Shifting Supports in Esö and Szentes (2007)

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Abstract

The common-support assumption of future type distributions is well adopted in the dynamic mechanism design literature. This assumption can become restrictive when the support is bounded. In this paper, we extend Esö and Szentes [7] by allowing shifting supports and formally show that both their analytical methodology and key insights are robust to this generalization.

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1 Introduction

Starting from the seminal work of Baron and Besanko [1], the literature of optimal dynamic mechanism design has grown rapidly.[1] Common supports—i.e., the support of an agent's period-$(t+1)$ type distributions are uniform across any type realization history—is typically assumed in the literature, such as Courty and Li [5], Esö and Szentes [7] and [8], Krähmer and Strausz [9], [10], and [11], Deb and Said [6], Li and Shi [12], and Bergemann, Castro, and Weintraub [2], among many others.[2] In two-period environments, this means that the support of the distribution of the agent's second-stage type is common across his first-stage types.

While the common-support assumption is appropriate in many applications, this is not always the case. Consider the following widely-used AR(1) process to model information dynamics: The agent receives a private signal v in the first stage about his second-stage true valuation V for a good

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[1]See, for example, Bergemann and Pavan [3] and Bergemann and Välimäki [4], for an excellent suvery of this literature.

[2]The only exception, to our best knowledge, is Pavan, Segal, and Toikka [15] who allow very flexible history-dependent distributions of future information in their analysis.

with $V = v + z$, where z is a shock which is independent of v.[3] If we assume that the supports of $H_v(\cdot)$—the cumulative distribution function of V corresponding to type v—are common across v's, then the only possibility is that the support is the whole real line,[4] which may not look realistic, because, in practice, the valuation of the good is likely to be nonnegative and/or bounded above. Therefore, for this example, it is more reasonable to assume that $H_v(\cdot)$ has a bounded support, which necessarily means that shifting supports must prevail.

The goal of this paper is to allow shifting supports in the model of Esö and Szentes [7] and formally show that both their analytical methodology and key insights can be extended.[5] Specifically, the revenue-maximizing mechanism takes the same form, and the orthogonalized future private information does not yield any information rents for the agent: the principal can achieve the same revenue as she could in the environment in which all agents' orthogonalized second-stage information is public. Our findings thus help to enlarge the scope of technically tractable dynamic models.

While following the procedure of Esö and Szentes [7], three notable issues arise when allowing shifting supports: 1) pinning down the second-stage optimal strategy following a lie in the first stage; 2) characterizing the first-stage off-equilibrium payoff; and 3) establishing the incentive compatibility of the candidate optimal mechanism. To elaborate, in Esö and Szentes [7], the agent will fully correct his first-stage lie by reporting a second-stage (orthogonalized) signal such that combining with the lie, it leads to the same *ex post* valuation (i.e., second-stage valuation). What makes the full lie-correction feasible is the common-support assumption for the *ex post* valuation. With shifting supports, clearly, a full lie-correction is not always feasible. We show that the agent's second-stage optimal strategy is to correct his lie as much as possible, because of the first-order stochastic dominance assumption in Esö and Szentes [7]. Due to the corner solution issue, this new strategy, in turn, may generate complications in characterizing the off-equilibrium payoff, which further impacts the way to establish the incentive compatibility of a proposed mechanism. Nevertheless, we find that the off-equilibrium payoff resembles that in Esö and Szentes [7], but with a different lie-correcting strategy.

The rest of the paper is organized as follows. Section 2 sets up the model. The main analysis is presented in Section 3. Section 4 concludes and Section 5 collects technical proofs.

[3] More generally, this applies to $V = \rho v + z$, where $\rho > 0$ is a constant.

[4] To see this, notice that if the infimum (supremum) of the common support of $H_v(\cdot)$ is finite, then this must imply that the supports of v and z are bounded below (above). However, this can never lead to the assumption that the supports of $H_v(\cdot)$ are the same for all v's.

[5] There are other approaches to establish the optimal mechansim in this extended setting. For example, the general analysis in Pavan, Segal, and Toikka [15] can also be applied to establish our results (please refer to discussions in Section 4).

2 The Model

We consider the following two-stage mechanism design problem built on Eső and Szentes [7].[6] There is a risk-neutral buyer (he) for a single individual good sold by a risk-neutral seller (She). The seller's valuation for the good is normalized as 0 and her goal is to maximize her revenue. The buyer's payoff is his valuation for the good (if he obtains it) less his payment. At the first stage, the buyer does not know exactly his valuation for the good, which will be realized at the second stage; however, the buyer has a private information (first-stage type) determining his true valuation. Specifically, the buyer's true valuation V, realized in the second stage, is a random draw from the cumulative distribution $H_v(\cdot)$, where v is his first-stage private type.

Our paper departs from Eső and Szentes [7] by allowing "shifting" supports. The CDF $H_v(\cdot)$ has a positive density $h_v(\cdot)$ everywhere over the support $[\underline{V}_v, \overline{V}_v]$, so the support of $H_v(\cdot)$ can vary with v. Suppose that $\underline{V}_v$ and $\overline{V}_v$ are continuously differentiable functions of v. Assume also that $H_v(V)$ is twice continuously differentiable in v and V. The buyer's first-stage type v follows a CDF $F(\cdot)$ with strictly positive density $f(\cdot)$ over the support $[\underline{v}, \overline{v}]$. $F(\cdot)$ and $H_v(\cdot)$ are public information, $\forall v$. The buyer's first-stage and second-stage types v and V need not be independent.

The timing of the game is: At time 0, Nature draws the buyer's type v according to $F(\cdot)$ and v is the buyer's private information. At time 1, the seller offers the contract and she commits to it; the buyer decides whether to accept the contract; if the buyer quits, he obtains his reservation payoff which is normalized as 0. At time 2, the buyer's private true valuation for the good, V, is randomly drawn from $H_v(\cdot)$, and then the contract is executed.

Similar to Eső and Szentes [7], we perform the following orthogonalization. Given v and V, construct s as $s = H_v(V)$, for any $V \in [\underline{V}_v, \overline{V}_v]$. It is easy to see that s follows the uniform distribution $U[0,1]$ and thus is independent of v. As is shown in Eső and Szentes [7], s and V contain the same information and the buyer can recover his valuation V through $u(v,s) = H_v^{-1}(s)$. Thus, treating the buyer's second-stage type as s is equivalent to the original information environment. In what follows, we call s as the second-stage *signal* and V as the second-stage *type*.

We make the following four assumptions[7] as in Eső and Szentes [7].

Assumption 1 $\frac{1-F(v)}{f(v)}$ *is weakly decreasing in* v.

Assumption 2 $\frac{\partial H_v(V)}{\partial v} < 0$, $\forall V \in (\underline{V}_v, \overline{V}_v)$.

Assumption 3 $\frac{\partial H_v(V)}{\partial v} / h_v(V)$ *is increasing in* V.

[6] Our analysis can be easily extended to the multi-agent case. For notation simplicity, we focus on the single-agent environment.

[7] As shown in Eső and Szentes [7], Assumption 3 is equivalent to $u_{12} \leq 0$ and Assumption 4 is equivalent to $u_{11}/u_1 \leq u_{12}/u_2$. (For a two-variable function, we use subscripts 1 and 2 to represent the partial derivative with respect to the first and second argument, respectively. Notations for partial derivatives of functions with three variables are similar.)

Assumption 4 $\frac{\partial H_v(V)}{\partial v}/h_v(V)$ *is increasing in* v.

Remark: Assumption 2 implies that for any v and $\hat{v}$ with $\hat{v} < v$, H_v first-order stochastic dominates $H_{\hat{v}}$; thus, in particular, $\overline{V}_v \geq \overline{V}_{\hat{v}}$ and $\underline{V}_v \geq \underline{V}_{\hat{v}}$. This implies that in general the support of H_v changes when v changes. In Eső and Szentes [7], the supports of H_v are common for all v.

Assumption 1 of Eső and Szentes [7] (i.e., our Assumption 3) becomes restrictive when the common support is bounded. To see this, suppose that the support of the second-stage valuation V is common (i.e., independent of the first-stage type) and denote the support as $[\underline{V}, \overline{V}]$. With the density function $h_v(V)$ being strictly positive everywhere on $[\underline{V}, \overline{V}]$, their Assumption 1 implies that the CDF satisfies $\partial H_v(V)/\partial v = 0$ for all $V \in [\underline{V}, \overline{V}]$,[8] which means that the distribution of V cannot depend on the first-stage type v.

3 Analysis of Shifting Supports

3.1 Benchmark

The goal of this paper is to formally show that under shifting supports, the key insights in the dynamic mechanism design literature still holds. We first study the benchmark case, in which the seller observes the buyer's second-stage signal while the buyer's first-stage type is still private. According to the revelation principle, one can restrict to direct mechanisms in which the buyer truthfully reports his first-stage type. Specifically, suppose that the buyer reports his first-stage type as v and his second-stage signal is s, then a direct mechanism can be expressed as $\{X(v,s), T(v,s)\}$, where $X(v,s)$ is the probability that the good is sold to the buyer, and $T(v,s)$ is the buyer's payment.

In the benchmark case, IC (incentive compatibility) means that the buyer will report his first-stage type truthfully. If a buyer with type v reports $\hat{v}$ in the first stage, then his expected payoff is

$$\pi(v, \hat{v}) = \int_0^1 [u(v,s)X(\hat{v},s) - T(\hat{v},s)]ds.$$

IC requires that $\pi(v,v) \geq \pi(v,\hat{v}), \forall v, \hat{v}$. Standard arguments such as envelope theorem (cf. Milgrom and Segal [13]) lead to the following result, the proof of which is omitted.[9]

Proposition 1 *In the benchmark case, IC implies that*

$$\pi(v,v) = \pi(\underline{v},\underline{v}) + \int_{\underline{v}}^{v} \int_0^1 u_1(z,s)X(z,s)dsdz, \forall v. \quad (1)$$

[8] To see this, notice that with common support, $H_v(\underline{V}) = 0$ and $H_v(\overline{V}) = 1$ for all v. Therefore, $\frac{\partial H_v(\underline{V})}{\partial v}/h_v(\underline{V}) = 0$ and $\frac{\partial H_v(\overline{V})}{\partial v}/h_v(\overline{V}) = 0$. Then, their Assumption 1 implies that $\frac{\partial H_v(V)}{\partial v}/h_v(V) = 0$ for all $V \in [\underline{V}, \overline{V}]$, which further implies that $\partial H_v(V)/\partial v = 0$ for all $V \in [\underline{V}, \overline{V}]$. We thank an anonymous reviewer for this motivation and observation.

[9] Since $|X(\hat{v},s)| \leq 1$, using Lebesgue's dominated convergence theorem, one is easy to verify that all assumptions in Theorem 2 in Milgrom and Segal [13] are satisfied.

Therefore, as is standard in the literature and similar to Esö and Szentes [7], the seller's payoff can be expressed as

$$\int_{\underline{v}}^{\bar{v}} \int_0^1 [u(v,s) - \frac{1-F(v)}{f(v)} u_1(v,s)] X(v,s) ds dF(v) - \pi(\underline{v}, \underline{v}). \tag{2}$$

Denote the adjusted virtual value as $W(v,s) = u(v,s) - \frac{1-F(v)}{f(v)} u_1(v,s)$. Proposition 1 immediately implies the following result.[10]

Corollary 1 *In the benchmark case, the optimal allocation rule is*

$$X^*(v,s) = \begin{cases} 1, \text{ if } W(v,s) \geq 0, \\ 0, \text{ otherwise.} \end{cases} \tag{3}$$

The seller's optimal revenue is

$$R^* = \int_{\underline{v}}^{\bar{v}} \int_0^1 \max\{W(v,s), 0\} ds dF(v). \tag{4}$$

Similar to Corollary 1 in Esö and Szentes [7], $X^*(v,s)$ has the following properties.

Corollary 2 *i) $X^*(v,s)$ is weakly increasing in both v and s.*
ii) If $v > \hat{v}$, $s \leq \hat{s}$, and $u(v,s) \geq u(\hat{v},\hat{s})$, then $X^*(v,s) \geq X^*(\hat{v},\hat{s})$.

The proof of i) in Corollary 2 is exactly the same as that of Corollary 1 in Esö and Szentes [7]. The proof of ii) is in the appendix.

3.2 The Original Problem

In this section, we study the original problem in which the first-stage type and the second-stage signal are the buyer's private information. According to the revelation principle (Myerson [14]), one can restrict to direct mechanisms denoted as (with a little abuse of notation) $\{X(v,s), T(v,s)\}$, where $X(v,s)$ is the probability that the good is sold to the buyer, and $T(v,s)$ is the buyer's payment.

3.2.1 Second Stage

Suppose that the buyer's first-stage type and second-stage signal is v and s, respectively. If the buyer truthfully reported his first-stage type v and reports $\hat{s}$ in the second stage, his expected second-stage payoff is

$$\tilde{\pi}(s, \hat{s}; v) = u(v,s) X(v,\hat{s}) - T(v,\hat{s}).$$

[10] The proof is similar to that of Proposition 1 in Esö and Szentes [7], which follows by checking IC of the proposed mechanism in (3) (implied by the monotonicity of $W(v,s)$ in v).

The second-stage IC means that truth-telling is optimal, i.e., $\tilde{\pi}(s, s; v) \geq \tilde{\pi}(s, \hat{s}; v), \forall \hat{s}$.

Again, standard arguments such as envelope theorem lead to the following result (the proof is omitted).[11]

Lemma 1 *In the original problem, suppose that the buyer reports his first-stage type v truthfully. Then the second-stage IC holds if and only if the following two conditions hold:*
i)

$$\tilde{\pi}(s, s; v) = \tilde{\pi}(0, 0; v) + \int_0^s u_2(v, t) X(v, t) dt, \forall s, \forall v; \tag{5}$$

ii) $X(v, s)$ is increasing in s, $\forall v$.

3.2.2 Optimal Deviation Following a "Lie"

Lemma 1 provides a necessary and sufficient condition for the second-stage IC, assuming truthful first stage. To characterize the first-stage IC, it is important to know the buyer's optimal deviation strategy after a "lie" in the first stage. To this end, consider a buyer with first-stage type v and second-stage signal s who reported $\hat{v}$ in the first stage. Esö and Szentes [7] show that the optimal deviation strategy is to correct the lie in the way that the *ex post* valuations are the same; i.e., the buyer will report $\hat{s}$ such that $u(\hat{v}, \hat{s}) = u(v, s)$. However, in general, this does not apply here, because of shifting supports—the buyer cannot find any $\hat{s} \in [0, 1]$ such that $u(\hat{v}, \hat{s}) = u(v, s)$, if $u(v, s) \notin [\underline{V}_{\hat{v}}, \overline{V}_{\hat{v}}]$.

We characterize the optimal deviation strategy following a lie in the first stage in the following lemma. (All proofs are relegated to the appendix.)

Lemma 2 *In an incentive-compatible two-stage mechanism, a buyer with type v who reported $\hat{v}$ in the first stage and observed signal s in the second stage will report $\hat{s} = \tilde{\sigma}(v, \hat{v}, s)$ such that:*

If $\hat{v} < v$, then

$$\tilde{\sigma}(v, \hat{v}, s) = \begin{cases} 1, \text{ if } u(v, s) > u(\hat{v}, 1), \\ \sigma(v, \hat{v}, s), \text{ otherwise.} \end{cases}$$

If $\hat{v} > v$, then

$$\tilde{\sigma}(v, \hat{v}, s) = \begin{cases} 0, \text{ if } \quad u(v, s) < u(\hat{v}, 0), \\ \sigma(v, \hat{v}, s), \text{ otherwise.} \end{cases}$$

Here, $\sigma(v, \hat{v}, s) \in [0, 1]$ is the unique signal such that $u(v, s) = u(\hat{v}, \sigma(v, \hat{v}, s))$.

Lemma 2 generalizes the observation in Lemma 4 of Esö and Szentes [7] to the shifting-supports case. It says that the buyer "tries his best" to correct his first-stage lie: If he is able to find a signal $\hat{s}$ in $[0, 1]$ to fully mask his lie in the first stage (i.e., $u(\hat{v}, \hat{s}) = u(v, s)$), then he will do so by reporting $\sigma(v, \hat{v}, s)$; if he is unable to fully mask his lie, then he tries his best to partially mask it by "hitting

[11] Note that by definition, $u(v,s) = H_v^{-1}(s)$. Thus, $u_2 > 0$, so that we have monotonicity ii) in Lemma 1).

the boundary"—when he under-reports ($\hat{v} < v$), first-order stochastic dominance implies that he is unable to fully correct his lie when his true valuation $u(v, s)$ is outside the support of $H_{\hat{v}}$ (i.e., $u(v, s) > u(\hat{v}, 1)$), thus the best he can do is to minimize the difference so that reporting $\hat{s} = 1$ is optimal; similarly, when he over-reports ($\hat{v} > v$), first-order stochastic dominance implies that he is unable to fully correct his lie when his true valuation $u(v, s)$ is outside the support of $H_{\hat{v}}$ (i.e., $u(v, s) < u(\hat{v}, 0)$), thus the best he can do is to minimize the difference so that reporting $\hat{s} = 0$ is optimal.

3.2.3 Optimal Deviation Payoff

Now we are ready to characterize the buyer's optimal deviation payoff. If a buyer with type v reports $\hat{v}$ in the first stage, his optimal expected payoff can be written as

$$\pi(v, \hat{v}) = \int_0^1 [u(v, s)X(\hat{v}, \tilde{\sigma}(v, \hat{v}, s)) - T(\hat{v}, \tilde{\sigma}(v, \hat{v}, s))]ds.$$

Lemma 2 enables us to characterize the buyer's optimal deviation payoff in a more tractable way than the above equation, which is crucial to establish our main result, as checking whether a mechanism is IC in the original problem highly relies on the comparison between payoffs on and off the equilibrium path.

Lemma 3 *In an incentive-compatible two-stage mechanism, if a buyer with type v reports $\hat{v}$ in the first stage, then his first-stage expected payoff can be expressed as:*[12]

$$\pi(v, \hat{v}) = \pi(\hat{v}, \hat{v}) + \int_0^1 \int_{\hat{v}}^{v} u_1(y, s)X(\hat{v}, \tilde{\sigma}(y, \hat{v}, s))dyds.$$

The form in the above lemma resembles that in Lemma 5 in Esö and Szentes [7], but with a different optimal lie-correcting strategy.

3.2.4 First-Stage IC

First-stage IC means that $\pi(v, v) \geq \pi(v, \hat{v})$, $\forall v, \hat{v}$, which further implies that

$$\pi(v, v) \geq \int_0^1 [u(v, s)X(\hat{v}, s) - T(\hat{v}, s)]ds,$$

where the RHS of the above inequality, denoted as $\hat{\pi}(v, \hat{v})$, is the expected first-stage payoff when the buyer truthfully reports his second-stage signal even if he lied in the first stage. First-stage IC implies that $\hat{\pi}(v, v) = \pi(v, v) \geq \hat{\pi}(v, \hat{v})$, $\forall v, \hat{v}$. As explained in footnote 9, standard arguments such as envelope theorem lead to the following result.

[12] $\int_a^b$ denotes $-\int_b^a$ if $a > b$.

Proposition 2 *In the original problem, the first-stage IC implies that*

$$\pi(v,v) = \pi(\underline{v},\underline{v}) + \int_{\underline{v}}^{v}\int_0^1 u_1(y,s)X(y,s)dsdy, \forall v. \tag{6}$$

Therefore, the seller's revenue can be expressed as

$$\int_{\underline{v}}^{\overline{v}}\int_0^1 [u(v,s) - \frac{1-F(v)}{f(v)}u_1(v,s)]X(v,s)dsdF(v) - \pi(\underline{v},\underline{v}). \tag{7}$$

Notice that (2) and (7) are exactly the same, so that the seller's revenues in the benchmark case and the original problem have the same expressions. However, this does not imply that these two problems are equivalent, as an IC mechanism in the benchmark case may not be IC in the original problem.

3.3 Implementability of the Optimum

In this subsection, we present our main result that the revenue upper bound of the benchmark case can be achieved in the original problem, which generalizes the one in Esö and Szentes [7] to the shifting-supports case.

Recall that we have not introduced the individual rationality (IR) conditions for the buyer. We refer to the following one as the IR constraints.

$$\pi(v,v) \geq 0, \forall v. \tag{8}$$

Notice that Proposition 2 itself does not imply that the benchmark case and the original problem are equivalent. It only states that *if* an allocation rule is implementable in the original problem, it can be done so without any revenue loss compared to the benchmark case. Recall that the revenue-maximizing allocation rule in the benchmark case is characterized in (4) of Corollary 1. Therefore, if the allocation rule $X^*(v,s)$ in (3) can be shown to be IC in the original problem, then by Proposition 2, the optimal revenue in Corollary 1 can also be achieved in the original problem. This is formally shown in the next proposition.

Proposition 3 *$X^*(v,s)$ in (3) is implementable in the original problem subject to (8). As a result, the highest revenue achievable in the original problem is also*

$$R^* = \int_{\underline{v}}^{\overline{v}}\int_0^1 \max\{W(v,s),0\}dsdF(v).$$

The critical step of the proof of the above proposition uses Lemma 3. We would like to point out that as mentioned right before Lemma 3, the verification of IC is crucial to establish our main result Proposition 3, which highly relies on the characterization of the off-equilibrium payoff $\pi(v,\hat{v})$

in Lemma 3 and its tractability in terms of comparison to the equilibrium payoff.

4 Concluding Remarks

In this paper, we study optimal mechanism design in the setting of Esö and Szentes [7], allowing shifting supports. Several notable issues arise, such as characterizing the optimal second-stage strategy following a first-stage lie, pinning down the off-equilibrium payoff (Lemma 3), and establishing the incentive compatibility of the candidate mechanism. We find that these issues can be overcome. As a result, the methodology and key insights in the literature can be extended to environments with shifting supports. Our findings, on the one hand, justify the adoption of common support in the literature as a convenient assumption; on the other hand, help to enlarge the scope of dynamic models which are technically tractable.

Courty and Li [5] provide another condition of shifting supports (see footnote 7 there): Supports of different types overlap sufficiently. Our paper complements their condition by allowing supports of different types to move smoothly. In particular, the supports of different types need not overlap in our paper; for example, the intersection of $[\underline{V}_{\underline{v}}, \overline{V}_{\underline{v}}]$ and $[\underline{V}_{\overline{v}}, \overline{V}_{\overline{v}}]$ can be empty.

As a final remark, there are other approaches to tackle the issue of shifting supports in Esö and Szentes [7]. For instance, one can first extend the supports of the second-stage type with respect to the first-stage type to a common one, so that the extended support is independent of the first-stage type—e.g., let the extended support be $[\underline{V}, \overline{V}] = \cup_{v\in[\underline{v},\overline{v}]}[\underline{V}_v, \overline{V}_v]$; define a suitable extended mechanism on such an extended support such that the agent will truthfully report his second-stage valuation even after a lie in the first stage;[13] and then apply the ideas and techniques in Esö and Szentes [7] and Pavan, Segal, and Toikka [15] to characterize the envelope condition and establish the sufficiency for incentive compatibility.[14]

5 Appendix

Proof of ii) in Corollary 2: There are two cases to consider. Case 1: $u(v, s) = u(\hat{v}, \hat{s})$, then it is exactly iii) in Corollary 1 in Esö and Szentes [7], so that $X^*(v, s) \geq X^*(\hat{v}, \hat{s})$. Case 2: $u(v, s) > u(\hat{v}, \hat{s})$. By Assumption 2, differentiability of $u(\cdot, \cdot)$, and $u(\hat{v}, s) \leq u(\hat{v}, \hat{s})$, there exists a

[13] The idea of extending the mechanism (in an IC way) is related to Skreta [16], who studies the static mechanism design problem where each agent's type space is a measurable subset of the real line (so the type space need not be convex or finite). The shifting-support case in our paper parallels the static model with "holes" in the type space (e.g., Skreta [16]). To see this, suppose that type v reported v' in the first stage and his second-stage true valuation is V. If $V \in [\underline{V}_{v'}, \overline{V}_{v'}]$, then obviously reporting V is his optimal second-stage report. If $V \notin [\underline{V}_{v'}, \overline{V}_{v'}]$, the agent cannot report V, as his report must be within $[\underline{V}_{v'}, \overline{V}_{v'}]$. However, one can define a hypothetical mechanism on the extended support $[\underline{V}, \overline{V}] = \cup_{v\in[\underline{v},\overline{v}]}[\underline{V}_v, \overline{V}_v]$ (i.e., the support is the common one $[\underline{V}, \overline{V}]$ for every first-stage type). This hypothetical mechanism is constructed in an IC way such that even if the agent lied in the first stage, it is still optimal for him to truthfully report his second-stage valuation; it also coincides with the original mechanism when restricting to the original type space. Extending the original mechanism (and the support) in such a way is similar to that for static mechanisms in Skreta [16].

[14] We are grateful to an anonymous reviewer who kindly pointed out this alternative approach.

$\tilde{v} \in [\hat{v}, v)$ such that $u(\tilde{v}, s) = u(\hat{v}, \hat{s})$. Case 1 implies that $X^*(\tilde{v}, s) \geq X^*(\hat{v}, \hat{s})$; i) of Corollary 2 implies that $X^*(v, s) \geq X^*(\tilde{v}, s)$. Therefore, $X^*(v, s) \geq X^*(\tilde{v}, s) \geq X^*(\hat{v}, \hat{s})$. $\square$

Proof of Lemma 2: Assume that the buyer with type v reported $\hat{v}$ in the first stage and observes signal s in the second stage. His goal is to choose an $\hat{s} \in [0, 1]$ to maximize his second-stage payoff $\tilde{\pi}(s, \hat{s}; v, \hat{v})$, i.e., $\max_{\hat{s} \in [0,1]} u(v, s)X(\hat{v}, \hat{s}) - T(\hat{v}, \hat{s})$. We focus on the case when $\hat{v} < v$. The other case $\hat{v} > v$ is similar. As for the case $\hat{v} < v$, we have two cases to consider.

Case 1: $u(\hat{v}, 1) \geq u(v, s)$. In this case, there exists a unique $\hat{s} \in [0, 1]$ such that $u(\hat{v}, \hat{s}) = u(v, s)$. Then Lemma 4 in Esö and Szentes [7] implies that reporting $\sigma(v, \hat{v}, s)$ is optimal.

Case 2: $u(\hat{v}, 1) < u(v, s)$. In this case, we will show that reporting $\hat{s} = 1$ is optimal. In fact, IC in the second stage requires that $\tilde{\pi}(s, s; v, v) \geq \tilde{\pi}(s, \hat{s}; v, v)$ for any $\hat{s} \in [0, 1]$, i.e.,

$$
\begin{aligned}
u(v, s)X(v, s) - T(v, s) &\geq u(v, s)X(v, \hat{s}) - T(v, \hat{s}) \\
&= [u(v, s) - u(v, \hat{s})]X(v, \hat{s}) + u(v, \hat{s})X(v, \hat{s}) - T(v, \hat{s}).
\end{aligned}
$$

That is equivalent to

$$
\tilde{\pi}(s, s; v, v) - \tilde{\pi}(\hat{s}, \hat{s}; v, v) \geq [u(v, s) - u(v, \hat{s})]X(v, \hat{s}). \tag{9}
$$

Note that

$$
\begin{aligned}
u(v, s)X(\hat{v}, \hat{s}) - T(\hat{v}, \hat{s}) &= [u(v, s) - u(\hat{v}, \hat{s})]X(\hat{v}, \hat{s}) + u(\hat{v}, \hat{s})X(\hat{v}, \hat{s}) - T(\hat{v}, \hat{s}) \\
&= [u(v, s) - u(\hat{v}, \hat{s})]X(\hat{v}, \hat{s}) + \tilde{\pi}(\hat{s}, \hat{s}; \hat{v}, \hat{v}).
\end{aligned}
$$

For any $\hat{s}' < \hat{s}'' \in [0, 1]$, recall from Lemma 1 that $X(\hat{v}, \hat{s}'') \geq X(\hat{v}, \hat{s}')$. Then,

$$
\begin{aligned}
& u(v, s)X(\hat{v}, \hat{s}'') - T(\hat{v}, \hat{s}'') - [u(v, s)X(\hat{v}, \hat{s}') - T(\hat{v}, \hat{s}')] \\
= \; & \underbrace{[u(v, s) - u(\hat{v}, \hat{s}'')]}_{>0 \text{ in case 2}} X(\hat{v}, \hat{s}'') + \tilde{\pi}(\hat{s}'', \hat{s}''; \hat{v}, \hat{v}) - [u(v, s) - u(\hat{v}, \hat{s}')]X(\hat{v}, \hat{s}') - \tilde{\pi}(\hat{s}', \hat{s}'; \hat{v}, \hat{v}) \\
\geq \; & [u(v, s) - u(\hat{v}, \hat{s}'')]X(\hat{v}, \hat{s}') + \tilde{\pi}(\hat{s}'', \hat{s}''; \hat{v}, \hat{v}) - [u(v, s) - u(\hat{v}, \hat{s}')]X(\hat{v}, \hat{s}') - \tilde{\pi}(\hat{s}', \hat{s}'; \hat{v}, \hat{v}) \\
= \; & -[u(\hat{v}, \hat{s}'') - u(\hat{v}, \hat{s}')]X(\hat{v}, \hat{s}') + \tilde{\pi}(\hat{s}'', \hat{s}''; \hat{v}, \hat{v}) - \tilde{\pi}(\hat{s}', \hat{s}'; \hat{v}, \hat{v}) \geq 0,
\end{aligned}
$$

where the last inequality follows from (9). Therefore, $u(v, s)X(\hat{v}, \hat{s}) - T(\hat{v}, \hat{s})$ is an increasing function in $\hat{s}$. This then implies that reporting $\hat{s} = 1$ is optimal. $\square$

Proof of Lemma 3: Fixing $\hat{v}$, there are two cases. Case 1: $\tilde{\sigma}(v, \hat{v}, s) = \sigma(v, \hat{v}, s)$. Take partial derivative of $\pi(v, \hat{v})$ with respect to v:[15]

$$\frac{\partial \pi(v,\hat{v})}{\partial v} = \int_0^1 \left[\begin{array}{c} u_1(v,s)X(\hat{v},\tilde{\sigma}(v,\hat{v},s)) \\ + \left(u(v,s)\frac{\partial X(\hat{v},\tilde{\sigma}(v,\hat{v},s))}{\partial \tilde{\sigma}(v,\hat{v},s)} - \frac{\partial T(\hat{v},\tilde{\sigma}(v,\hat{v},s))}{\partial \tilde{\sigma}(v,\hat{v},s)}\right) \frac{\partial \tilde{\sigma}(v,\hat{v},s)}{\partial v} \end{array} \right] ds.$$

Since $\tilde{\sigma}(v,\hat{v},s) = \sigma(v,\hat{v},s)$, it is an interior solution, which implies

$$u(v,s)\frac{\partial X(\hat{v},\tilde{\sigma}(v,\hat{v},s))}{\partial \tilde{\sigma}(v,\hat{v},s)} - \frac{\partial T(\hat{v},\tilde{\sigma}(v,\hat{v},s))}{\partial \tilde{\sigma}(v,\hat{v},s)} = 0.$$

Case 2: $\tilde{\sigma}(v,\hat{v},s) = 0$ or 1 (corner solution). Take $\tilde{\sigma}(v,\hat{v},s) = 1$ as an example; the other case is similar. In this case,

$$\pi(v,\hat{v}) = \int_0^1 [u(v,s)X(\hat{v},1) - T(\hat{v},1)]ds,$$

so

$$\frac{\partial \pi(v,\hat{v})}{\partial v} = \int_0^1 [u_1(v,s)X(\hat{v},1) - T(\hat{v},1)]ds.$$

Hence, in both cases,

$$\frac{\partial \pi(v,\hat{v})}{\partial v} = \int_0^1 u_1(v,s)X(\hat{v},\tilde{\sigma}(v,\hat{v},s))ds,$$

which further implies Lemma 3. □

Proof of Proposition 3: One only needs to show that $X^*(v,s)$ is implementable in the original problem. In fact, construct the payment rule using (5) and (6) as follows:

$$T^*(v,s) = u(v,s)X^*(v,s) - \int_0^s u_2(v,z)X^*(v,z)dz - \tilde{\pi}^*(0,0;v),$$

where

$$\tilde{\pi}^*(0,0;v) = \int_{\underline{v}}^v \int_0^1 u_1(y,s)X^*(y,s)dsdy - \int_0^1 \int_0^s u_2(v,z)X^*(v,z)dzds.$$

One is easy to verify that under $\{X^*(v,s), T^*(v,s)\}$,

$$\tilde{\pi}(s,s;v) = \tilde{\pi}^*(0,0;v) + \int_0^s u_2(v,z)X^*(v,z)dz, \tag{10}$$

$$\pi(v,v) = \underbrace{\pi(\underline{v},\underline{v})}_{=0} + \int_{\underline{v}}^v \int_0^1 u_1(y,s)X^*(y,s)dsdy. \tag{11}$$

Now what is left is to check that $\{X^*(v,s), T^*(v,s)\}$ is IC in the original problem. By Corollary 2, $X^*(v,s)$ is increasing in s. Together with (10), the second-stage IC follows from Lemma 1.

Checking first-stage IC is a bit involved. To this end, suppose that the buyer with type v reports

[15] The (almost everywhere) differentiability of X and T in $\tilde{\sigma}$ follows from the second-stage IC.

$\hat{v} < v$ at the first stage. The case when he reports a type higher than v is similar. Our goal is to show that $\pi(v, \hat{v}) - \pi(v, v) \leq 0$. In fact,

$$\pi(v, \hat{v}) - \pi(v, v) = [\pi(v, \hat{v}) - \pi(\hat{v}, \hat{v})] - [\pi(v, v) - \pi(\hat{v}, \hat{v})]. \tag{12}$$

Recall from Lemma 3 that

$$\pi(v, \hat{v}) = \pi(\hat{v}, \hat{v}) + \int_0^1 \int_{\hat{v}}^v u_1(y, s) X^*(\hat{v}, \tilde{\sigma}(y, \hat{v}, s)) dy ds.$$

Together with (11), the RHS of (12) can be rewritten as

$$\int_0^1 \int_{\hat{v}}^v u_1(y, s) X^*(\hat{v}, \tilde{\sigma}(y, \hat{v}, s)) dy ds - \int_0^1 \int_{\hat{v}}^v u_1(y, s) X^*(y, s) dy ds. \tag{13}$$

For any $s \in [0, 1]$ and any $y \in [\hat{v}, v]$, we have $\tilde{\sigma}(y, \hat{v}, s) \geq s$ and $u(y, s) \geq u(\hat{v}, \tilde{\sigma}(y, \hat{v}, s))$. ii) of Corollary 2 then implies

$$X^*(\hat{v}, \sigma(y, \hat{v}, s)) \leq X^*(y, s), \ \forall y \in [\hat{v}, v], \forall s \in [0, 1],$$

which further implies that (13) is non-positive. In other words, $\pi(v, \hat{v}) - \pi(v, v) \leq 0$, so that the first-stage IC holds.

We have shown above that $\{X^*(v, s), T^*(v, s)\}$ is IC in the original problem. Obviously, it also satisfies the IR constraint (8). This then immediately implies that the highest revenue is R^* in Proposition 3. □

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