\gamma-1}}(1-\gamma)+\left[\gamma-(1-\rho)\right].
\end{aligned}$$

The RHS decreases with ρ, which means it is no greater than

$$\left\{(1-\rho)^{-\frac{1}{\gamma-1}}(1-\gamma)+[\gamma-(1-\rho)]\right\}|_{\rho=0}=0.$$

Therefore, we have $\xi'(x)\leq 0$, which further means $\xi(x)\leq 0$ as $\xi(0)\leq 0$. We thus have

$$TS^{R,A}(\rho)-TS(\rho)\leq 0,$$

which says reimbursing the all-pay component of all suppliers leads to lower total surplus.

Recall that Proposition 9 says that reimbursing all pay component of all suppliers leads to higher suppliers' payoffs if $\varsigma(q)=q^{\gamma},\gamma>1$. This implies that the procurer must enjoy a higher payoff when she does not reimburse the all-pay component of the suppliers.

Appendix B

In this section, we describe how to generalize our two-step procedure for identifying symmetric pure strategy equilibrium in an environment with $N\geq 2$ suppliers and a general score function $s(q,p)$ with $s_q(q,p)>0$, $s_p(q,p)<0$.

Implication of the first observation

We treat equilibrium $(q(c_i),p(c_i))$ as an incentive-compatible general direct mechanism that generates scores decreasing in types, then the Myerson [12] approach allows us to write down payment rule $p(c_i)$ as a function of the allocation rule $q(c_i)$. The following is the details.

Given $(q(c_i),p(c_i))$ that leads to scores that are decreasing in types, consider a type c_i supplier i's problem of making an announcement $\tilde{c}_i$ to maximize his expected payoff when the other supplier is truthful:

$$\max_{\tilde{c}_i}\pi_i(\tilde{c}_i,c_i)=[1-F(\tilde{c}_i)]^{N-1}[p(\tilde{c}_i)-(1-\rho)c_i\varsigma(q(\tilde{c}_i))]-\rho c_i\varsigma(q(\tilde{c}_i)).$$

If $(q(\cdot),p(\cdot))$ is a pure strategy equilibrium, we would have $\tilde{c}_i^*=c_i$, i.e. the incentive compatibility (IC) condition:

$$c_i=\arg\max_{\tilde{c}_i}\pi_i(\tilde{c}_i,c_i)=[1-F(\tilde{c}_i)]^{N-1}[p(\tilde{c}_i)-(1-\rho)c_i\varsigma(q(\tilde{c}_i))]-\rho c_i\varsigma(q(\tilde{c}_i)). \tag{26}$$

The IC condition (26), together with the envelope theorem, thus leads to

$$\frac{d\pi_i(c_i,c_i)}{dc_i}=-\{[1-F(c_i)]^{N-1}(1-\rho)\varsigma(q(c_i))+\rho\varsigma(q(c_i))\}. \tag{27}$$

Note that at equilibrium with scores monotonic in types, we must have $\pi_i(\bar{c}, \bar{c}) = 0$ for both $\rho = 0$ and $\rho = 1$. For both cases, the least efficient type $\bar{c}$ never wins and never incurs cost to produce. Note we must have $q(\bar{c}) = 0$ for $\rho = 1$.

Using the envelope condition of (27), we have

$$\pi_i(c_i, c_i) = \int_{c_i}^{\bar{c}} \{[1 - F(t)]^{N-1}(1-\rho)\varsigma(q(t)) + \rho\varsigma(q(t))\}dt. \tag{28}$$

By the definition of $\pi_i(\tilde{c}_i, c_i)$, we have

$$\begin{aligned} \pi_i(c_i, c_i) &= [1 - F(c_i)]^{N-1}[p(c_i) - (1-\rho)c_i\varsigma(q(c_i))] - \rho c_i \varsigma(q(c_i)) \\ &= \int_{c_i}^{\bar{c}} \{[1 - F(c_i)]^{N-1}(1-\rho)\varsigma(q(c_i)) + \rho\varsigma(q(c_i))\}dt, \end{aligned}$$

which leads to price rule of

$$p(c_i) = \frac{\int_{c_i}^{\bar{c}} \{[1 - F(t)]^{N-1}(1-\rho)\varsigma(q(t)) + \rho\varsigma(q(t))\}dt + \rho c_i \varsigma(q(c_i))}{[1 - F(c_i)]^{N-1}} + (1-\rho)c_i\varsigma(q(c_i)) > 0. \tag{29}$$

Implication of the second observation

Suppose $(q(c_i), p(c_i))$ is a symmetric pure strategy equilibrium. By the implications of the first observation, we must have $q'(c_i) < 0$ (monotone allocation rule is implied by incentive compatibility condition) and (29) holds. Define

$$s(c_i) = s(q(c_i), p(c_i)). \tag{30}$$

We now look at the implications of the second observation, which says that equilibrium $(q(c_i), p(c_i))$ must solve the following optimization problem:

$$\begin{aligned} (q(c_i), p(c_i)) &= \arg\max_{(q,p)} [1 - F(c_i)]^{N-1}[p - (1-\rho)c_i\varsigma(q)] - \rho c_i \varsigma(q), \\ s.t. &: \ s(q, p) = s(c_i). \end{aligned}$$

The corresponding Lagrangian is

$$\max_{(q,p,\lambda)} L(q, p, \lambda) = [1 - F(c_i)]^{N-1}[p - (1-\rho)c_i\varsigma(q)] - \rho c_i \varsigma(q) + \lambda[s(q, p) - s(c_i)].$$

The first order conditions are as follows:

$$\begin{aligned}\frac{L(q,p,\lambda)}{\partial q} &= -[1-F(c_i)]^{N-1}(1-\rho)c_i\varsigma'(q) - \rho c_i\varsigma'(q) + \lambda s_q(q,p) = 0,\\ \frac{L(q,p,\lambda)}{\partial p} &= [1-F(c_i)]^{N-1} + \lambda s_p(q,p) = 0,\\ \frac{L(q,p,\lambda)}{\partial \lambda} &= s(q,p) - s(c_i) = 0.\end{aligned}$$

We thus have

$$\varsigma'(q(c_i)) = -\frac{s_q(q(c_i),p(c_i))}{s_p(q(c_i),p(c_i))}\frac{1}{c_i\{(1-\rho)+\frac{\rho}{[1-F(c_i)]^{N-1}}\}}, \tag{31}$$

where $p(c_i)$ is given by (29).

Together with boundary condition $q(\bar{c}) = 0$, we can pin down the candidate quality bid $q(c_i)$. If there is no solution exists, then there is no symmetric pure strategy equilibrium.

Suppose $q(c_i)$ is a solution of (31) with boundary condition $q(\bar{c}) = 0$, and $p(c_i)$ is the corresponding price bid given by (29). If the corresponding score bid $s(c_i) = s(q(c_i), p(c_i))$ is decreasing in types, then we have $(q(c_i), p(c_i))$ constitutes a symmetric pure strategy equilibrium. Otherwise, there is no symmetric pure strategy equilibrium.

An application to the ratio score

As an application, we next show that with the ratio form score function $s = s(q,p) = \frac{q}{p}$, there might be no symmetric pure strategy equilibrium for the case of an all-pay quality bid, i.e. $\rho = 1$. The equilibrium analysis in this application applies to a score function of $s = s(q,p) = \frac{q^{\mu}}{p^{\tau}}$ with μ, $\tau > 0$.

In this case, (31) is written as

$$\varsigma'(q) = \frac{p}{q}\frac{[1-F(c_i)]^{N-1}}{c_i}. \tag{32}$$

Given (29), we have

$$\begin{aligned}\varsigma'(q(c_i)) &= \frac{1}{q(c_i)}\frac{\int_{c_i}^{\bar{c}}\varsigma(q(t))dt + c_i\varsigma(q(c_i))}{c_i},\\ \text{i.e., } c_i q(c_i)\varsigma'(q(c_i)) &= \int_{c_i}^{\bar{c}}\varsigma(q(t))dt + c_i\varsigma(q(c_i)).\end{aligned}$$

Taking derivative wrt. c_i both sides, we have

$$q(c_i)\varsigma'(q(c_i)) + c_i q(c_i)\varsigma''(q(c_i))q'(c_i) = 0.$$

We thus have

$$q'(c_i) = -\frac{\varsigma'(q(c_i))}{c_i\varsigma''(q(c_i))} < 0, \tag{33}$$

with boundary condition $q(\bar{c}) = 0$.

Assume $\varsigma(q) = q^\gamma, \gamma > 1$. We have

$$\begin{aligned} (\gamma - 1)\,\frac{q'(c_i)}{q(c_i)} &= -\frac{1}{c_i} < 0, \\ \text{i.e. } \frac{d\ln q(c_i)}{dc_i} &= \frac{d[-\ln c_i]}{dc_i}, \end{aligned}$$

with boundary condition $q(\bar{c}) = 0$.

We thus have

$$\ln q(c_i) = -\frac{1}{\gamma - 1}\ln c_i + A,$$

which gives

$$q(c_i) = (\frac{1}{c_i})^{\frac{1}{\gamma-1}}\exp(A).$$

Note we cannot have $q(\bar{c}) = 0$ if $\bar{c}$ is finite. Therefore, there exists no symmetric pure strategy equilibrium for this case.

However, if $\varsigma'(0) > 0$, then we indeed have a solution for (33) with boundary condition $q(\bar{c}) = 0$, which can be identified as below.

(33) can be rewritten as

$$\frac{d\varsigma'(q(c_i))}{dc_i} = \frac{d[-\ln c_i]}{dc_i},$$

which entails

$$\varsigma'(q(c_i)) = (\frac{1}{c_i})\exp(A).$$

Using the boundary condition of $q(\bar{c}) = 0$, we have

$$\exp(A) = \bar{c}\varsigma'(0),$$

which leads to

$$q(c_i) = (\varsigma')^{-1}(\frac{\bar{c}\varsigma'(0)}{c_i}).$$

To confirm such identified $(q(c_i), p(c_i))$ is indeed an equilibrium, we now only need to verify that $s(c_i) = s(q(c_i), p(c_i)) = \frac{q(c_i)}{p(c_i)}$ is decreasing in types. Using (32) and (29) for $\rho = 1$, we have

$$\begin{aligned}
[\ln s(c_i)]' &= \frac{q'(c_i)}{q(c_i)} - \frac{p'(c_i)}{p(c_i)} = \frac{1}{q(c_i)}[q'(c_i) - \frac{q(c_i)}{p(c_i)}p'(c_i)] \\
&= \frac{1}{q(c_i)}[q'(c_i) - (\frac{c_i\varsigma'(q(c_i))}{[1-F(c_i)]^{N-1}})^{-1}(\frac{c_i\varsigma'(q(c_i))q'(c_i)}{[1-F(c_i)]^{N-1}} + (N-1)p(c_i)f(c_i))] \\
&= -\frac{1}{q(c_i)}(\frac{c_i\varsigma'(q(c_i))}{[1-F(c_i)]^{N-1}})^{-1}(N-1)p(c_i)f(c_i) \\
&< 0,
\end{aligned}$$

which means $s'(c_i) < 0$.

An interesting observation is that with the ratio score, regardless of the number of players, all type distributions with the same support $[\underline{c}, \bar{c}]$ yield the same equilibrium quality bidding strategies $q(c_i)$, $c_i \in [\underline{c}, \bar{c}]$.

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